Computational Finance, Lecture 1

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Introduction and motivation

There are many indeterministic (random) effects like oil prices, customer behaviour, political situation ans so on, which one has to take into account when making financial decisions. Using mathematical modelling as a tool for making those decisions is very common.

If an investor makes a decision about buying or selling a financial instrument, it is customary to consider the expected return and risk of the investment. Investment decision is usually made based on those quantities and investor's risk tolerance, so there is a different "right price" for each investor. It turns out that such simple decision rules are not valid when one wants to trade with contracts whose value depends on other traded instruments. A well-known class of such contracts are financial options.

Let us look at some examples of option contracts and have a glimpse on reasons why the prices of such contracts can not be determined just by the expected value and the risk of corresponding investment.

Definitions and examples

We adopt a non-standard, but quite general definition of financial options

Definitsioon 1 *An* option *is a contract giving it's holder the right to receive in the future a payment which amount is determined by the behaviour of the stock market up to the moment of executing the contract.*

Option contracts are classified according to several characteristics including

- possible execution times (a fixed date vs a time interval),
- the number of underlying assets,
- how the value of option depends on the asset prices (depending on the price at the execution time vs a path dependent value of the asset prices).

In order to clarify the meaning of the definition, let us look at some examples.

Example 2 The right to buy 100 Nokia shares for 500 Euros exactly after 3 months (say, December 3rd, 2018) .

This is an *European* (with fixed execution date) *Call* (the right to buy) option, which is equivalent to the right to receive after three months the sum of $100 \cdot \max(S(T) - 5, 0)$ Euros, where $S(T)$ denotes the price of a Nokia share at the specified date.

Example 3 The right to sell one Amazon.com share during next 6 months for \$1950.

This is *American* (with a free execution time) *Put* (the right to sell) option.

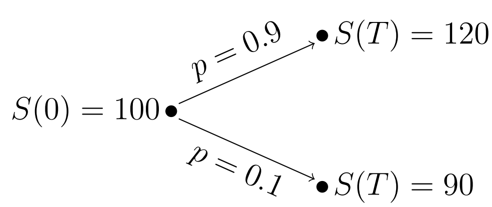
Example 4 The right to exchange after one year 10000 USD for Euros with the rate that is the average of the daily exchange rates in the one year period.

The last an *Asian* option that is an example of path dependent options.

Strange things about pricing options.

If an investor makes a decision about buying or selling a financial instrument, it is customary to consider the expected return and risk of the investment. Investment decision is usually made based on those quantities and investor's risk tolerance, so there is a different "right price" for each investor. It turns out that usual thinking models do not help to determine the right price for an option contract. In order to clarify this point, let us consider some simple toy models for stock price behaviour.

First, suppose that the at a future time T there are only two possible stock prices:



Assume for simplicity that the risk free interest rate is 0 (meaning that it is not possible to ear interest by depositing money in bank and it is possible to borrow money so that you have to pay back exactly the sum you borrowed). Let us consider and option to buy a at time T 10 shares of stock for 99. Now the value of this option at time T is 210 if $S(T) = 120$ (since you can buy 10 shares for 99 when the market price is 120) and it is worthless, if $S(T) = 0$. So buying the option seems to be a good investment possibility, if the price is not too high: the expected value of the option at time T is

$$0.9 \cdot 210 + 0.1 \cdot 0 = 189.$$

Note also that if the probability of $S(T) = 120$ is 0.1, then the expected value is only 21, so the expected value of the option depends strongly on the market probabilities.

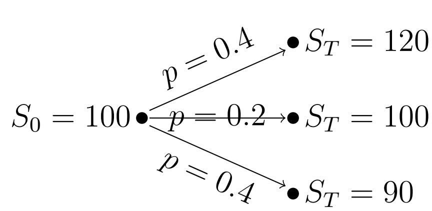
So it is natural to think that the fair price of the option should also depend on market probabilities.

But before deciding to buy the option we can consider alternative investment possibilities. It turns out that we can achieve exactly the same outcome by forming an investment portfolio consisting from a loan of 630 and 7 shares of stock: if $S_T = 120$, then the value of the portfolio is $7 \cdot 120 - 630 = 210$ and if $S_T = 90$, then the value is $7 \cdot 90 - 630 = 0$. The cost of forming the portfolio at $t = 0$ is 700-630=70 (we get 630 from the loan and have to add 70 of our own money to buy 7 shares for 100 each). So it is clear that at this market no sensible person pays more than 70 for the option. Moreover, if there exists any person willing to buy the option for more than 70, there is a possibility for anyone to earn money at the market without any risk of losing anything: one should just sell the option for the price and use 70 to set up the portfolio to cover the liabilities at time T. Such opportunities are called arbitrage opportunities and usually it is assumed that there are no arbitrage opportunities at the market.

If we allow portfolios with a negative number of shares (short selling) we can argue, that the price of the option can not be less than 70. Otherwise we could buy the option, short sell 7 shares of stock for 100 each, deposit 630 in a bank and use the remaining money as we want. At time T we can in both cases use the bank deposit and the money we get by exercising the option to buy back 7 shares of the stock we borrowed earlier. Even when short selling is not allowed, any owner of 7 shares who wants to keep the shares, could make some extra many by temporarily selling the shares, buying the option and at time T buying the shares back, so there should be a large demand for the option if it sold for less than 70, hence the price of option should increase.

So the price of the option is completely determined by the market model. Moreover, the arguments we used did not depend on the probabilities of the up and down movements, so the option price does not depend on expected value and the risk of the option contract.

Let us consider now a similar market model with three different stock prices at time T:



It is easy to check that the price of the same option considered for the previous market model can not be larger than 70 (since the same portfolio as before requires 70 of initial investment and is worth at least as much as the option at time T for all possible values of $S(T)$. It is also possible to show that the price of the option can not be less than 10 (consider setting up the portfolio with one option, -7 shares of stock and a bank deposit of 690). Moreover, it is also possible to show that from the arbitrage principle it follows only that the value of the option is between 10 and 70 and if the option is traded for a price between those values, then arbitrage is not possible. This means that it is not possible to form a portfolio from the option, a stock holding and a loan/deposit so that the set-up cost of the portfolio is 0 but the value at time T is never negative and is positive with a positive probability (can you show it?). Consider now a second option that pays 20, if $S(T) = 100$ and 0 otherwise. If we consider this option separately from the first one, then from the arbitrage arguments if follows only that the price of the option is between 0 and 20. But if it is possible to both buy and sell any number of contracts of one of those two options then the price of the other one is completely determined. So there are strong consistency requirements between the prices of different options.

Exercise 1 *Suppose one can freely trade (both buy or short sell) the second option that pays 20, if $S(T) = 100$ and 0 otherwise and that the option price is 5 at time $t = 0$. Find a portfolio consisting of a number of the contracts of the second option, a stock holding and a loan/bank deposit (with 0 interest rate) so that in the case of the last market model the portfolio replicates (has exactly the same value for all possible stock prices at the time T) as the first option (the right to buy 10 shares at time T for 99 per share). Using this portfolio, determine the price of the first option at time $t = 0$. (Hint: the portfolio is determined by 3 unknowns – the number of option holdings, the number of shares and the loan. Write down 3 equations for the portfolio value to be equal to the pay-off of the first option at the time $t = T$ and solve for the unknowns. The value of the portfolio at time 0 has then to be equal to the price of the first option.)*

Based on the two simple models we can make the following conclusions:

- Naive pricing approaches (based on the expected return and risk) do not work.
- In the case of some market models the option price is determined completely by the model (and the "no arbitrage" condition)
- There are market models, for which the option prices are not determined completely but prices of different options have to be consistent with each other.