Game Theory

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Game Theory

- **Game Theory is the mathematical study of strategic situations, i.e. where there is more than one decision-maker, and each decision-maker can affect the outcome.**
- **My choice may affect your problem.**
- **We need a method that takes everyone's choices into account.**

Some Applications of Game Theory

- **The study of military strategies.**
- **Auctions.**
- **The study of oligopolies (Oligopoly is a market form wherein a market is dominated by a small number of big sellers (oligopolists)),** ***e.g. Boeing & Airbus, Visa and Mastercard***

Military Strategies

- **In the early 1950s, the USA and USSR were involved in a nuclear arms race.**
- **Assume that each country can build an arsenal of nuclear bombs, or can refrain from doing so.**

- **Each country's favorite outcome is that it has bombs and the other country does not.**
- **The next best outcome is that neither country has any bombs.**
- **The next best outcome is that both countries have bombs (what matters is relative strength, and bombs are costly to build).**
- **The worst outcome is that only the other country has bombs.**

What is a Game?

- **A game consists of**
  - **a set of players (USA and USSR)**
  - **a set of strategies for each player (Build Bomb or Don’t Build)**
  - **the payoffs to each player for every possible choice of strategies by the players. (If you make a bomb your strength increases, but bombs are costly to build)**

What is a Game?

- A **game** (Duopoly) consists of
  - a set of **players** (Firm 1 and Firm 2)
  - a set of **strategies** for each player (Quantity: Q_1 and Q_2)
  - the **payoffs** to each player for every possible choice of strategies by the players. (Profits earned by Firm 1 and Firm 2)

Two-Player Games

- **A game with just two players is a two-player game.**
- **We will study only games in which there are two players, each of whom can choose between only two actions.**

An Example of a Two-Player Game

- **The players are called A and B.**
- **Player A has two actions, called “Up” and “Down”.**

An Example of a Two-Player Game

- **Player B has two actions, called "Left" and "Right".**
- **The table showing the payoffs to both players for each of the four possible action combinations is the game's payoff matrix.**

An Example of a Two-Player Game

		Player B	
		L	**R**
Player A	**U**	**(3,9)**	**(1,8)**
	D	**(0,0)**	**(2,1)**

This is the game's payoff matrix.

Player A's payoff is shown first.
Player B's payoff is shown second.

An Example of a Two-Player Game

Player B

		L	R
Player A	U	(3,9)	(1,8)
	D	(0,0)	(2,1)

A play of the game is a pair such as (U,R) where the 1st element is the action chosen by Player A and the 2nd is the action chosen by Player B.

An Example of a Two-Player Game

		Player B L	Player B R
Player A	U	(3,9)	(1,8)
Player A	D	(0,0)	(2,1)

This is the game's payoff matrix.

E.g. if A plays **U**p and B plays **R**ight then A's payoff is **1** and B's payoff is **8**.

An Example of a Two-Player Game

		Player B	
		L	**R**
Player A	**U**	**(3,9)**	**(1,8)**
	D	**(0,0)**	**(2,1)**

This is the game's payoff matrix.

And if A plays **D**own and B plays **R**ight then A's payoff is **2** and B's payoff is **1**.

An Example of a Two-Player Game

		Player B	
		L	**R**
Player A	**U**	**(3,9)**	**(1,8)**
	D	**(0,0)**	**(2,1)**

What plays are we likely to see for this game?

An Example of a Two-Player Game

		Player B	
		L	**R**
Player A	**U**	**(3,9)**	**(1,8)**
	D	**(0,0)**	**(2,1)**

Is (U,R) a likely play?

An Example of a Two-Player Game

Player B

Is (U,R) a likely play?

		L	R
Player A	U	(3,9)	(1,8)
	D	(0,0)	(2,1)

If B plays Right then A's best reply is Down since this improves A's payoff from 1 to 2. So (U,R) is not a likely play.

An Example of a Two-Player Game

		Player B	
		L	**R**
Player A	**U**	**(3,9)**	**(1,8)**
	D	**(0,0)**	**(2,1)**

Is (D,R) a likely play?

An Example of a Two-Player Game

Player B

		L	R
Player A	U	(3,9)	(1,8)
	D	(0,0)	(2,1)

Is (D,R) a likely play?

If B plays Right then A's best reply is Down.

An Example of a Two-Player Game

Player B

Is (D,R) a likely play?

Player A		L	R
	U	(3,9)	(1,8)
	D	(0,0)	(2,1)

If B plays Right then A's best reply is Down.
If A plays Down then B's best reply is Right.
So (D,R) is a likely play.

An Example of a Two-Player Game

Player B

		L	R
Player A	U	(3,9)	(1,8)
	D	(0,0)	(2,1)

Is (D,L) a likely play?

An Example of a Two-Player Game

Player B

L → R

Is (D,L) a likely play?

Player A		L	R
	U	(3,9)	(1,8)
	D	(0,0)	(2,1)

If A plays Down then B's best reply is Right, so (D,L) is not a likely play.

An Example of a Two-Player Game

Player B

		L	R
Player A	U	(3,9)	(1,8)
	D	(0,0)	(2,1)

Is (U,L) a likely play?

An Example of a Two-Player Game

		Player B L	Player B R
Player A	U	(3,9)	(1,8)
Player A	D	(0,0)	(2,1)

Is (U,L) a likely play?

If A plays Up then B's best reply is Left.

An Example of a Two-Player Game

Is (U,L) a likely play?

		Player B	
		L	R
Player A	U	(3,9)	(1,8)
	D	(0,0)	(2,1)

If A plays Up then B's best reply is Left.
If B plays Left then A's best reply is Up.
So (U,L) is a likely play.

Nash Equilibrium

- **A play of the game where each strategy is a best reply to the other is a**
  Nash equilibrium.
- **Our example has two Nash equilibria; (U,L) and (D,R).**

An Example of a Two-Player Game

		Player B	
		L	**R**
Player A	**U**	**(3,9)**	**(1,8)**
	D	**(0,0)**	**(2,1)**

(U,L) and **(D,R)** are both Nash equilibria for the game.

An Example of a Two-Player Game

		Player B	
		L	R
Player A	U	(3,9)	(1,8)
	D	(0,0)	(2,1)

(U,L) and (D,R) are both Nash equilibria for the game. But which will we see? Notice that (U,L) is preferred to (D,R) by both players. Must we then see (U,L) only?

The Prisoner's Dilemma

		Clyde S	Clyde C
Bonnie	S	(-1,-1)	(-10,0)
	C	(0,-10)	(-5,-5)

Strategies: Silence and Confess.

What plays are we likely to see for this game?

The Prisoner's Dilemma

		Clyde: S	Clyde: C
Bonnie	S	(-1,-1)	(-10,0)
	C	(0,-10)	(-5,-5)

If Bonnie plays Silence then Clyde's best reply is Confess.

The Prisoner's Dilemma

Clyde

		S	C
Bonnie	S	**(-1,-1)**	**(-10,0)**
	C	**(0,-10)**	**(-5,-5)**

If Bonnie plays Silence then Clyde's best reply is Confess.
If Bonnie plays Confess then Clyde's best reply is Confess.

The Prisoner's Dilemma

		Clyde S	Clyde C
Bonnie	S	(-1,-1)	(-10,0)
	C	(0,-10)	(-5,-5)

So no matter what Bonnie plays, Clyde's best reply is always Confess. Confess is a **dominant strategy** for Clyde.

The Prisoner's Dilemma

Clyde

		S	C
Bonnie	S	(-1,-1)	(-10,0)
	C	(0,-10)	(-5,-5)

Similarly, no matter what Clyde plays, Bonnie's best reply is always Confess. Confess is a **dominant strategy** for Bonnie also.

Who Plays When?

- **In both examples the players chose their strategies simultaneously.**
- **Such games are simultaneous play games.**

Who Plays When?

- **But there are other games in which one player plays before another player.**
- **Such games are sequential play games.**
- **The player who plays first is the leader. The player who plays second is the follower.**

A Sequential Game Example

- **Sometimes a game has more than one Nash equilibrium and it is hard to say which is more likely to occur.**
- **When a game is sequential it is sometimes possible to argue that one of the Nash equilibria is more likely to occur than the other.**

A Sequential Game Example

		Player B	
		L	R
Player A	U	(3,9)	(1,8)
	D	(0,0)	(2,1)

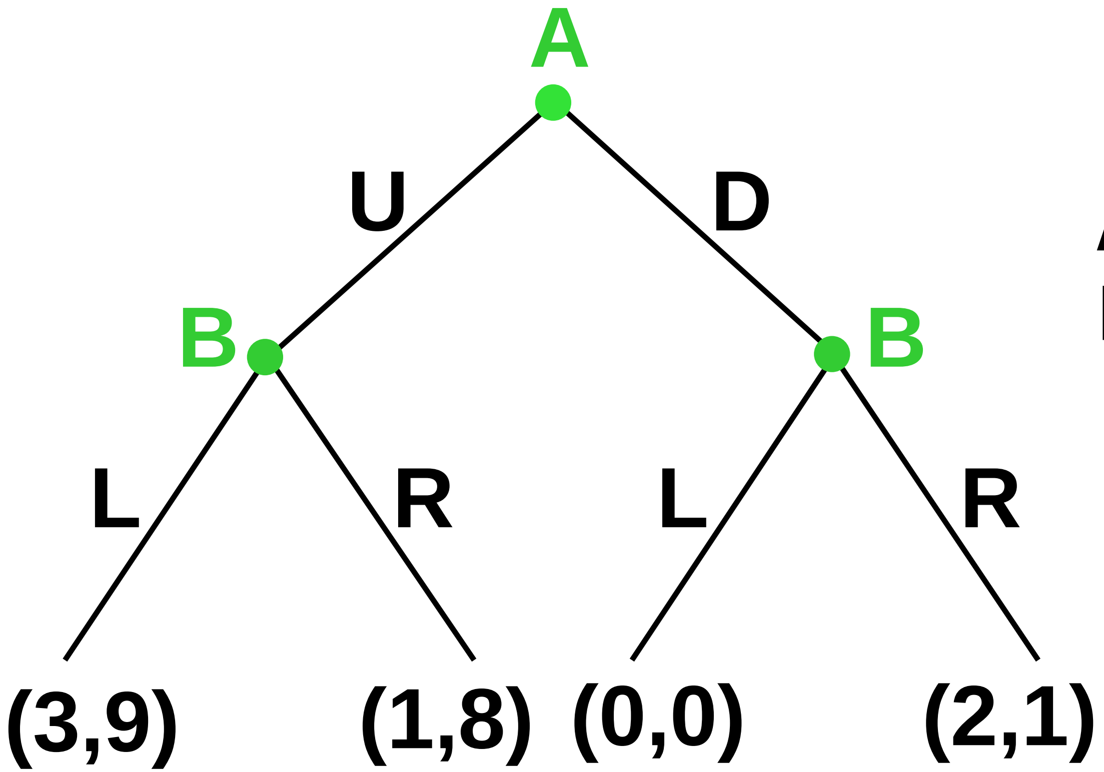
(U,L) and (D,R) are both NE when this game is played **simultaneously** and we have no way of deciding which equilibrium is more likely to occur.

A Sequential Game Example

		Player B	
		L	R
Player A	U	(3,9)	(1,8)
	D	(0,0)	(2,1)

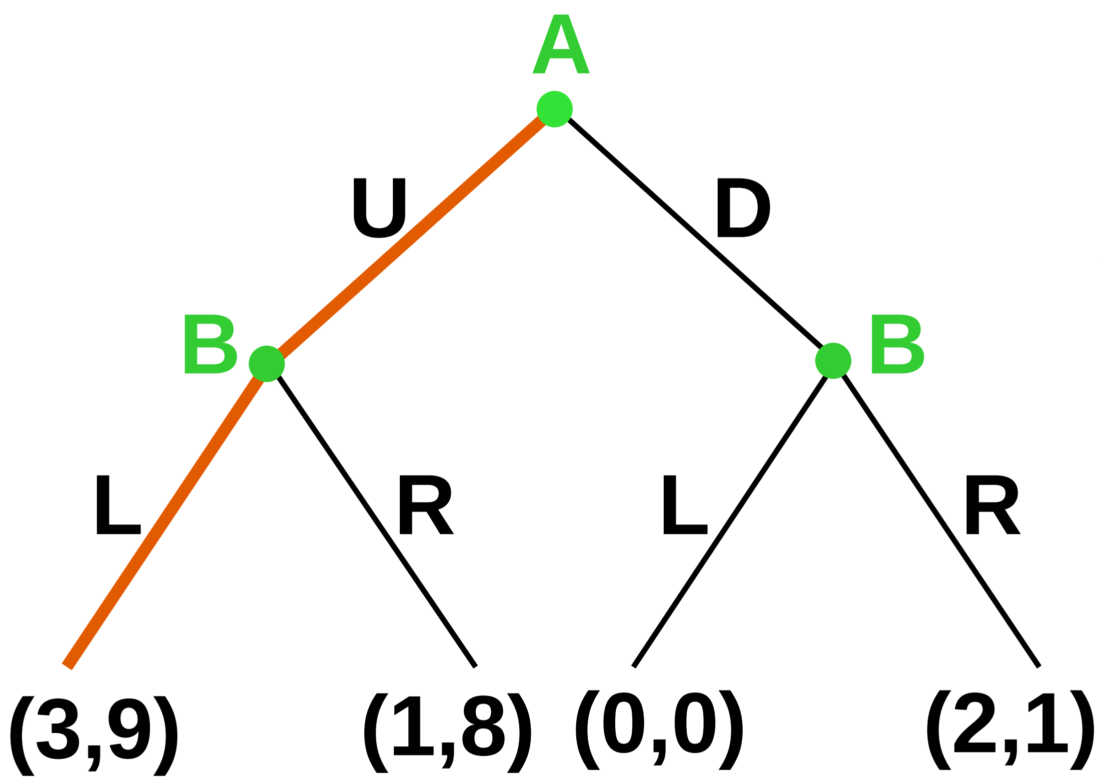
Suppose instead that the game is played **sequentially**, with A leading and B following. We can rewrite the game in its **extensive form**.

A Sequential Game Example



A plays first.
B plays second.

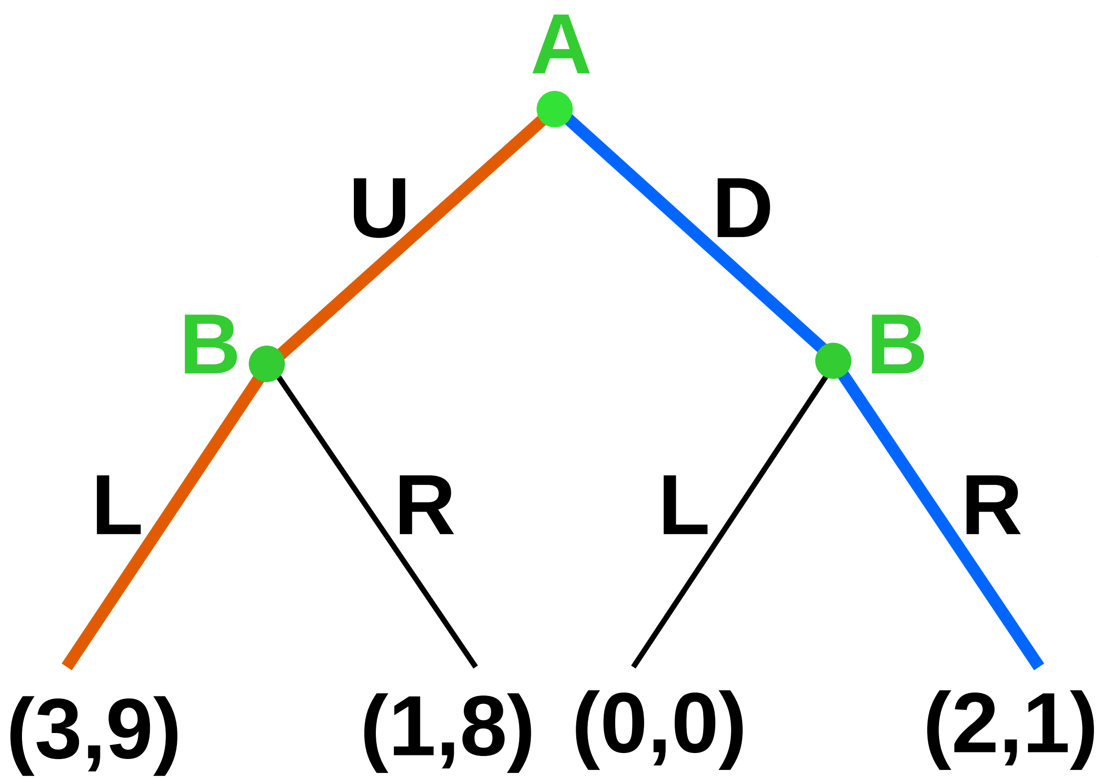
A Sequential Game Example



A plays first.
B plays second.

(U,L) is a Nash equilibrium.

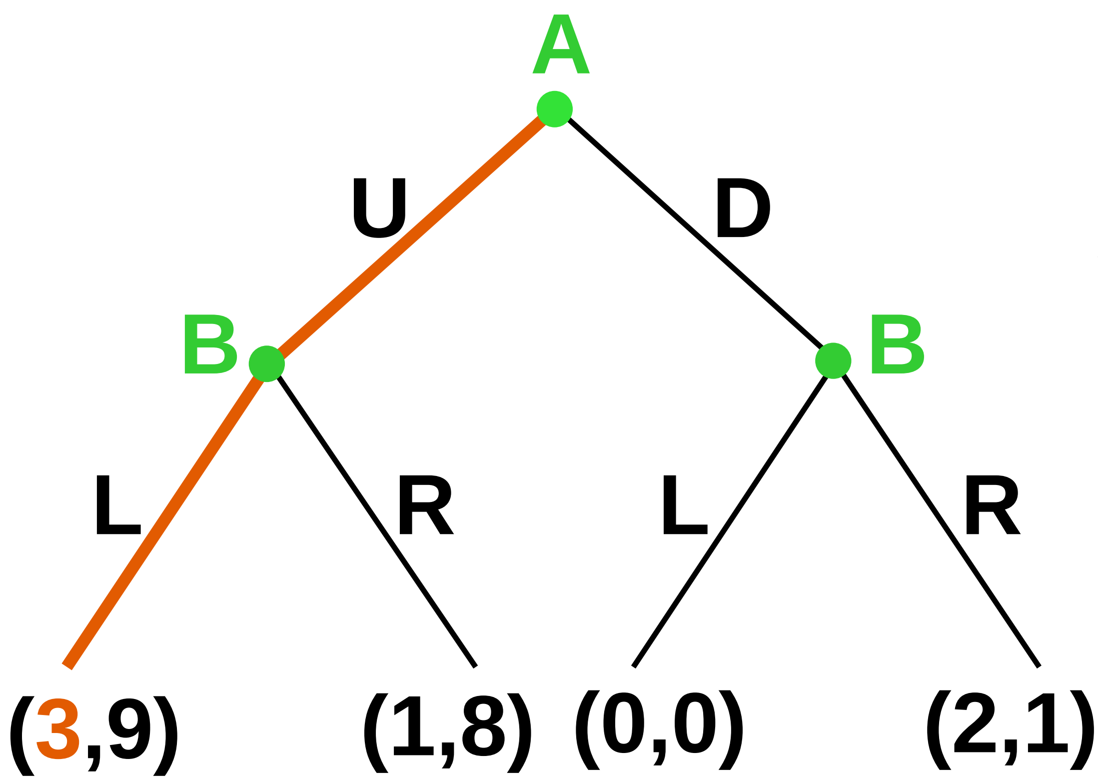
A Sequential Game Example



A plays first.
B plays second.

(U,L) is a Nash equilibrium. So is **(D,R)**.
Is one equilibrium more likely to occur?

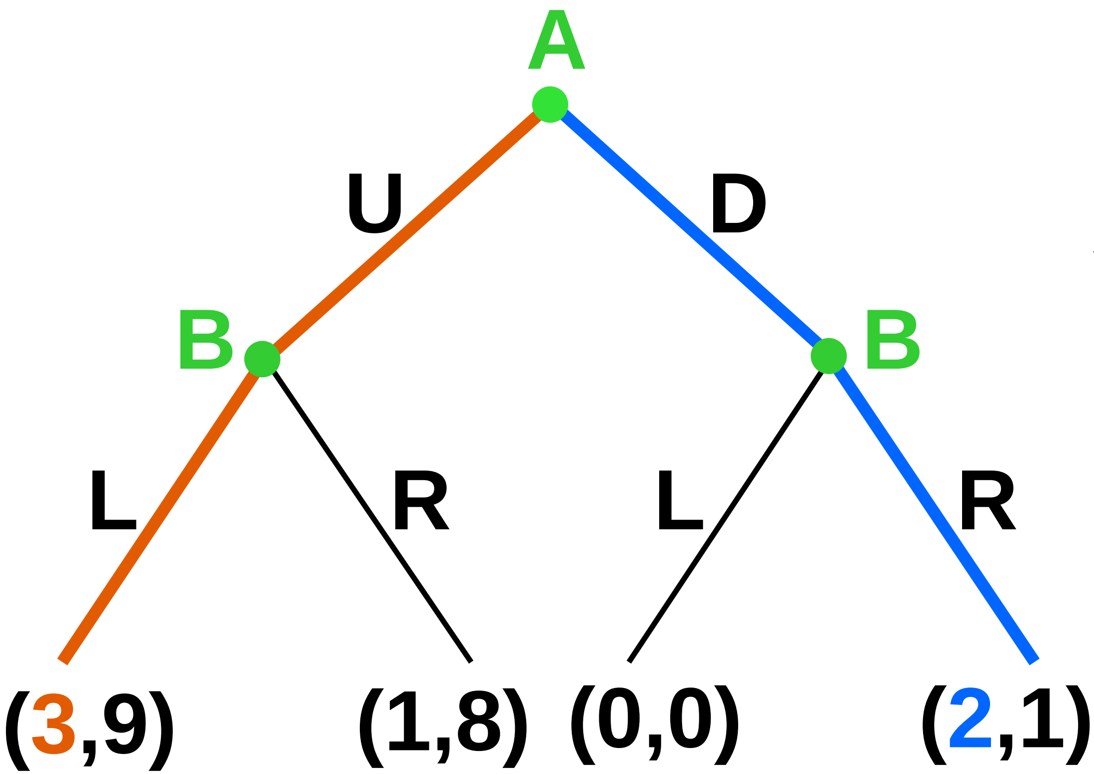
A Sequential Game Example



A plays first.
B plays second.

If A plays **U** then B follows with **L**; A gets 3.

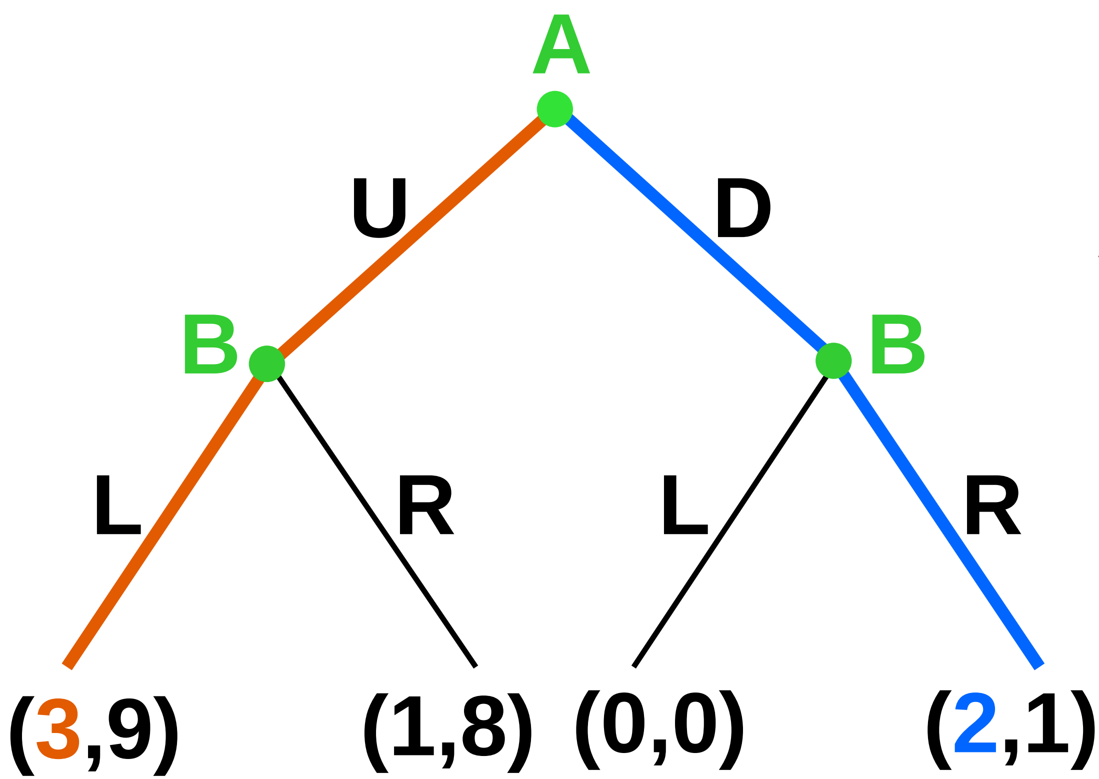
A Sequential Game Example



A plays first.
B plays second.

If A plays **U** then B follows with **L**; A gets 3.
If A plays **D** then B follows with **R**; A gets 2.

A Sequential Game Example



A plays first.
B plays second.

So (U,L) is the likely NE.

If A plays **U** then B follows with **L**; A gets 3.
If A plays **D** then B follows with **R**; A gets 2.

A Sequential Game Example

		Player B	
		L	R
Player A	U	(3,9)	(1,8)
	D	(0,0)	(2,1)

This is our original example once more. Suppose again that play is simultaneous. We discovered that the game has two Nash equilibria; (U,L) and (D,R).

A Sequential Game Example

		Player B	
		L	R
Player A	U	(3,9)	(1,8)
	D	(0,0)	(2,1)

Player A has been thought of as choosing to play either U or D, but no combination of both; *i.e.* as playing **purely** U or D. U and D are Player A's **pure strategies**.

A Sequential Game Example

		Player B	
		L	R
Player A	U	(3,9)	(1,8)
	D	(0,0)	(2,1)

Similarly, L and R are Player B's **pure strategies**.

A Sequential Game Example

		Player B	
		L	R
Player A	U	(3,9)	(1,8)
	D	(0,0)	(2,1)

Consequently, (U,L) and (D,R) are **pure strategy Nash equilibria**. Must every game have at least one pure strategy Nash equilibrium?

Pure Strategies

Player B

		L	R
Player A	U	(1,2)	(0,4)
	D	(0,5)	(3,2)

Here is a new game. Are there any pure strategy Nash equilibria?

Pure Strategies

		Player B	
		L	**R**
Player A	**U**	**(1,2)**	**(0,4)**
	D	**(0,5)**	**(3,2)**

Is (U,L) a Nash equilibrium?

Pure Strategies

Player B

Player A		L	R
	U	(1,2)	(0,4)
	D	(0,5)	(3,2)

Is (U,L) a Nash equilibrium? No.

Is (U,R) a Nash equilibrium?

Pure Strategies

		Player B: L	Player B: R
Player A	U	(1,2)	(0,4)
	D	(0,5)	(3,2)

Is (U,L) a Nash equilibrium? No.
Is (U,R) a Nash equilibrium? No.
Is (D,L) a Nash equilibrium?

Pure Strategies

		Player B	
		L	**R**
Player A	**U**	**(1,2)**	**(0,4)**
	D	**(0,5)**	**(3,2)**

Is (U,L) a Nash equilibrium? No.
Is (U,R) a Nash equilibrium? No.
Is (D,L) a Nash equilibrium? No.
Is (D,R) a Nash equilibrium?

Pure Strategies

		Player B	
		L	**R**
Player A	**U**	**(1,2)**	**(0,4)**
	D	**(0,5)**	**(3,2)**

Is (U,L) a Nash equilibrium? No.
Is (U,R) a Nash equilibrium? No.
Is (D,L) a Nash equilibrium? No.
Is (D,R) a Nash equilibrium? No.

Pure Strategies

Player B

		L	R
Player A	U	(1,2)	(0,4)
	D	(0,5)	(3,2)

So the game has no Nash equilibria in pure strategies. Even so, the game does have a Nash equilibrium, but in **mixed strategies**.

Mixed Strategies

- **Instead of playing purely Up or Down, Player A selects a probability distribution (π_U,1-π_U), meaning that with probability π_U Player A will play Up and with probability 1-π_U will play Down.**
- **Player A is mixing over the pure strategies Up and Down.**
- **The probability distribution (π_U,1-π_U) is a mixed strategy for Player A.**

Mixed Strategies

- **Similarly, Player B selects a probability distribution (π_L,1-π_L), meaning that with probability π_L Player B will play Left and with probability 1-π_L will play Right.**
- **Player B is mixing over the pure strategies Left and Right.**
- **The probability distribution (π_L,1-π_L) is a mixed strategy for Player B.**

Mixed Strategies

		Player B	
		L	**R**
Player A	**U**	**(1,2)**	**(0,4)**
	D	**(0,5)**	**(3,2)**

This game has no Nash equilibrium in pure strategies, but it does have a Nash equilibrium in mixed strategies. How is it computed?

Mixed Strategies

		Player B	
		L, π_L	**R,** $1-\pi_L$
Player A	**U,** π_U	**(1,2)**	**(0,4)**
	D, $1-\pi_U$	**(0,5)**	**(3,2)**

Mixed Strategies

Player B

		L, π_L	**R,** $1-\pi_L$
Player A	**U,** π_U	**(1,2)**	**(0,4)**
	D, $1-\pi_U$	**(0,5)**	**(3,2)**

A's expected value of choosing Up is ??

Mixed Strategies

		Player B	
		L, π_L	**R,** $1-\pi_L$
Player A	**U,** π_U	**(1,2)**	**(0,4)**
	D, $1-\pi_U$	**(0,5)**	**(3,2)**

A's expected value of choosing Up is π_L.

A's expected value of choosing Down is ??

Mixed Strategies

Player B

		L, π_L	**R**, $1-\pi_R$
Player A	**U**, π_U	**(1,2)**	**(0,4)**
	D, $1-\pi_U$	**(0,5)**	**(3,2)**

A's expected value of choosing Up is π_L.
A's expected value of choosing Down is $3(1 - \pi_L)$.

Mixed Strategies

		Player B	
		L, π_L	**R,** $1-\pi_L$
Player A	**U,** π_U	**(1,2)**	**(0,4)**
	D, $1-\pi_U$	**(0,5)**	**(3,2)**

A's expected value of choosing Up is π_L.
A's expected value of choosing Down is $3(1 - \pi_L)$.
If $\pi_L > 3(1 - \pi_L)$ then A will choose only Up, but there is no NE in which A plays only Up.

Mixed Strategies

		Player B	
		L, π_L	**R,** $1-\pi_L$
Player A	**U,** π_U	**(1,2)**	**(0,4)**
	D, $1-\pi_U$	**(0,5)**	**(3,2)**

A's expected value of choosing Up is π_L.
A's expected value of choosing Down is $3(1 - \pi_L)$.
If $\pi_L < 3(1 - \pi_L)$ then A will choose only Down, but there is no NE in which A plays only Down.

Mixed Strategies

		Player B	
		L, π_L	**R,** $1-\pi_L$
Player A	**U,** π_U	**(1,2)**	**(0,4)**
	D, $1-\pi_U$	**(0,5)**	**(3,2)**

If there is a NE necessarily $\pi_L = 3(1 - \pi_L) \Rightarrow \pi_L = 3/4$; *i.e.* the way B mixes over Left and Right must make A indifferent between choosing Up or Down.

Mixed Strategies

		Player B	
		L, 3/4	**R,** 1/4
Player A	**U,** π_U	**(1,2)**	**(0,4)**
	D, $1-\pi_U$	**(0,5)**	**(3,2)**

If there is a NE necessarily $\pi_L = 3(1 - \pi_L) \Rightarrow \pi_L = 3/4$; *i.e.* the way B mixes over Left and Right must make A indifferent between choosing Up or Down.

Mixed Strategies

		Player B	
		L, 3/4	**R,** 1/4
Player A	**U,** π_U	**(1,2)**	**(0,4)**
	D, $1-\pi_U$	**(0,5)**	**(3,2)**

Mixed Strategies

Player B

	L, 3/4	**R,** 1/4
U, π_U	**(1,2)**	**(0,4)**
Player A **D,** $1-\pi_U$	**(0,5)**	**(3,2)**

B's expected value of choosing Left is ??

Mixed Strategies

		Player B	
		L, 3/4	**R,** 1/4
Player A	**U,** π_U	**(1,2)**	**(0,4)**
	D, $1-\pi_U$	**(0,5)**	**(3,2)**

B' s expected value of choosing Left is $2\pi_U + 5(1 - \pi_U)$.

B' s expected value of choosing Right is ??

Mixed Strategies

		Player B	
		L, 3/4	**R,** 1/4
Player A	**U,** π_U	**(1,2)**	**(0,4)**
	D, $1-\pi_U$	**(0,5)**	**(3,2)**

B's expected value of choosing Left is $2\pi_U + 5(1 - \pi_U)$.

B's expected value of choosing Right is $4\pi_U + 2(1 - \pi_U)$.

Mixed Strategies

		Player B	
		L, 3/4	**R,** 1/4
Player A	**U,** π_U	**(1,2)**	**(0,4)**
	D, $1-\pi_U$	**(0,5)**	**(3,2)**

B' s expected value of choosing Left is $2\pi_U + 5(1 - \pi_U)$.
B' s expected value of choosing Right is $4\pi_U + 2(1 - \pi_U)$.
If $2\pi_U + 5(1 - \pi_U) > 4\pi_U + 2(1 - \pi_U)$ then B will choose only Left, but there is no NE in which B plays only Left.

Mixed Strategies

		Player B	
		L, 3/4	**R,** 1/4
Player A	**U,** π_U	**(1,2)**	**(0,4)**
	D, $1-\pi_U$	**(0,5)**	**(3,2)**

B's expected value of choosing Left is $2\pi_U + 5(1 - \pi_U)$.
B's expected value of choosing Right is $4\pi_U + 2(1 - \pi_U)$.
If $2\pi_U + 5(1 - \pi_U) < 4\pi_U + 2(1 - \pi_U)$ then B plays only Right, but there is no NE where B plays only Right.

Mixed Strategies

		Player B	
		L, 3/4	**R,** 1/4
Player A	**U,** 3/5	**(1,2)**	**(0,4)**
	D, 2/5	**(0,5)**	**(3,2)**

If there is a NE then necessarily

$2\pi_U + 5(1 - \pi_U) = 4\pi_U + 2(1 - \pi_U) \Rightarrow \pi_U = 3/5;$

i.e. the way A mixes over Up and Down must make B indifferent between choosing Left or Right.

Mixed Strategies

		Player B	
		L, 3/4	**R,** 1/4
Player A	**U,** 3/5	**(1,2)**	**(0,4)**
	D, 2/5	**(0,5)**	**(3,2)**

The game's only Nash equilibrium consists of A playing the mixed strategy (3/5, 2/5) and B playing the mixed strategy (3/4, 1/4).

Mixed Strategies

		Player B	
		L, 3/4	**R,** 1/4
Player A	**U,** 3/5	**(1,2)** 9/20	**(0,4)**
	D, 2/5	**(0,5)**	**(3,2)**

The payoff will be (1,2) with probability

$3/5 \times 3/4 = 9/20.$

Mixed Strategies

Player B

	L, 3/4	**R,** 1/4
U, 3/5	**(1,2)** 9/20	**(0,4)** 3/20
D, 2/5	**(0,5)**	**(3,2)**

Player A

The payoff will be (0,4) with probability $3/5 \times 1/4 = 3/20$.

Mixed Strategies

		Player B	
		L, 3/4	**R,** 1/4
Player A	**U,** 3/5	**(1,2)** 9/20	**(0,4)** 3/20
	D, 2/5	**(0,5)** 6/20	**(3,2)**

The payoff will be (0,5) with probability

2/5 × 3/4 = 6/20.

Mixed Strategies

Player B

	L, 3/4	**R,** 1/4
Player A **U,** 3/5	**(1,2)** 9/20	**(0,4)** 3/20
Player A **D,** 2/5	**(0,5)** 6/20	**(3,2)** 2/20

The payoff will be (3,2) with probability

$2/5 \times 1/4 = 2/20$.

Mixed Strategies

Player B

		L, 3/4	**R,** 1/4
Player A	**U,** 3/5	**(1,2)** 9/20	**(0,4)** 3/20
	D, 2/5	**(0,5)** 6/20	**(3,2)** 2/20

A's NE expected payoff is

$$1 \times 9/20 + 3 \times 2/20 = 3/4.$$

Mixed Strategies

		Player B	
		L, 3/4	**R,** 1/4
Player A	**U,** 3/5	**(1,2)** 9/20	**(0,4)** 3/20
	D, 2/5	**(0,5)** 6/20	**(3,2)** 2/20

A's NE expected payoff is

$$1 \times 9/20 + 3 \times 2/20 = 3/4.$$

B's NE expected payoff is

$$2 \times 9/20 + 4 \times 3/20 + 5 \times 6/20 + 2 \times 2/20 = 16/5.$$

How Many Nash Equilibria?

- **A game with a finite number of players, each with a finite number of pure strategies, has at least one Nash equilibrium.**
- **So if the game has no pure strategy Nash equilibrium then it must have at least one mixed strategy Nash equilibrium.**