

Computational Finance, Fall 2017

Computer Lab 9

If we consider an European option with exercise time T and payoff function p and assume the validity of Black-Scholes market model, then the option price at time t is given by $v(S(t), t) = u(\ln(S(t)), t)$, where u is the solution of the problem

$$\frac{\partial u}{\partial t}(x, t) + \alpha(x, t) \frac{\partial^2 u}{\partial x^2}(x, t) + \beta(x, t) \frac{\partial u}{\partial x}(x, t) - r u(x, t) = 0, \quad x \in \mathbf{R}, 0 \leq t < T \quad (1)$$

satisfying the final condition

$$u(x, T) = p(e^x), \quad x \in \mathbf{R}.$$

Here

$$\alpha(x, t) = \frac{\sigma^2(e^x, t)}{2},$$

$$\beta(x, t) = r - D - \frac{\sigma^2(e^x, t)}{2}.$$

For solving the equation for u numerically, we introduce two boundaries x_{min} and x_{max} and specify boundary conditions $u(x_{min}, t) = \phi_1(t)$, $u(x_{max}, t) = \phi_2(t)$ at those points. Next, we introduce the points $x_i = x_{min} + i\Delta x$, $i = 0, \dots, n$ and $t_k = k\Delta t$, $k = 0, \dots, m$ and denote by U_{ik} approximate values of $u(x_i, t_k)$. Here $\Delta x = \frac{x_{max} - x_{min}}{n}$ and $\Delta t = \frac{T}{m}$. In the case of the explicit finite difference method we compute the values U_{ik} as follows:

$$U_{im} = p(e^{x_i}), \quad i = 0, \dots, n$$

$$U_{0,k-1} = \phi_1(t_{k-1}), \quad U_{n,k-1} = \phi_2(t_{k-1}), \quad k = m, m-1, \dots, 1,$$

$$U_{i,k-1} = a_{ik}U_{i-1,k} + b_{ik}U_{ik} + c_{ik}U_{i+1,k}, \quad i = 1, \dots, n-1, \quad k = m, m-1, \dots, 1,$$

where

$$a_{ik} = \frac{\Delta t}{\Delta x^2} \left(\alpha_{ik} - \frac{\beta_{ik}}{2} \Delta x \right),$$

$$b_{ik} = 1 - 2 \frac{\Delta t}{\Delta x^2} \alpha_{ik} - r \Delta t,$$

$$c_{ik} = \frac{\Delta t}{\Delta x^2} \left(\alpha_{ik} + \frac{\beta_{ik}}{2} \Delta x \right).$$

If σ is a constant, then the coefficients a, b and c are also constants and the numerical scheme simplifies to

$$U_{i,k-1} = a U_{i-1,k} + b U_{ik} + c U_{i+1,k}, \quad i = 1, \dots, n-1, \quad k = m, m-1, \dots, 1.$$

The stability condition is in this case $b \geq 0$.

Exercise 1. Write a function that for given values of $n, \rho > 1, r, D, S_0, T, \sigma$ and for given functions p, ϕ_1 and ϕ_2 takes m to be equal to the smallest integer satisfying the stability constraint and returns the values U_{i0} , $i = 0, \dots, n$ of the approximate solution (option prices) obtained by explicit finite difference method and the corresponding stock prices $S_i = e^{x_i}$ in the case $x_{min} = \ln \frac{S_0}{\rho}$, $x_{max} = \ln(\rho S_0)$. Test the correctness of your code by comparing the results to the exact values obtained by Black-Scholes formula in the case $r = 0.03, \sigma = 0.5, D = 0.05, T = 0.5, E = 97, S_0 = 100, p(s) = \max(s - E, 0), \phi_1(t) = p(e^{x_{min}}), \phi_2(t) = p(e^{x_{max}})$.

Exercise 2. Let $r = 0.02, \sigma = 0.6, D = 0.03, T = 0.5, E = 99, S_0 = 100, p(s) = \max(s - E, 0)$. If we use the explicit method of previous exercise, then even if we let n go to infinity there is going to be a finite error between the exact option price at $t = 0, S(0) = S_0$ and the

corresponding approximate value. This error is caused by introducing artificial boundaries x_{min} and x_{max} and the boundary conditions specified at those boundaries. Use the boundary conditions $\phi_1(t) = p(e^{x_{min}})$, $\phi_2(t) = p(e^{x_{max}})$ and determine the value of the resulting error for $\rho = 1.5, 2, 2.5$. In order to see the resulting error you should do several computations with fixed ρ and increasing values of n (assuming m is determined from the stability condition, n should be increased by multiplying it by 2 each time). Use the knowledge that for large enough n the part of the error depending on the choice of n behaves approximately like $\frac{const.}{n^2}$ (so the difference of the last two computations divided by 3 is an estimate of this part of the error for the last computation) for determining how far your last computation is from the limiting value.

It is important to understand that a single computation by a finite difference method can be used for computing prices of the same option at different time moments for any reasonable value of the current stock price at the time moments of interest. Of course our numerical method does not give us a formula for the function v , but only a table of values corresponding to certain stock prices and time moments. So to find a value of the function for a given value of s and for given time moment, we should determine the closest values of stock prices and time moments for which we have results and to use them to compute the value we need. One commonly used approach is **bilinear interpolation**, which works as follows.

Suppose that we know the values of a function $u(x, t)$ at the corners of a rectangle $[x_1, x_2] \times [t_1, t_2]$, denote the known values by $u(x_1, t_1) = U_{11}$, $u(x_1, t_2) = U_{12}$, $u(x_2, t_1) = U_{21}$, $u(x_2, t_2) = U_{22}$. Denote also $\Delta x = x_2 - x_1$, $\Delta t = t_2 - t_1$. Then the value of u at the point (x, t) , where $x_1 \leq x < x_2$, $t_1 \leq t < t_2$ can be computed approximately by the bilinear interpolation formula

$$u(x, t) \approx \frac{U_{11}(x_2 - x)(t_2 - t) + U_{12}(x_2 - x)(t - t_1) + U_{21}(x - x_1)(t_2 - t) + U_{22}(x - x_1)(t - t_1)}{\Delta x \Delta t}.$$

It is known that if the function u has continuous second derivatives in x and t variables, then the approximation error is bounded by $const. \cdot (\Delta x^2 + \Delta t^2)$.

Homework 4. (deadline November 8, 2017) Modify the solver of Exercise 1 so that instead of S_0 and ρ it takes values of x_{min} and x_{max} as parameters and returns the matrix U as the answer. Use the explicit finite difference method with $r = 0.02$, $\sigma = 0.6$, $D = 0.05$, $T = 0.4$, $p(s) = \max(20 - \text{abs}(s - 70), 0)$, $\phi_1(t) = p(e^{x_{min}})$, $\phi_2(t) = p(e^{x_{max}})$, $x_{min} = \ln 20$, $x_{max} = \ln 160$, $n = 90$ to compute the matrix U . The matrix should be computed only once when solving the homework problem. Use this matrix to find approximate option prices:

1. at time $t = 0$, when $S(0) = 80$ without interpolation (the value is in the matrix U , you have to compute the indexes exactly),
2. Use U to compute approximately the value of the option at time $t = 0.15$, when $S(0.15) = 77$ by using the bilinear interpolation formula given above.