Computational Finance, Fall 2016
Computer Lab 11

The aim of the lab is to implement the basic implicit method for computing European option prices. For this we consider the problem

$$\begin{aligned}
\frac{\partial u}{\partial t}(x,t) + \alpha(x,t)\frac{\partial^2 u}{\partial x^2}(x,t) + \beta(x,t)\frac{\partial u}{\partial x}(x,t) - r\,u(x,t) &= 0,\ x \in (x_{min}, x_{max}), 0 \le t < T,\\
u(x_{min}, t) &= \phi_1(t), 0 \le t < T,\\
u(x_{max}, t) &= \phi_2(t), 0 \le t < T,\\
u(x,T) &= p(e^x),\ x \in (x_{min}, x_{max}).
\end{aligned}$$

We introduce the points $x_i = x_{min} + i\Delta x,\ i = 0,\dots,n$ and $t_k = k\Delta t,\ k = 0,\dots,m$ and denote by $U_{i,k}$ the approximate values of $u(x_i, t_k)$. Here $\Delta x = \frac{x_{max} - x_{min}}{n}$ and $\Delta t = \frac{T}{m}$. In the case of the basic implicit finite difference method we compute the values U_{ik} as follows: using the final condition we set

$$U_{im} = p(e^{x_i}),\ i = 0,\dots,n$$

and for determining the values of $U_{ik},\ i = 0,\dots,n,\ k = m-1,\dots,0$ we solve for each value of k (starting with $k = m-1$) a three-diagonal system

$$\begin{aligned}
U_{0k} &= \phi_1(t_k),\\
a_{ik}U_{i-1,k} + b_{ik}U_{ik} + c_{ik}U_{i+1,k} &= U_{i,k+1},\ i = 1,\dots,n-1,\\
U_{nk} &= \phi_2(t_k)
\end{aligned}$$

for the unknown values of $U_{ik},\ i = 0,\dots,n$. Here

$$\begin{aligned}
a_{ik} &= -\frac{\alpha(x_i,\tau_k)\Delta t}{\Delta x^2} + \frac{\beta(x_i,t_k)\Delta t}{2\Delta x},\\
b_{ik} &= 1 + \frac{2\alpha(x_i,t_k)\Delta t}{\Delta x^2} + r\Delta t,\\
c_{ik} &= -\frac{\alpha(x_i,t_k)\Delta t}{\Delta x^2} - \frac{\beta(x_i,t_k)\Delta t}{2\Delta x}.
\end{aligned}$$

Exercise 1. Write a function that for given values of m, n, x_{min}, x_{max}, T and for given functions p,σ, ϕ_1 and ϕ_2 returns the values $U_{i0},\ i = 0,\dots,n$ of the approximate solution (option prices) obtained by solving the transformed BS equation by the implicit finite difference method. Use this method for computing approximate values of the option price in the case $r = 0.02$, $\sigma(s,t) = 0.5,\ D = 0.03,\ T = 0.5,\ E = 100,\ S0 = 98,\ p(s) = 2 \cdot |E - s|,\ \rho = 2,\ x_{min} = \ln\frac{S0}{\rho},\ x_{max} = \ln(\rho\, S0),\ n = 20,\ m = 100$. Use $\phi_1(t) = p(e^{x_{min}}),\ \phi_2(t) = p(e^{x_{max}})$

Exercise 2. Repeat the previous exercise in the case of boundary conditions derived from special solutions.

Practical Homework 5. (Deadline 16.11.2016) When we use a finite difference method for option pricing, there are two sources of errors: 1) truncation error, caused by introduction of x_{min} and x_{max} and boundary conditions; 2) discretisation error, caused by approximating partial derivatives with finite difference approximations. In this homework we consider a possibility to estimate the discretisation error. Namely, when we derived explicit and basic implicit methods, we saw that discretisation error is $O(\Delta t + \Delta x^2)$. This estimate can be made more precise: for large enough m and n the discretisation error behaves like *const.* $\cdot\ (O(\Delta t + \Delta x^2))$ if we consider error at a specific point (x,t). Although we do not know the constant *const.*, this enables us to say that if we multiply m by 4 and n by 2, the discretisation error gets approximately 4 times smaller and therefore, by computing answer1 with $m = m_0,\ n = n_0$ and answer2 by $m = 4\dot{m}_0, n = 2\dot{n}_0$, we can estimate the discretisation error of the last result by computing $(answer1 - answer2)/3$ (detailed explanations for that will be given in the lecture). In the case of explicit method, if m is computed from the stability condition, it is automatically multiplied by a number that is close to 4 whenever we multiply n by two, so we can use the same estimate. This is called Runge's error estimate and is widely used by practitioners.

Use Runge's estimate to compute option prices in the case $r = 0.02$, $\sigma(s,t) = 0.6 - \frac{0.2}{1+0.02(s-50)^2}$, $D = 0.03$, $T = 0.5$, $S0 = 45$, $\rho = 2$, $x_{min} = \ln\frac{S0}{\rho}$, $x_{max} = \ln(\rho\, S0)$, the pay-off function

$$p(s) = \begin{cases} 0, & s \leq 40, \\ 0.01 \cdot (s-40)^2, & 40 < s \leq 50, \\ 2 \cdot (s-45), & s > 50 \end{cases}$$

and boundary conditions corresponding to suitable special solutions so that discretisation error is less than 0.02 by both the explicit method and the basic implicit method, starting with $n = 10$ and, in the case of basic implicit method, $m = 5$.