LATTICE THEORY
FALL 2017

To get a good mark, a student has to know the following notions:

preorder, order relation, equivalence relation, poset, chain, comparable elements of a poset, covering, upper bound, lower bound, join, meet, lattice, complete lattice, bounded poset, sublattice, convex sublattice, interval, join-homomorphism, meet-homomorphism, homomorphism of lattices, isomorphism of lattices, lattice congruence, kernel of a lattice homomorphism, quotient lattice, down-closed subset of a poset, ideal of a lattice, filter of a lattice, principal ideal, principal filter, minimality condition for a poset, descending chain condition for a poset, fixed point, closure system, closure operator, closed subset, algebraic closure operator, compact element of a complete lattice, algebraic lattice, upper semilattice, ideal of an upper semilattice, completion of a poset, join-dense subset of a poset, modular lattice, length of a chain, length of a finite lattice, join irreducible element, irredundant representation of an element, distributive lattice, prime ideal, prime filter, congruence extension property, principal congruence, complement of an element, complemented lattice, Boolean lattice, semimodular lattice, lattice of finite length, height function on a lattice of finite length, independent set of atoms, geometric lattice, atomistic lattice, geometry.

The questions for the oral part of the exam are the following (with references to lecture notes):

(1) Prove that the two definitions of a lattice (order theoretic and algebraic) are equivalent. (Theorem 1.12)
(2) Prove that if a lattice L has the smallest element and L satisfies (ACC) then L is complete. (Theorem 2.5)
(3) Prove that the lattice of ideals of an upper semilattice with zero is algebraic. (Theorem 2.26)
(4) Prove that every algebraic lattice is isomorphic to the lattice of ideals of an upper semilattice with zero. (Theorem 2.26)
(5) Prove that if P is a poset then the set $\mathsf{DM}(P)$ is a complete lattice with respect to inclusion. (Theorem 2.29)
(6) Prove the result about the properties of $\mathsf{DM}(P)$. (Theorem 2.39)
(7) Prove the universal property of $\mathsf{DM}(P)$. (Theorem 2.41)
(8) Prove the characterization theorem of modular lattices. (Theorem 3.4)
(9) Prove the Kurosh–Ore theorem for modular lattices. (Theorem 3.18)
(10) Prove 3 implications in the characterization theorem of distributive lattices. (Theorem 4.6)
(11) Prove that a lattice is distributive if and only if it does not contain sublattices isomporphic to N_5 and M_3. (Theorem 4.7)
(12) Prove the Prime Ideal Property for distributive lattices. (Theorem 4.15)
(13) Prove the Stone Theorem about distributive lattices. (Theorem 4.19)

(14) Prove the Birkhoff Theorem about finite distributive lattices. (Theorem 4.20)
(15) Prove that all maximal chains of a distributive lattice L have length $|\mathcal{J}(L)|$. (Theorem 4.23)
(16) Prove that distributive lattices have the Congruence Extension Property. (Theorem 5.7)
(17) Prove that every nonempty ideal of a distributive lattice is a congruence class. (Theorem 5.13)
(18) Prove that a bounded lattice is a Boolean lattice if and only if it satisfies de Morgan laws and every element has precisely one complement. (Theorem 6.8)
(19) Prove that the congruences of a Boolean lattice are in one-to-one correspondence with ideals. (Theorem 6.10)
(20) Prove that in a semimodular lattice of finite length arbitrary two maximal chains have the same length. (Theorem 7.6)
(21) Prove a characteriation of semimodular lattices using the height function. (Theorem 7.9)
(22) Prove a theorem about independent sets of atoms in semimodular lattices. (Theorem 7.11)
(23) Prove that the set of closed subsets of a geometry is a geometric lattice. (Theorem 8.7)
(24) Prove that every geometric lattice is isomorphic to the lattice of closed subsets of some geometry. (Theorem 8.8)

The written part of the exam gives 20% of the mark, it takes place from 10:00 to 10:30, and a student is supposed to know definitions and to be able to give some examples (a la give an example of a modular lattice which is not distributive). At the end of the written part a student chooses randomly a question for the oral part from the list above. The oral part will give 80% of the mark and it will take place between 14:00 and 16:00 on the same day. A student has to present a full proof on the blackboard and he/she should be able to answer the questions about the proof. A one A4 page of handwritten notes can be used during the oral presentation on blackboard.