

Computational Finance, Fall 2016

Computer Lab 15

The aim of the lab is to learn to compute prices of Asian options.

Let us assume that the volatility σ in the Black-Scholes market model depends only on the current stock price S .

Let $v(s, I, t)$ be the function giving the price of an Asian option (depending on arithmetic average) with exercise time T and payoff $p(s, A_T)$. For finding approximate option prices we introduce artificial boundaries I_{max}, S_{max} , choose natural numbers n_s, n_I, m and look for approximate values of v at points (s_i, I_j, t_k) , where

$$s_i = i\Delta s = i\frac{S_{max}}{n_s}, \quad I_j = j\Delta I = j\frac{I_{max}}{n_I}, \quad t_k = k\Delta t = k\frac{T}{m}.$$

Denote those approximate values by V_{ijk} . A finite difference approximation gives the following equations for the unknown values:

$$a_i V_{i-1,jk} + b_i V_{ijk} + c_i V_{i+1,jk} = \frac{s_i \Delta t}{\Delta I} V_{i,j+1,k} + V_{i,j,k+1},$$

where $i = 1, 2, \dots, n_s - 1$, $j = 0, 1, \dots, n_I - 1$, $k = 0, \dots, m - 1$ and (using the notation $\rho = \frac{\Delta t}{\Delta s^2}$)

$$\begin{aligned} a_i &= \frac{\rho}{2} (-s_i^2 \sigma^2(s_i) + (r - D)s_i \Delta s), \\ b_i &= \left(1 + \rho s_i^2 \sigma^2(s_i) + \frac{s_i \Delta t}{\Delta I} + r \Delta t \right), \\ c_i &= -\frac{\rho}{2} (s_i^2 \sigma^2(s_i) + (r - D)s_i \Delta s). \end{aligned}$$

The equations together with equations for V_{0jk} and $V_{n_s,j,k}$ (specified below) can be viewed as a three-diagonal system for finding the values of V_{ijk} , $i = 0, \dots, n_s$, if the values corresponding to the level $t = t_{k+1}$ and the values $V_{i,j+1,k}$, $i = 0, \dots, n_s$ have been found earlier. So we solve the system in backward direction with respect to k (for $k = m - 1, m - 2, \dots, 0$) for each k , we solve it for $j = n_I - 1, n_I - 2, \dots, 0$.

The values of V_{ijm} can be found from the final condition:

$$V_{ijm} = p(s_i, \frac{I_j}{T}), \quad i = 0, \dots, n_s, \quad j = 0, \dots, n_I.$$

At the boundary $S = 0$ we have the exact boundary condition $v(0, I, t) = p(0, \frac{I}{T})e^{-r(T-t)}$, hence

$$V_{0jk} = p(0, \frac{I_j}{T})e^{-r(T-t_k)}, \quad k = 0, \dots, m - 1.$$

In order to determine all values of V_{ijk} uniquely, we have to specify boundary conditions for the boundary $I = I_{max}$ and the boundary $s = S_{max}$. Denote by $\phi_1(s, t)$ and $\phi_2(I, t)$ the functions describing the boundary conditions, then

$$V_{i,n_I,k} = \phi_1(s_i, t_k), \quad V_{n_s,j,k} = \phi_2(I_j, t_k).$$

Exercise Let us consider an average price put option (payoff function $p(s, A_T) = \max(E - A_T, 0)$, where E is the exercise price specified in the option contract). Assume $r = 0.05$, $D = 0$, $\sigma = 0.5$, $T = 0.5$, $E = 100$, $S(0) = 95$. Define $I_{max} = ET$, $S_{max} = 190$, $\phi_1(s, t) = 0$,

$$\phi_2(I, t) = \max\{Ee^{-r(T-t)} + \frac{e^{-D(T-t)}}{(r-D)T}(e^{-(r-D)(T-t)} - 1)S_{max} - \frac{e^{-r(T-t)}}{T}I, 0\}.$$

Write a function, that for given values n_s, n_I, m finds an approximate option price at $t = 0$ using the finite difference method described above.

Let us discuss one possibility to specify suitable artificial boundary conditions ϕ_1 and ϕ_2 . Recall that the functions of the form

$$v(s, I, t) = C_1 e^{-r(T-t)} + e^{-D(T-t)} \left(C_2 - C_3 \frac{e^{-(r-D)(T-t)}}{r-D} \right) s + C_3 e^{-r(T-t)} I$$

are solutions of the Asian option pricing PDE in the case $r \neq D$; if $r = D$ then corresponding special solutions are

$$v(s, I, t) = C_1 e^{-r(T-t)} + e^{-r(T-t)} (C_2 + C_3(T-t)) s + C_3 e^{-r(T-t)} I.$$

Consider payoff functions of the form

$$p(s, a) = \max\{k_1 + k_2 s + k_3 a, 0\}.$$

Let v_{spec} be the special solution satisfying $v_{spec}(s, I, T) = k_1 + k_2 s + k_3 I/T$ then we define

$$\phi_1(s, t) = \max\{v_{spec}(s, I_{max}, t), 0\}$$

and

$$\phi_2(I, t) = \max\{v_{spec}(S_{max}, I, t), 0\}.$$

Practical Homework 7 (Deadline December 14, 2016) Let us consider the Asian option with the exercise time $T = 0.5$ and payoff function

$$p(s, a) = \max(0.5s - a + 20, 0),$$

where the average is the arithmetic average over the interval $[0, 0.5]$. Assume $r = 0.02$, $D = 0$, $\sigma(s) = 0.6$, $T = 0.5$, $S(0) = 45$. Define $I_{max} = 100$, $S_{max} = 180$. Use a single computation to find an approximate value of the option (set up at time $t = 0$) at time $t = 0$ and at time $t = 0.15$ if $S(0.15) = 50$ and $\int_0^{0.15} S(t) dt = 15$ by using the finite difference method described in this Lab with $n_s = 72$, $n_I = 50$, $m = 100$.