Computational Finance, Fall 2018
Computer Lab 11

The aim of the lab is to implement the basic implicit method and Crank-Nicolson method for computing European prices of European options. For this we consider the problem

$$\begin{aligned}
\frac{\partial u}{\partial t}(x,t) + \alpha(x,t)\frac{\partial^2 u}{\partial x^2}(x,t) + \beta(x,t)\frac{\partial u}{\partial x}(x,t) - r\,u(x,t) &= 0,\ x \in (x_{min}, x_{max}), 0 \le t < T,\\
u(x_{min}, t) &= \phi_1(t), 0 \le t < T,\\
u(x_{max}, t) &= \phi_2(t), 0 \le t < T,\\
u(x,T) &= p(e^x),\ x \in (x_{min}, x_{max}).
\end{aligned}$$

We introduce the points $x_i = x_{min} + i\Delta x,\ i = 0,\dots,n$ and $t_k = k\Delta t,\ k = 0,\dots,m$ and denote by $U_{i,k}$ the approximate values of $u(x_i, t_k)$. Here $\Delta x = \frac{x_{max}-x_{min}}{n}$ and $\Delta t = \frac{T}{m}$. In the case of the basic implicit finite difference method we compute the values U_{ik} as follows: using the final condition we set

$$U_{im} = p(e^{x_i}),\ i = 0,\dots,n$$

and for determining the values of $U_{ik},\ i = 0,\dots,n,\ k = m-1,\dots,0$ we solve for each value of k (starting with $k = m-1$) a three-diagonal system

$$\begin{aligned}
U_{0k} &= \phi_1(t_k),\\
a_{ik}U_{i-1,k} + b_{ik}U_{ik} + c_{ik}U_{i+1,k} &= U_{i,k+1},\ i = 1,\dots,n-1,\\
U_{nk} &= \phi_2(t_k)
\end{aligned}$$

for the unknown values of $U_{ik},\ i = 0,\dots,n$. Here

$$\begin{aligned}
a_{ik} &= -\frac{\alpha(x_i,\tau_k)\Delta t}{\Delta x^2} + \frac{\beta(x_i,t_k)\Delta t}{2\Delta x},\\
b_{ik} &= 1 + \frac{2\alpha(x_i,t_k)\Delta t}{\Delta x^2} + r\Delta t,\\
c_{ik} &= -\frac{\alpha(x_i,t_k)\Delta t}{\Delta x^2} - \frac{\beta(x_i,t_k)\Delta t}{2\Delta x}.
\end{aligned}$$

In the case of Crank-Nicolson method we solve at each time step the system

$$\begin{aligned}
U_{0k} &= \phi_1(t_k),\\
a_{ik}U_{i-1,k} + b_{ik}U_{ik} + c_{ik}U_{i+1,k} &= d_{ik}U_{i-1,k+1} + e_{ik}U_{i,k+1} + f_{ik}U_{i+1,k+1},\ i = 1,\dots,n-1,\\
U_{nk} &= \phi_2(t_k),
\end{aligned}$$

where

$$\begin{aligned}
a_{ik} &= -\frac{\alpha(x_i,t_k+\frac{\Delta t}{2})\Delta t}{2\Delta x^2} + \frac{\beta(x_i,t_k+\frac{\Delta t}{2})\Delta t}{4\Delta x},\\
b_{ik} &= 1 + \frac{\alpha(x_i,t_k+\frac{\Delta t}{2})\Delta t}{\Delta x^2} + \frac{r\Delta t}{2},\\
c_{ik} &= -\frac{\alpha(x_i,t_k+\frac{\Delta t}{2})\Delta t}{2\Delta x^2} - \frac{\beta(x_i,t_k+\frac{\Delta t}{2})\Delta t}{4\Delta x},\\
d_{ik} &= -a_{ik},\ f_{ik} = -c_{ik},\\
e_{ik} &= 1 - \frac{\alpha(x_i,t_k+\frac{\Delta t}{2})\Delta t}{\Delta x^2} - \frac{r\Delta t}{2}.
\end{aligned}$$

Exercise 1. Write a function that for given values of m, n, x_{min}, x_{max}, T and for given functions p,σ, ϕ_1 and ϕ_2 returns the values $U_{i0},\ i = 0, \ldots, n$ of the approximate solution (option prices) obtained by solving the transformed BS equation by the implicit finite difference method. Use this method for computing approximate values of the option price in the case $r = 0.02$, $\sigma(s,t) = 0.5$, $D = 0.03$, $T = 0.5$, $E = 100$, $S0 = 98$, $p(s) = 2 \cdot |E - s|$, $\rho = 2$, $x_{min} = \ln \frac{S0}{\rho}$, $x_{max} = \ln(\rho\, S0)$, $n = 20$, $m = 100$. Use $\phi_1(t) = p(e^{x_{min}}),\ \phi_2(t) = p(e^{x_{max}})$

Exercise 2. Repeat the previous exercise in the case of boundary conditions derived from special solutions.

Exercise 3. Implement CN method for solving the transformed BS equation. Use it for finding the an approximate price at $t = 0$ for the option described in problem 1 in the case of constant boundary conditions, $\rho = 4$, $m = 100$, $n = 200$. Also find the actual error of the answer (Hint: the exact price can be computed as the sum of put and call options).

Practical Homework 5. (Deadline 22.11.2018) When we use a finite difference method for option pricing, there are two sources of errors: 1) truncation error, caused by introduction of x_{min} and x_{max} and boundary conditions; 2) discretization error, caused by approximating partial derivatives with finite difference approximations. In this homework we consider a possibility to estimate the discretization error. Namely, when we derived explicit and basic implicit methods, we saw that discretization error is $O(\Delta t + \Delta x^2)$. Of cause, for this estimate to be valid, the solution of the PDE has to satisfy some differentiability conditions, which do not always hold in the case of option pricing, but for this lab, we assume that the estimate holds. This estimate can be made more precise: for large enough m and n the discretization error behaves like $const. \cdot (\Delta t + \Delta x^2)$ if we consider error at a specific point (x, t). Although we do not know the constant $const.$, this enables us to say that if we multiply m by 4 and n by 2, the discretization error gets approximately 4 times smaller and therefore, by computing answer1 with $m = m_0,\ n = n_0$ and answer2 by $m = 4\dot{m}_0, n = 2\dot{n}_0$, we can estimate the discretization error of the last result by computing $(answer1 - answer2)/3$ (detailed explanations for that will be given in the lecture). In the case of explicit method, if m is computed from the stability condition, it is automatically multiplied by a number that is close to 4 whenever we multiply n by two, so we can use the same estimate. This is called Runge's error estimate and is widely used by practitioners.

Use Runge's estimate to compute approximately the option price in the case $r = 0.02$, $\sigma(s,t) = 0.4 + \frac{0.2}{1+0.02(s-65)^2}$, $D = 0.02$, $T = 0.5$, $S0 = 63$, $\rho = 3$, $x_{min} = \ln \frac{S0}{\rho}$, $x_{max} = \ln(\rho\, S0)$, the pay-off function

$$p(s) = \begin{cases} 10 - \frac{s}{8}, & s \le 60, \\ \frac{(s-90)^2+300}{480}, & 60 < s \le 120, \\ \frac{s}{8} - 12.5, & s > 120 \end{cases}$$

and boundary conditions corresponding to suitable special solutions so that discretization error is less than 0.02 by both the explicit method and the basic implicit method, starting with $n = 10$ and, in the case of basic implicit method, $m = 5$.