Computational Finance, Fall 2017
Computer Lab 3

The aim of the Lab is to learn to simulate the paths of solutions of stochastic differential equations corresponding to common stock market models.

Sometimes (especially for applying Monte-Carlo methods) it is important to know how to simulate the stock price trajectories corresponding to a market model. A simple and quite universal (but often not the best) way to generate the trajectories of solutions of stochastic differential equations is Euler-Maruyama method, where differentials are replaced by differences over small time intervals (t_{i-1}, t_i) and all other values on the right hand side are taken at the time moment t_{i-1}. The same idea applies if we have one equation or many equations.

For Black-Scholes market model

$$dS(t) = S(t)\big(\,\mu(t)\,dt + \sigma(S(t),t)\,dB(t)\,\big)$$

this leads to an approximation

$$S(t_i) - S(t_{i-1}) \approx S(t_{i-1})\big(\,\mu(t_{i-1})\,h_i + \sigma(S(t_{i-1}),t_{i-1})\,(B(t_i) - B(t_{i-1}))\,\big),$$

where $0 = t_0 < t_1 < \ldots < t_m = T$ is a partition of the interval $[0,T]$ into (usually equal) subintervals and $h_i = t_i - t_{i-1}$. If the time intervals are equal, we can use the single value $h = \frac{T}{m}$ instead of h_i. Using this approximation, the knowledge that $B(t_i) - B(t_{i-1}) \sim N(0, \sqrt{h_i})$ and a given value of $S_0 = S(0)$ we can compute approximate values $S_1, S_2, \ldots, S_m$ of $S(t_1), S(t_2), \ldots, S(t_m)$ by

$$S_i = S_{i-1}\left(1 + \mu(t_{i-1})\,h_i + \sigma(S_{i-1}, t_{i-1})\sqrt{h_i}X_i\right),\ i = 1, \ldots, m,$$

where X_i are independent random variables from the standard normal distribution.

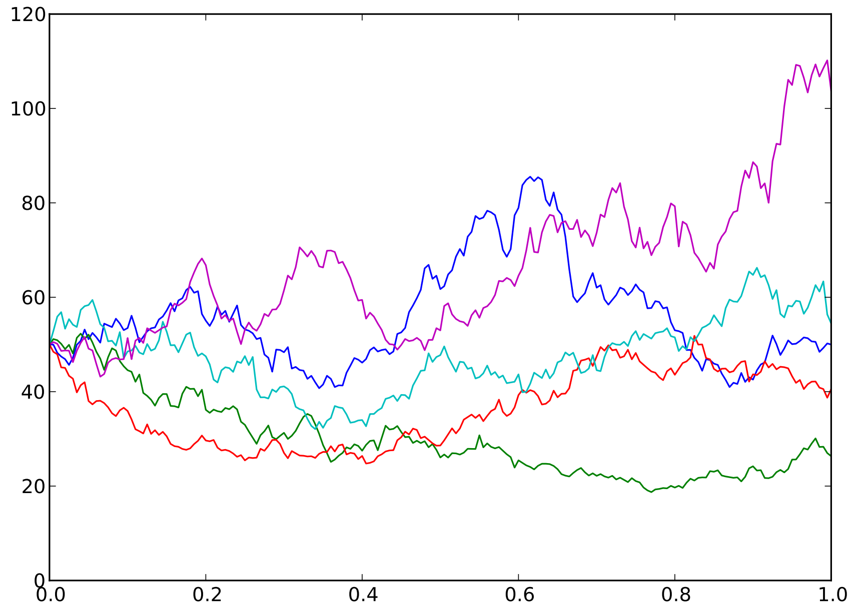
Exercise 1. Write a function `BSgraph(S0,n,m,mu,sigma,T)` that plots the graph of n trajectories of the stock price on the interval $[0,T]$, corresponding to the Black-Scholes market model with constant parameters μ and σ. For computing the values of the stock prices divide the interval $[0,T]$ into m equal subintervals (ie. use the time points $t_i = \frac{i \cdot T}{m},\ i = 0, 1, \ldots, m$) and use the Euler's method. This is again a good example when it is possible to used vectorized commands to make the code shorter (and to speed up computations). Define a $(m+1) \times n$ matrix S to store the values of the stock prices, in each column the values of a different trajectory and in i-th row the values corresponding to the time moment $t_i,\ i = 0, 1, 2, \ldots, m$. All the trajectories start from the value S_0 that is given in the parameter $S0$ of the function, so we can write tha value in the 0-th row of the matrix by using the command `S[0,:]=S0`. Now, according to the Euler-Maruyama method we compute the value of each trajectori at the next time moment by using the values of the stock prices at the previous time moment and a different random number for each trajectory. We can generate a vector of different random numbers for each trajectory by the command `np.random.randn(n)` and so for each $i = 1, 2, ..., m$ we can compute the i-th of the matrix with a single line of the code by `S[i,:] =S[i-1,:]*(1+`.... We have to do this step by step (since the values of the previous row have to be computed before we can compute the next one, so using a for cycle over the values of i is necessary.

Since this function does not have to return a value (it draws the graph instead), there does not have to be any return commands at the end of the function. So the function should end with `pl.plot(...)` and `pl.show()` commands (assuming the package pylab has been imported with alias pl).

If the code is written correctly, then by entering the command

```
BSgraph(S0=50,n=5,m=200,mu=0.1,sigma=0.5,T=1)
```

a picture similar to the following shoud be generated:



Exercise 2. In the case of pricing European options by Monte-Carlo methods, we do not want to look at the trajectories of the stock prices but need only to generate values of $S(T)$. Define a function `ST(S0,n,m,mu,sigma,T)` that returns a vector of n randomly generated values of $S(T)$ according to BS market model with constant μ and non-constant volatility given by a function $\sigma(s,t)$. Compute the mean value and standard deviation of 100000 generated stock prices at time $t=T$ in the case $\mu=0.05$, $m=100$, $S_0=10$, $T=1$ and $\sigma(s,t)=\frac{e^{-0.1\cdot t}}{1+0.005s^2}$. (For checking answers: mean and standard deviation should be approximately 10.5 and 6.1)

Exercise 3. Often we have several stochastic processes in a market model. Let us consider a model with stochastic interest rate:

$$\begin{aligned} dS(t) &= S(t)(r(t)\,dt + 0.5dB_1(t)), \\ dr(t) &= (0.05 - r(t))\,dt + 0.02\,dB_2(t), \end{aligned}$$

where B_1 and B_2 are independent Brownian motions. Use Euler-Maruyama method for defining a function that for given n and m outputs n generated values of $S(0.5)$ in the case $S(0)=100,\ r(0)=0.04$.

Homework problem 1. (Deadline September 28, 2017) It is well known that the Black-Scholes model is not perfect and that, if a the model holds, then the volatility can not be constant for most stocks. A possible alternative is to allow the variance of the stock price changes to be stochastic, too. Let us consider the model

$$\begin{aligned} dS(t) &= S(t)(0.05\,dt + \sqrt{V(t)}\,dB_1(t)), \\ dV(t) &= 0.5\cdot(0.25 - V(t))\,dt + 0.1V(t)\,dB_2(t), \end{aligned}$$

where B_1 and B_2 are independent Brownian motions. Define a function `Sgen` that for given m and n generates n stock prices $S(T)$ in the case $T=0.5$, $S(0)=40$, $V(0)=0.3$ by using Euler-Maruyama method with m time steps. Using the function with $m=40, n=100000$, compute approximately the expected value of $\sqrt{|S(T)-35|}$.