Computational Finance, Fall 2016
Computer Lab 6

The aim of the Lab is to learn to apply Monte-Carlo method for computing option prices.

Often it can be shown (or it is assumed in the case of certain market model) that the price of an European option can be expressed as the expected value

$$V = E[e^{-rT} p(S(T))],$$

where $S(T)$ is generated according to a certain stochastic differential equation. In such case we can compute V approximately by generating n values of the random variable $S(T)$: $S(T)_1, S(T)_2, \ldots, S(T)_n$ and computing the arithmetic average of the function under the expectation:

$$V \approx \bar{V}_n = \frac{e^{-rT}}{n} \sum_{i=1}^{n} p(S(T)_i).$$

From Central Limit Theorem it follows that

$$P\left(|V - \bar{V}_n| \leq \frac{-\Phi^{-1}(\frac{\alpha}{2})\mathrm{std}(Y)}{\sqrt{n}}\right) \approx 1 - \alpha$$

for large values of n. Here Φ is the cumulative distribution function of the standard normal distribution and $Y = e^{-rT}p(S(T))$

Exercise 1. If we assume that the Black-Scholes market model with constant volatility holds, then we have to generate $S(T)$ according to the stochastic differential equation

$$dS(t) = S(t)((r - D)\, dt + \sigma\, dB(t).$$

From the lecture we know that the solution of the equation is

$$S(t) = S(0)e^{(r-D-\frac{\sigma^2}{2})t+\sigma B(t)},$$

so $S(T) = S(0)e^{(r-D-\frac{\sigma^2}{2})T+\sigma X}$, where $X \sim N(0, \sqrt{T})$. Write a function MC1, that for given values of $S(0), r, D, \sigma, T, \alpha$ and n and for given payoff function p computes an approximate option value and its error estimate holding with the probability $(1 - \alpha)$ by Monte-Carlo method, using n generated stock prices. Verify the correctness of the function by Black-Scholes formulas for put and call options in the case $S(0) = 100$, $E = 100$, $\sigma = 0.6$, $T = 0.5$, $r = 0.02$, $D = 0.03$, $\alpha = 0.05$ and $n = 10000$. How often the actual error is larger than the error estimate if you use MC1 1000 times?

Very often it is not possible to generate $S(T)$ values that correspond exactly to the stochastic differential equation; then it is necessary to use some approximation methods. One such method is the Euler's method, where we divide the interval $[0, T]$ into m equal subintervals and use the approximations (int the case of Black-Scholes market model)

$$S_{i+1} = S_i(1 + (r - D)\, \Delta t + \sigma(S_i, t_i)X_i), i = 0, \ldots, m - 1,$$

where S_i are approximations to $S(i\Delta t)$, $\Delta t = \frac{T}{m}$ and $X_i \sim N(0, \sqrt{\Delta t})$. Instead of $S(T)$ we use S_m, thus we use Monte-Carlo method to compute an approximate value of $\hat{V}$, where

$$\hat{V}_m = E[e^{-rT} p(S_m)].$$

Since S_m for a fixed m does not have exactly the same distribution as $S(T)$, we have in general $\hat{V}_m \neq V$ and therefore Monte-Carlo method converges to a value that is different from the option price.

Exercise 2. Write a function MC2 that computes approximate option prices so that the stock prices are generated according to Euler's method. Determine how large is the difference between $\hat{V}_m$ and the correct option price in the case of European call option, using the same parameters as in the previous exercise for $m = 2, 4, 8, 16$. In order to see the difference, large enough value for n should be used (if possible, the corresponding MC error should be at least 5 times smaller than the computed difference).

It is known that if p is continuous and has bounded first derivative (ie it is Lipshitz continuous), then

$$|V - \hat{V}_m| = \frac{C}{m} + o(\frac{1}{m}),$$

where C is a constant that does not depend on m and $m \cdot o(\frac{1}{m}) \to 0$ as $m \to \infty$. Thus, if we use S_m instead of $S(T)$ and use Monte-Carlo method to compute approximate values $\bar{V}_{m,n}$ of $\hat{V}_m$, then the total error can be estimated as

$$|V - \bar{V}_{m,n}| \leq |V - \hat{V}_m| + |\hat{V}_m - \bar{V}_{m,n}| \leq \frac{C}{m} + o(\frac{1}{m}) + |\hat{V}_m - \bar{V}_{m,n}|.$$

The last term is the error of the Monte-Carlo method and can be estimated easily. So, in order to compute the option price V with a given error ε, we should choose large enough m (so that the absolute value of the term $\frac{C}{m}$ is small enough, for example less than $\frac{\varepsilon}{2}$) and then use MC method with large enough n so that the MC error estimate is also small enough (less than $\frac{\varepsilon}{2}$). There is one trouble: we do not know C. One possibility to estimate C is as follow:

1. Choose some values for m_0, n_0 for m and n. They should not be too small, but very large values take too much computation time. Usually reasonable values for m_0 start from 10 and reasonable values for n_0 start from 100000.
2. Use MC method twice to compute $\bar{V}_{m_0,n_0}$ and $\bar{V}_{2m_0,n_0}$
3. Estimate the value of C: if m_0 is large enough, then

   $$|C| \leq \bar{C} = 2m_0 \cdot (|\bar{V}_{m_0,n_0} - \bar{V}_{2m_0,n_0}| + |\hat{V}_{m_0} - \bar{V}_{m_0,n_0}| + |\hat{V}_{2m_0} - \bar{V}_{2m_0,n_0}|).$$

   The last two terms are errors of the MC method, for which we have estimates available.
4. Choose m_1 such that $\frac{\bar{C}}{m_1} \leq \frac{\varepsilon}{2}$ and n_1 such that MC error of $\bar{V}_{m_1,n_1}$ is less than $\frac{\varepsilon}{2}$. Then we assume that $\bar{V}_{m_1,n_1}$ is an approximation of the true option price which satisfies the desired error estimate.

In order to get reasonable estimates for the right value of m, the value of n_0 should be such that the difference of the values of the two first computations is larger than the sum of the corresponding MC errors. In n_0 is too small, then we overestimate the value of m_1 and the final computation may take too much time.

Exercise 3. Apply the previous procedure for computing the option price with MC2 so that the total error is less than 0.06 with probability 0.9. Compare your answer with the exact option price. Did you get the answer with the required accuracy?