Simulation Methods in Fin Mat, Lab 3

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Black-Scholes market model

$$dS(t) = S(t) \cdot (\mu \cdot dt + \sigma \cdot dB(t))$$

$B(t)$ - the standard Brownian motion
σ - the volatility. Can be a function of $S(t)$ and t mu - trend

No-Arbitrage pricing: the price of an European option with exercise time T and pay-off p is given by

$$Price(t, S_t) = E[e^{-r(T-t)}p(S(T)) \mid S(t) = S_t]$$

where the expectation is taken uder the (risk-neutral) measure for which the stock price equation is

$$dS(t) = S(t)((r - D)\, dt + \sigma dB(t)).$$

r - the risk free interest rate, D - dividend rate

Using MC to find option prices in the case of constant volatility

If σ is constant, then the solution of

$$dS(t) = S(t)((r - D)\,dt + \sigma dB(t))$$

satisfies

$$S(T) = S(t)e^{(r-D-\frac{\sigma^2}{2})(T-t)+\sigma\,(B(T)-B(t))}$$

Since $B(T) - B(t) \sim N(0, T - t)$, we can write

$$Price(t, S_t) = E[e^{-r(T-t)}p(S(T))$$

where

$$S(T) = S_t e^{(r-D-\frac{\sigma^2}{2})(T-t)+\sigma\,\sqrt{T-t}X})],\ X \sim N(0,1)$$

How to generate $S(T)$ if σ is not constant?

Euler-Maruyama method

Consider a SDE (Stochastic Differential Equation)

$$dY(t) = \alpha(Y(t), t)\, dt + \beta(Y(t), t)\, dB(t), \;\; Y(0) = Y_0$$

Euler-Maruyama method: Use the intuitive meaning of SDE: over small interval (t_i, t_{i+1}) the change is Y is approximately the equal to α, times the change in t plus β times the change in the Brownian motion.

Mathematically:

$$Y(t_{i+1}) - Y(t_i) \approx \alpha(Y(t_i), t_i) \cdot (t_{i+1} - t_i) + \beta(Y(t_i), t_i) \cdot (B(t_{i+1}) - B(t_i))$$

Use notations $t_i = i\Delta t$, where $\Delta t = \frac{i \cdot T}{m}$. Define

$$Y_{i+1} = Y_i + \alpha(Y_i, t_i)\Delta t + \beta(Y_i, t_i)\sqrt{\Delta t} X_i, \;\; i = 0, 1, \ldots, m-1,$$

where $X_i \sim N(0, 1)$. Similar formulas for systems of SDE.

Euler-Maruyama method (continued)

$$dY(t) = \alpha(Y(t), t)\,dt + \beta(Y(t), t)\,dB(t), \;\; Y(0) = Y_0$$

Euler-Maruyama method

$$Y_{i+1} = Y_i + \alpha(Y_i, t_i)\Delta t + \beta(Y_i, t_i)\sqrt{\Delta t}X_i, \;\; i = 0, 1, \ldots, m-1$$

BS market model

$$dS(t) = S(t) \cdot (\mu \cdot dt + \sigma \cdot dB(t))$$

Euler-Maruyama method

$$S_{i+1} = S_i \cdot (1 + \mu(S_i, t_i)\Delta t + \sigma(S_i, t_i)\sqrt{\Delta t}\,X_i), i = 0, 1, \ldots, m-1$$

Use S_m instead of $S(T)$

Strong and weak convergence

- The trajectory of Brownian motion determines a trajectory of the stock price
- A given trajectory $B(t),\ t \in [0, T]$ determines the exact values $S(t_i),\ i = 0, \ldots, m$
- can also compute approximate values $S_i, i = 0, \ldots, m$ according to Euler's method with $\sqrt{\Delta t}\, X_i = B(t_i) - B(t_{i-1})$
- Strong convergence characterizes how close are those values in the average, how quicly the differences in prices goes to 0 as m increases
- Weak convergence is related to how close are the distributions of $S(T)$ and S_m. Closeness of distribution is much weaker condition than closeness of values of random variables

Illustration