Märkusi 4. praktikumi kohta

Mida tähendas fraas "mitme muutuja liitfunktsiooni kõrgemat järku täisdiferentsiaali leidmine on paras peavalu"?

Funktsiooni $F = F(x, y)$ täisdiferentsiaal punktis X on defineeritud kui

$$dF(X) = \frac{\partial F}{\partial x}(X)dx + \frac{\partial F}{\partial y}(X)dy.$$

Kui nüüd $F = F(x, y) = f(u, v)$, kus $u = u(x, y)$, $v = v(x, y)$, siis

$$\begin{aligned}
dF &= \left(\frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x}\right) dx + \left(\frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y}\right) dy \\
&= \frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy\right) + \frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy\right) \\
&= \frac{\partial f}{\partial u}du + \frac{\partial f}{\partial v}dv \\
&= df,
\end{aligned}$$

st esimest järku täisdiferentsiaal on invariantne muutujavahetuse suhtes, $dF = df$. Kõrgemat järku täisdiferentsiaalide korral see enam ei kehti, st üldjuhul $d^2F \neq d^2f$. Leiame valemi d^2F arvutamiseks

$$\begin{aligned}
d^2F = d(dF) &= d\left(\frac{\partial f}{\partial u}\frac{\partial u}{\partial x}dx + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x}dx + \frac{\partial f}{\partial u}\frac{\partial u}{\partial y}dy + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y}dy\right) \\
&= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial u}\frac{\partial u}{\partial x}dx + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x}dx + \frac{\partial f}{\partial u}\frac{\partial u}{\partial y}dy + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y}dy\right) dx \\
&\quad + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial u}\frac{\partial u}{\partial x}dx + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x}dx + \frac{\partial f}{\partial u}\frac{\partial u}{\partial y}dy + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y}dy\right) dy \\
&= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial u}\frac{\partial u}{\partial x}\right) dx^2 + \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial v}\frac{\partial v}{\partial x}\right) dx^2 + \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial u}\frac{\partial u}{\partial y}\right) dx\,dy + \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial v}\frac{\partial v}{\partial y}\right) dx\,dy \\
&\quad + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial u}\frac{\partial u}{\partial x}\right) dx\,dy + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial v}\frac{\partial v}{\partial x}\right) dx\,dy + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial u}\frac{\partial u}{\partial y}\right) dy^2 + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial v}\frac{\partial v}{\partial y}\right) dy^2 \\
&= \left(\frac{\partial^2 f}{\partial u^2}\frac{\partial u}{\partial x} + \frac{\partial^2 f}{\partial u \partial v}\frac{\partial v}{\partial x}\right)\frac{\partial u}{\partial x}dx^2 + \frac{\partial f}{\partial u}\frac{\partial^2 u}{\partial x^2}dx^2 + \left(\frac{\partial^2 f}{\partial u \partial v}\frac{\partial u}{\partial x} + \frac{\partial^2 f}{\partial v^2}\frac{\partial v}{\partial x}\right)\frac{\partial v}{\partial x}dx^2 + \frac{\partial f}{\partial v}\frac{\partial^2 v}{\partial x^2}dx^2 \\
&\quad + \left(\frac{\partial^2 f}{\partial u^2}\frac{\partial u}{\partial x} + \frac{\partial^2 f}{\partial u \partial v}\frac{\partial v}{\partial x}\right)\frac{\partial u}{\partial y}dx\,dy + \frac{\partial f}{\partial u}\frac{\partial^2 u}{\partial x \partial y}dx\,dy \\
&\quad + \left(\frac{\partial^2 f}{\partial u \partial v}\frac{\partial u}{\partial x} + \frac{\partial^2 f}{\partial v^2}\frac{\partial v}{\partial x}\right)\frac{\partial v}{\partial y}dx\,dy + \frac{\partial f}{\partial v}\frac{\partial^2 v}{\partial x \partial y}dx\,dy \\
&\quad + \left(\frac{\partial^2 f}{\partial u^2}\frac{\partial u}{\partial y} + \frac{\partial^2 f}{\partial u \partial v}\frac{\partial v}{\partial y}\right)\frac{\partial u}{\partial x}dx\,dy + \frac{\partial f}{\partial u}\frac{\partial^2 u}{\partial x \partial y}dx\,dy \\
&\quad + \left(\frac{\partial^2 f}{\partial u \partial v}\frac{\partial u}{\partial y} + \frac{\partial^2 f}{\partial v^2}\frac{\partial v}{\partial y}\right)\frac{\partial v}{\partial x}dx\,dy + \frac{\partial f}{\partial v}\frac{\partial^2 v}{\partial x \partial y}dx\,dy \\
&\quad + \left(\frac{\partial^2 f}{\partial u^2}\frac{\partial u}{\partial y} + \frac{\partial^2 f}{\partial u \partial v}\frac{\partial v}{\partial y}\right)\frac{\partial u}{\partial y}dy^2 + \frac{\partial f}{\partial u}\frac{\partial^2 u}{\partial y^2}dy^2 + \left(\frac{\partial^2 f}{\partial u \partial v}\frac{\partial u}{\partial y} + \frac{\partial^2 f}{\partial v^2}\frac{\partial v}{\partial y}\right)\frac{\partial v}{\partial y}dy^2 + \frac{\partial f}{\partial v}\frac{\partial^2 v}{\partial y^2}dy^2 \\
&= \frac{\partial^2 f}{\partial u^2}\left(\left(\frac{\partial u}{\partial x}\right)^2 dx^2 + 2\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}dx\,dy + \left(\frac{\partial u}{\partial y}\right)^2 dy^2\right)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{\partial^2 f}{\partial u \partial v}\left(2\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}dx^2 + 2\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}dx\,dy + 2\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}dx\,dy + 2\frac{\partial u}{\partial y}\frac{\partial v}{\partial y}dy^2\right)\\
&+\frac{\partial^2 f}{\partial v^2}\left(\left(\frac{\partial v}{\partial x}\right)^2 dx^2 + 2\frac{\partial v}{\partial x}\frac{\partial v}{\partial y}dx\,dy + \left(\frac{\partial v}{\partial y}\right)^2 dy^2\right)\\
&+\frac{\partial f}{\partial u}\left(\frac{\partial^2 u}{\partial x^2}dx^2 + 2\frac{\partial^2 u}{\partial x \partial y}dx\,dy + \frac{\partial^2 u}{\partial y^2}dy^2\right) + \frac{\partial f}{\partial v}\left(\frac{\partial^2 v}{\partial x^2}dx^2 + 2\frac{\partial^2 v}{\partial x \partial y}dx\,dy + \frac{\partial^2 v}{\partial y^2}dy^2\right)\\
=&\left(\frac{\partial^2 f}{\partial u^2}du^2 + 2\frac{\partial^2 f}{\partial u \partial v}du\,dv + \frac{\partial^2 f}{\partial v^2}dv^2\right) + \frac{\partial f}{\partial u}d^2u + \frac{\partial f}{\partial v}d^2v\\
=&\, d^2 f + \frac{\partial f}{\partial u}d^2u + \frac{\partial f}{\partial v}d^2v
\end{aligned}
$$

Seega $d^2F = d^2f$ ainult juhul kus $d^2u = d^2v = 0$ ehk u ja v on lineaarsed.

Kuna jõudsime lahendada väga vähe ülesandeid Taylori valemi koha, panen mõned lahendused siia kirja.

Ülesanne 54. a) Olgu $f(x, y) = e^{x+y}$ ja $A = (1, -1)$. Leida 3-ndat järku Taylori valem punktis A, st leida esitus

$$
f(A+h) = f(A) + d_A f(h) + \frac{1}{2}d_A^2 f(h) + \frac{1}{6}d_A^3 f(h) + \alpha_3(h), \quad \lim_{h\to 0}\frac{\alpha_3(h)}{\|h\|^3} = 0.
$$

Kuna $\dfrac{\partial f}{\partial x} = \dfrac{\partial f}{\partial y} = e^{x+y}$, $\dfrac{\partial^2 f}{\partial x^2} = \dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{\partial^2 f}{\partial y^2} = e^{x+y}$ jne, siis

$$
\begin{aligned}
df(h) &= e^{x+y}(h_1 + h_2),\\
d^2 f(h) &= e^{x+y}(h_1^2 + 2h_1h_2 + h_2^2),\\
d^3 f(h) &= e^{x+y}(h_1^3 + 3h_1^2h_2 + 3h_1h_2^2 + h_2^3)
\end{aligned}
$$

ning

$$
\begin{aligned}
f(A) &= e^{1-1} = 1\\
d_A f(h) &= e^{1-1}(h_1 + h_2) = h_1 + h_2,\\
d_A^2 f(h) &= e^{1-1}(h_1^2 + 2h_1h_2 + h_2^2) = (h_1 + h_2)^2,\\
d_A^3 f(h) &= e^{1-1}(h_1^3 + 3h_1^2h_2 + 3h_1h_2^2 + h_2^3) = (h_1 + h_2)^3.
\end{aligned}
$$

Taylori valem avaldub kujul $f(1+h_1, -1+h_2) = 1 + h_1 + h_2 + \dfrac{(h_1+h_2)^2}{2} + \dfrac{(h_1+h_2)^3}{6} + \alpha_3(h)$ ehk

$$
f(x, y) = 1 + x + y + \frac{(x+y)^2}{2} + \frac{(x+y)^3}{6} + \alpha_3(x, y),
$$

kus $\displaystyle\lim_{\sqrt{(x-1)^2+(y+1)^2}\to 0}\frac{\alpha_3(x,y)}{((x-1)^2 + (y+1)^2)^{3/2}} = 0.$

Ülesanne. Joonistage funktsioonide $f(x, y) = e^{x+y}$ ja $f(x, y) = 1 + x + y + \dfrac{(x+y)^2}{2} + \dfrac{(x+y)^3}{6}$ vahe graafik punkti $(1, -1)$ ümbruses. Mis selgub?

Ülesanne 55. a) Kasutame Taylori valemit juhul $n = 1$, st

$$f(a + h_1, b + h_2) \approx f(a, b) + \frac{\partial f}{\partial x}(a, b)h_1 + \frac{\partial f}{\partial y}(a, b)h_2.$$

Selles ülesandes $f(x, y) = x^y$, $(a, b) = (1, 2)$, $h_1 = 0{,}03$, $h_2 = 0{,}04$. Seega $f(a, b) = 1$, $\frac{\partial f}{\partial x}(a, b) = yx^{y-1}(1, 2) = 2$, $\frac{\partial f}{\partial y}(a, b) = x^y \ln x(1, 2) = 0$ ning

$$1{,}03^{2{,}04} \approx 1 + 2 \cdot 0{,}03 + 0 \cdot 0{,}04 = 1{,}06.$$