Computational Finance, Fall 2017
Computer Lab 15

The aim of the lab is to learn to compute prices of Asian options.

Let us assume that the volatility σ in the Black-Scholes market model depends only on the current stock price S.

Let $v(s, I, t)$ be the function giving the price of an Asian option (depending on arithmetic average) with exercise time T and payoff $p(s, A_T)$. For finding approximate option prices we introduce artificial boundaries I_{max}, S_{max}, choose natural numbers n_s, n_I, m and look for approximate values of v at points (s_i, I_j, t_k), where

$$s_i = i\Delta s = i\frac{S_{max}}{n_s},\ I_j = j\Delta I = j\frac{I_{max}}{n_I},\ t_k = k\Delta t = k\frac{T}{m}.$$

Denote those approximate values by V_{ijk}. A finite difference approximation gives the following equations for the unknown values:

$$a_i V_{i-1,jk} + b_i V_{ijk} + c_i V_{i+1,jk} = \frac{s_i \Delta t}{\Delta I} V_{i,j+1,k} + V_{ij,k+1},$$

where $i = 1, 2, \ldots, n_s - 1,\ j = 0, 1, \ldots, n_I - 1,\ k = 0, \ldots, m-1$ and (using the notation $\rho = \frac{\Delta t}{\Delta s^2}$)

$$a_i = \frac{\rho}{2}(-s_i^2\sigma^2(s_i) + (r - D)s_i\Delta s),$$

$$b_i = \left(1 + \rho s_i^2\sigma^2(s_i) + \frac{s_i\Delta t}{\Delta I} + r\Delta t\right),$$

$$c_i = -\frac{\rho}{2}(s_i^2\sigma^2(s_i) + (r - D)s_i\Delta s).$$

The equations together with equations for V_{0jk} and $V_{n_s,j,k}$ (specified below) can be viewed as a three-diagonal system for finding the values of $V_{ijk},\ i = 0, \ldots, n_s$, if the values corresponding to the level $t = t_{k+1}$ and the values $V_{i,j+1,k},\ i = 0, \ldots, n_s$ have been found earlier. So we solve the system in backward direction with respect to k (for $k = m-1, m-2, ..., 0$) for each k, we solve it for $j = n_I - 1, n_I - 2, ..., 0$.

The values of V_{ijm} can be found from the final condition:

$$V_{ijm} = p(s_i, \frac{I_j}{T}),\ i = 0, \ldots, n_s,\ j = 0, \ldots, n_I.$$

At the boundary $S = 0$ we have the exact boundary condition $v(0, I, t) = p(0, \frac{I}{T})e^{-r\,(T-t)}$, hence

$$V_{0jk} = p(0, \frac{I_j}{T})e^{-r(T-t_k)},\ k = 0, \ldots, m-1.$$

In order to determine all values of V_{ijk} uniquely, we have to specify boundary conditions for the boundary $I = I_{max}$ and the boundary $s = S_{max}$. Denote by $\phi_1(s, t)$ and $\phi_2(I, t)$ the functions describing the boundary conditions, then

$$V_{i,n_I,k} = \phi_1(s_i, t_k),\ V_{n_s,j,k} = \phi_2(I_j, t_k).$$

Exercise Let us consider an average price put option (payoff function $p(s, A_T) = \max(E - A_T, 0)$, where E is the exercise price specified in the option contract). Assume $r = 0.05$, $D = 0$, $\sigma = 0.5$, $T = 0.5$, $E = 100$, $S(0) = 95$. Define $I_{max} = ET$, $S_{max} = 190$, $\phi_1(s, t) = 0$,

$$\phi_2(I, t) = \max\{Ee^{-r(T-t)} + \frac{e^{-D(T-t)}}{(r - D)T}(e^{-(r-D)(T-t)} - 1)S_{max} - \frac{e^{-r(T-t)}}{T}I, 0\}.$$

Write a function, that for given values n_s, n_I, m finds an approximate option price at $t = 0$ using the finite difference method described above.

Let us discuss one possibility to specify suitable artificial boundary conditions ϕ_1 and ϕ_2. Recall that the functions of the form

$$v(s, I, t) = C_1 e^{-r(T-t)} + e^{-D(T-t)} \left(C_2 - C_3 \frac{e^{-(r-D)(T-t)}}{r - D} \right) s + C_3 \, e^{-r(T-t)} I$$

are solutions of the Asian option pricing PDE in the case $r \neq D$; if $r = D$ then corresponding special solutions are

$$v(s, I, t) = C_1 e^{-r(T-t)} + e^{-r(T-t)} \left(C_2 + C_3(T - t) \right) s + C_3 \, e^{-r(T-t)} I.$$

Consider payoff functions of the form

$$p(s, a) = \max\{k_1 + k_2 s + k_3 a, 0\}.$$

Let v_{spec} be the special solution satisfying $v_{spec}(s, I, T) = k_1 + k_2 s + k_3 I/T$ then we define

$$\phi_1(s, t)) = \max\{v_{spec}(s, I_{max}, t), 0\}$$

and

$$\phi_2(I, t)) = \max\{v_{spec}(S_{max}, I, t), 0\}.$$

Practical Homework 7 (Deadline December 20, 2017) Let us consider the Asian option with the exercise time $T = 0.6$ and payoff function

$$p(s, a) = \max(10 - 2s + 1.5a, 0),$$

so that the owner gets the payment $p(S(T), A_T)$ at the exercise time $T = 0.6$, where the average A_T is the arithmetic average over the interval $[0, 0.6]$. Assume $r = 0.01$, $D = 0.02$, $\sigma(s) = 0.6$, $S(0) = 55$. Define $I_{max} = 100$, $S_{max} = 165$. Find an approximate value of the option at time $t = 0$ by using the finite difference method described in this Lab with $n_s = 90$, $n_I = 50$, $m = 100$.