Simulation Methods in Financial Mathematics
Computer Lab 5

Goal of the lab:

- To learn to use Euler's method for systems of stochastic differential equations and to compute option prices with a given accuracy when using a numerical method for generating stock prices.

The Euler's method for solving a stochastic differential equation (SDE) of the form

$$dY(t) = \alpha(t, Y(t))\, dt + \beta(t, Y(t))\, dB(t),\ Y(0) = Y_0$$

can be presented as

$$Y_{k+1} = Y_k + \alpha(t_k, Y_k)(t_{k+1} - t_k) + \beta(t_k, Y_k)X_k,\ k = 0, 1, \dots, m-1$$

where $X_k \sim N(0, \sqrt{t_{k+1} - t_k})$ are independent random variables and Y_k are the approximate values of $Y(t_k)$. Typically we take $t_k = k \cdot \frac{T}{m}$, which in turn means that $t_{k+1} - t_k = \Delta t = \frac{T}{m}$. More generally, the Euler's method for solving a system of N SDE's of the form

$$dY_i(t) = \alpha(t, Y_1(t), \dots, Y_N(t))\, dt + \beta(t, Y_1(t), \dots, Y_N(t))\, dB_i(t),\ Y_i(0) = Y_{i0}, i = 1, \dots, N$$

is

$$Y_{i,k+1} = Y_{ik} + \alpha(t_k, Y_{1k}, \dots, Y_{Nk})(t_{k+1} - t_k) + \beta(t_k, Y_{1k}, \dots, Y_{Nk})X_{ik},\ i = 1, \dots, N,\ k = 0, 1, \dots, m-1,$$

where the vectors $(X_{1k}, \dots, X_{Nk})$ are independent random vectors with the same n-dimensional normal distribution as $(B_1(t_{k+1}) - B_1(t_k), \dots, B_N(t_{k+1}) - B_N(t_k))$. In the particular case when all Brownian motions are idependent and $t_k = k \cdot \frac{T}{m}$, all values X_{ik} are iid random variables with the distribution $N(0, \sqrt{\frac{T}{m}})$.

We know several methods for generating stock prices and we know that for pricing options the weak convergence rate of the method used is important. It is known that if p is continuous and has bounded first derivative (ie it is Lipshitz continuous), then Euler's method and Milstein's method are weakly convergent with rate $q = 1$ and we also know one method that has weak convergence rate 2 for sufficiently nice pay-off functions. Next we consider, how this information can be used for computing the price of and option with a given accuracy when we have to use a numerical method for generating the stock prices.

Let V be the price of an European option with the expiration date T and pay-off function p, then

$$V = E(\exp(-rT)p(S(T))),$$

where $S(t),\ 0 \le t \le T$ follows certain stochastic differential equation (SDE). If the SDE can not be solved exactly, then instead of $S(T)$ we use S_m, thus we use Monte-Carlo method to compute an approximate value V_m of V, where

$$V_m = E[e^{-rT}p(S_m)].$$

Often we know the weak convergence rate of a numerical method. This means that

$$|V - V_m| = \frac{C}{m^q} + o(\frac{1}{m^q})$$

for some $q > 0$. Here C is a constant that does not depend on m and $m^q \cdot o(\frac{1}{m^q}) \to 0$ as $m \to \infty$. Actually, usually a more precise relation

$$V - V_m = \frac{C_1}{m^q} + o(\frac{1}{m^q}),$$

holds (and thus the previous estimate for the absolute value of the error holds with here $C = |C_1|$) and we use that later for estimating the coefficient C.

Thus, if we use S_m instead of $S(T)$ and use Monte-Carlo method with allowed error ε at a specified allowed error probability α, then the total error of the computed number $\hat{V}_{m,\varepsilon}$ is

$$|V - \hat{V}_{m,\epsilon}| \le |V - V_m| + |V_m - \hat{V}_{m,\epsilon}| \le \frac{C}{m^q} + o(\frac{1}{m^q}) + \epsilon.$$

The last term is the error of the Monte-Carlo method and can be chosen by us. So, in order to compute the option price V with a given error ε, we should choose large enough m (so that the term $\frac{C}{m^q}$ is small enough, for example less than $\frac{\varepsilon}{2}$) and then use MC method with allowed error $\epsilon = \frac{\varepsilon}{2}$). There is one trouble: we do not know C. One possibility to estimate C is as follow:

1. Choose some values for m_0, ϵ_0 for m and MC error ϵ. The value of m_0 should not be too small, but very large values take too much computation time; the value of the allowed error ϵ_0 should be sufficiently small (we discuss it in more detail in the next step). In practice we usually use $m_0 = 5$ or $m_0 = 10$.

2. Use MC method twice to compute $\hat{V}_{m_0,\epsilon_0}$ and $\hat{V}_{2m_0,\epsilon_0}$. The value ϵ_0 is small enough if the results differ significantly more than by ϵ_0. If we use too large value of ϵ_0, then we overestimate the value of C and hence the final value of m in the following steps and our final computations may take too much time.

3. Estimate the value of C. We use the inequality

$$|V_{m_0} - V_{2m_0}| \leq |V_{m_0} - \hat{V}_{m_0,\epsilon_0}| + |V_{2m_0} - \hat{V}_{2m_0,\epsilon_0}| + |\hat{V}_{m_0,\epsilon_0} - \hat{V}_{2m_0,\epsilon_0}|.$$

If we use the more precise information about V_m and V_{2m} and assume that the terms $o(\frac{1}{m_0^q})$ and $o(\frac{1}{(2m_0)^q})$ are practically zero, then it follows that

$$\begin{aligned} C &\leq \frac{(2m_0)^q}{2^q - 1} \cdot (|\hat{V}_{m_0,\epsilon_0} - \hat{V}_{2m_0,\epsilon_0}| + |V_{m_0} - \hat{V}_{m_0,\epsilon_0}| + |V_{2m_0} - \hat{V}_{2m_0,\epsilon_0}|) \\ &\leq \frac{(2m_0)^q}{2^q - 1} \cdot (|\hat{V}_{m_0,\epsilon_0} - \hat{V}_{2m_0,\epsilon_0}| + 2\epsilon_0) =: \bar{C}. \end{aligned}$$

4. Choose m_1 such that $\frac{\bar{C}}{m_1^q} \leq \frac{\varepsilon}{2}$ and compute $\bar{V}_{m_1,\frac{\varepsilon}{2}}$. The last result is an approximation of the true option price which satisfies the desired error estimate, if the starting value of m_0 was large enough so that the additional error terms of order $o(\frac{1}{m^q})$ are practically equal to zero. In this course we do not consider methods of determining if the starting value of m_0 was sufficiently large and take the result of the last computation to be the desired answer.

Tasks:

1. Find the value of an European call option with strike price $E = 98$ at time $t = 0$ with precision 0.1, when $\alpha = 0.05$, $r = 0.05$, $D = 0$, $T = 0.5$, $S(0) = 100$ and $\sigma(t, s) = 0.7 - 0.7\,e^{-0.01s}$. To solve the problem we have to choose an m so that the error due to m would be sufficiently small.

2. Future risk free interest rate is actually not a constant and can be considered to be a random variable. In the case Black-Scholes model with a random interest rate the prices of European options can be computed as

$$Price = E[\exp(-\int_0^T r(t)\,dt)p(S(T))],$$

where

$$dS(t) = S(t)((r(t) - D)\,dt + \sigma\,dB_1(t))$$

and $r(t)$ follows a suitable stochastic differential equation. We conside so called Cox-Ingersoll-Ross model

$$dr(t) = a(b - r(t))\,dt + \sigma_2\sqrt{r(t)}\,dB_2(t),$$

where $B_1(t)$ and $B_2(t)$ are independent Brownian motions. So we have a system of stochastic differential equations for S and r. When computing the option prices we can replace the integral $\int_0^T r(t)$ with the product of the mean value or the interest rate and T. So, for computing the option price we should write a function that for a given values of parameters $D, \sigma, \sigma_2, T, a, b, m$ and n generates n pairs of the future stock prices $S(T)$ and mean values of the interest rates corresponding to the sama trajectory. Use Euler's method with m time steps for solving the system of SDEs for S and r.

3. **Homework** (Deadline 14.03.2017). Assume that the price at $t = 0$ of an European option with pay-off function p is given by

$$Price = E[e^{-0.05 \cdot T} p(S_1(T), S_2(T)],$$

where T is the exercise time (or duration of the option), r is the risk free interest rate and $S_i(T),\ i = 1, 2$ correspond to the solution of the system of stochastic differential equaitons

$$\begin{aligned} dS_1(t) &= S_1(t) \cdot (0.05\,dt + 0.5 \cdot dB_1(t)), \\ dS_2(t) &= S_2(t) \cdot (0.05\,dt + (0.3 + \frac{0.4}{1 + 0.01(S_1(t) - 50)^2}) \cdot dB_2(t)), \end{aligned}$$

satisfying the initial conditions $S_1(0) = 50,\ S_2(0) = 40$. Here B_1 and B_2 are independent Brownian motions.

Find the price of the European option with the exercise time $T = 0.75$ and the pay-off function $p(s_1, s_2) = \max(45 - s_1, s_2 - 40, 0)$ with total error less than 0.02 at the confidence level $\alpha = 0.05$.