TRANSITION TO ADVANCED MATHEMATICS

Book of Exercises

Fall 2019

Contents

1. Concept, definition, theorem, assumption and conclusion

1. Generalize and restrict the following concepts:

a) parallelogram;
b) triangle;
c) geometry;
d) rational number;
e) root;

2. Give examples of properties that characterize

a) all triangles;
b) only some triangles;
c) that do not apply to any triangle.

3. Explain whether the following definitions are correct or not. If not, then change them so they would be.

a) A parallelogram is called a square if all its angles are right angles.
b) A natural number that is bigger than one and is divisible only by one and itself is called a prime number.
c) The line between the vertex and the opposite side of a triangle is called the height of the triangle.
d) An empty set is a set that does not contain anything except the number zero.
e) The speed of constant motion is the ratio of the distance and the time.
f) The absolute value of a number is the number itself if the number is positive and the inverse number if the number is negative.
g) Quadratic equation is an equation that can be transformed to the form $ax^2+bx+c=0$.

4. Which of the rules that apply to definitions have been broken?

a) Human is an animal who builds houses.
b) Laughable is something we laugh at.
c) Vice is the opposite of benefit.
d) Barrel is a vessel that is used for holding liquids.
e) Dog is man's best friend.
f) Ignorance is the lack of knowledge.
g) Two polygons are called similar is they have the same shape.
h) Physics is a science that is taught by the physics teacher.

5. Is the following classification correct? If not, the why it is not?

a) People are divided into men, women and elders.
b) Integers are divided into negative integers, number zero and natural numbers.
c) Quantities can be equal or not equal.
d) Two lines can be parallel, intersecting or oblique lines.
e) Trees are divided into hardwood trees and conifers.

6. Determine the assumption and the conclusion.

a) If $2x-1=5$, then $x=3$.
b) If there is smoke, then there is also fire.
c) From $\log_2 x = 1$ follows that $x=2$.
d) $ab \neq 0$; $ab > 0$.
e) $\sin\alpha = \cos\beta$; $\alpha+\beta = 90^\circ$.
f) $abc = 0$ is a necessary condition for $a=b=c=0$.
g) The area of a right angled triangle is equal to the half of the product of its legs.
h) The base angle of the right angled, isosceles triangle is 45°.

7. Rewrite the sentences in the form *if ..., then ...*.

a) Diagonals of the rectangle are equal.
b) $x=-6 \Rightarrow |x| = 6$.
c) $x^2 = 0$ if and only if $x=0$.
d) The product of two even numbers is divisible by four.
e) The square of a prime number is not a prime.
f) From $a+b=a$ follows that $b=0$.
g) Every regular polygon is equilateral.
h) The area of a right angled triangle is equal to half of the product of its legs.
i) The base angle of the right angled, isosceles triangle is 45°.

8. It is easy to confuse propositions (a) "If A, then B" and (b) "If B, then A". Find two statements A and B such that the proposition (a) is true and proposition (b) is false.

9. Write the converses, inverses and contrapositives of the given sentences. Which of them are true?

a) If $a=b$ and $c=d$, then $a-c=b-d$.
b) If $y<5$, then $y\neq 6$.
c) If $x=-5$, then $x^2=25$.
d) If the square of an integer is even, then the integer is even.
e) The diagonals of the rhombus are perpendicular.
f) If the free term of a quadratic equation is zero, then one of its solutions is also zero.

10. Instead of the dotted line write *necessary and sufficient* or only *necessary* or only *sufficient*.

a) To conclude that a quadrangle is a rectangle it is .... that its diagonals are equal.
b) Regularity of a polygon is .... for the existence of a circle that passes through all the vertices of the polygon.
c) To conclude that the sum of natural numbers $n+m$ is divisible by k it is ...., that n and m are divisible by k.
d) To conclude that a quadrangle is a rectangle it is .... that all its angles are equal.
e) To conclude that $\alpha=\beta$ it is .... that $\sin\alpha = \sin\beta$.
f) To conclude that $a \parallel c$ it is .... that $a \parallel b$ and $b \parallel c$.

11. Let a and b be such real numbers that $a<0$ and $b<0$. Are the following conditions necessary, sufficient, necessary and sufficient or neither necessary nor sufficient to conclude that $a>b$?

a) $b < a$
b) $|a| < |b|$
c) $a < 3b$
d) $a^2 < b^2$
e) $\sqrt{b^2} > |a|$
f) $\frac{b}{a} > 1$
g) $|\frac{a}{b}| > 1$
h) $\ln|a| > \ln|b|$

12. In the following sentences replace the dotted line with the words *and* and *or*, such that the sentences would be true (a and b are real numbers).

a) If $a > 0 \ldots b > 0$ then $ab > 0$.
b) $ab = 0$, if $a = 0 \ldots b = 0$.
c) $\frac{a}{b} = 0$, if $a = 0 \ldots b \neq 0$.
d) If $a \neq 0 \ldots b \neq 0$ then $ab \neq 0$.
e) If $ab < 0$, then $(a < 0 \ldots b > 0) \ldots (a > 0 \ldots b < 0)$.
f) The area of the triangle increases if the base ... the height increase(s) in length.

13. Read the following sentences (x, y, a, b and α are real numbers).

a) $\forall y,\ y^4 + 4 > 0$
b) $\forall a, \forall b,\ a \cdot b = b \cdot a$
c) $\forall \alpha,\ \cos^2 \alpha = 1 - \sin^2 \alpha$
d) $\exists x,\ \log_a x = 0$
e) $\exists \alpha,\ \tan \alpha \neq \frac{\sin \alpha}{\cos \alpha}$
f) $\forall a, \forall b,\ |a \pm b| \leqslant |a| + |b|$
g) $\exists x,\ e^x = 1$

14. Write the following sentences by using symbols $\forall$ and $\exists$.

a) The square of every real number is nonnegative.
b) There exists an angle α such that its cosine is equal to the number 0.5.
c) For every two numbers, their sum does not change when changing the order of the addends.
d) Among every four random numbers there exist two numbers such that their difference is divisible by three.
e) For every integer x there exists an integer y, such that $xy = 1$.
f) It does not matter which integer you choose, there always exists a different integer that is bigger.

15. Determine the truth values of following sentences if n and m are integers.

a) $\forall n,\ n + 1 > n$.
b) $\exists n,\ 2n = 3n$.
c) $\exists n,\ n = -n$.
d) $\forall n,\ 3n \leqslant 4n$.
e) $\forall n,\ n^2 \geqslant 0$.
f) $\exists n,\ n^2 = 2$.
g) $\forall n,\ n^2 \geqslant n$.
h) $\exists n,\ n^2 < 0$.
i) $\forall n, \exists m,\ n^2 < m$.
j) $\exists n, \forall m,\ n < m^2$.
k) $\forall n, \exists m,\ n + m = 0$.
l) $\exists n, \forall m,\ nm = m$.

16. Determine the truth values of following sentences if x and y are real numbers.

a) $\exists x,\ x^3 = -1$.
b) $\exists x,\ x^4 < x^2$.
c) $\forall x,\ (-x)^2 = x^2$.
d) $\forall x,\ 2x > x$.
e) $\exists x,\ x^2 = 2$.
f) $\exists x,\ x^2 = -1$.
g) $\forall x,\ x^2 + 2 \geqslant 1$.
h) $\forall x,\ x^2 \neq x$.
i) $\forall x,\ \exists y,\ x^2 = y$.
j) $\exists x,\ \forall y,\ xy = 0$.

Negations of sentences containing quantifiers $\forall$ (for all) and $\exists$ (exists):
$\neg(\forall x P(x)) \equiv \exists x(\neg P(x))$, $\neg(\exists x P(x)) \equiv \forall x(\neg P(x))$,
$\neg(\forall x \exists y P(x,y)) \equiv \exists x \forall y(\neg P(x,y))$, $\neg(\exists x \forall y P(x,y)) \equiv \forall x \exists y(\neg P(x,y))$.

17. Form the negations of given sentences and determine their truth values.
a) 2015 is divisible by 3.
b) Every prime number is odd.
c) It is not true that 2 is a prime number.
d) Every quadratic equation has two real solutions.
e) There exist a positive number such that we can not take its logarithm.
f) $\exists x,\ x^2 + 1 > 0$.
g) $\forall \alpha,\ \cos^2 \alpha = \dfrac{1}{1 + \tan^2 \alpha}$.

18. Negate the following sentences.
a) There exists a number that is equal to its opposite number.
b) For every triangle the sides are proportional with sines of their opposite angles
c) Every triangle has a circumcircle.
d) There exists a triangle such that its medians do not intersect in one point.
e) The square of every real number is nonnegative.
f) There exists a real number x, such that $y^3 \neq x$ for all real numbers y.
g) There exists a real number a, such that $a + x = x$ for every real number x.

19. Negate the following sentences.
a) All women are mothers.
b) All dogs have fleas.
c) Some students have solved all exercises from this exercise book.
d) There is someone in this classroom who does not have a good attitude.
e) Every student in this class has taken a course on calculus.
f) There exists an honest politician.
g) All Americans eat cheeseburgers.

20. Negate the following sentences.
a) $\exists x \in \mathbb{R},\ e^x < 0$.
b) $\forall a, b \in \mathbb{N},\ a + b = 1$.
c) $\exists x \in \mathbb{N},\ x = x + 1$.
d) $\forall x \in \mathbb{N},\ x + x = 2x$.
e) $\exists x \in \mathbb{Z},\ \forall y \in \mathbb{Z},\ x > y$.
f) $\forall x \in \mathbb{Z},\ \exists y \in \mathbb{Z},\ x + y = 0$.
g) $\forall x \in \mathbb{R},\ x^2 > x$.
h) $\exists x \in \mathbb{Q},\ x^2 = 2$.
i) $\forall x \in \mathbb{R},\ \forall y \in \mathbb{N},\ x + y > 0$.
j) $\forall x \in \mathbb{R},\ \exists y \in \mathbb{R},\ xy = 1$.
k) $\exists x \in \mathbb{R},\ \forall y \in \mathbb{R},\ xy = y$.
l) $\forall x \in \mathbb{R},\ \exists y \in \mathbb{R},\ x = y^2$.

21*. There are two villages in the Twovillage island: Truthful and Dishonest. In the village Truthful live people who always tell the truth and in the village Dishonest live people who always lie. A foreigner encountered six locals in the Twovillage island. Since the foreigner could not figure out who is from which village, then the foreigner asked each of them a question: "How many of you are truthful?". First five gave the following answers: "Two of us are truthful", "None of us is truthful", "Three of us are truthful",”Only one of us is truthful” and "Three of us are truthful". From these answers the foreigner could not figure out how many of them were truthful, but as soon as he heard the last answer, he knew exactly how many of them were truthful.
Was the sixth answer true? How many truthful people were in his company?

22*. Marek and Kaido just became friends with Kristel and they would like to know when is her birthday. Kristel gives them a list of ten possible dates: 5. May, 16. May, 19. May, 17. June, 18. June, 14. July, 16. July, 14. August, 15. August and 17. August. Then Kristel tells Marek which month is her birthday and Kaido which day of the month is her birthday.
Marek says: "I do not know when is Kristel’s birthday, but I do know that Kaido do not know either."
Kaido answers: ”At first I did not know when is Kristel’s birthday, but now I know."
Marek says: "Now I know too, when is Kristel’s birthday."
When is Kristel’s birthday?

2. Propositional calculus

A	B	$\neg A$	$A \wedge B$	$A \vee B$	$A \Rightarrow B$	$A \Leftrightarrow B$
t	t	v	t	t	t	t
t	v	v	v	t	v	v
v	t	t	v	t	t	v
v	v	t	v	v	t	t

Precedence of logical operators: $\neg$, $\wedge$, $\vee$, $\Rightarrow$, $\Leftrightarrow$.

Definition. Propositional formulas are exactly those that can be constructed by following rules:

a) every propositional variable is a propositional formula;
b) truth values t and f are propositional formulas;
c) if $\mathcal{F}$ is a propositional formula, then $\neg\mathcal{F}$ is also a propositional formula;
d) if $\mathcal{F}$ and $\mathcal{G}$ are propositional formulas, then $\mathcal{F} \wedge \mathcal{G}$, $\mathcal{F} \vee \mathcal{G}$, $\mathcal{F} \Rightarrow \mathcal{G}$ and $\mathcal{F} \Leftrightarrow \mathcal{G}$ are also propositional formulas;
e) if $\mathcal{F}$ is a propositional formula, then $(\mathcal{F})$ is also a propositional formula.

Definition. A propositional formula $\mathcal{F}$ is called a tautology if it is true for every valuation of its variables. Formula $\mathcal{F}$ is called a contradiction if it is false for every valuation of its variables.

Definition. A propositional formula $\mathcal{F}$ is called satisfiable if it is true for at least one valuation of its variables. A formula $\mathcal{F}$ is called invalid if it is false for at least one valuation of its variables.

23. Check if the following sentences can be propositions in the propositional calculus. If for some sentence you can not figure out a unique answer, then try to reformulate the sentence in such way that it becomes a proposition.

a) Life as a work of art.
b) All swans are white.
c) Study, study, study.
d) To write this sentence we used at least ten words.
e) The biggest number does not exist.

24. Write the following propositions if A denotes the proposition "I like ice cream" and B denotes the proposition "You like chocolate".

a) $A \wedge B$
b) $\neg A$
c) $\neg B$
d) $A \vee B$
e) $A \vee \neg B$
f) $\neg(A \wedge B)$
g) $\neg A \vee \neg B$
h) $\neg A \wedge B$
i) $\neg(A \vee B)$
j) $\neg A \vee \neg B$

25. Let A =*"Mouse jumps"* and B =*"Cat hops"*. Write the following propositions in symbol form.

a) Mouse jumps or cat hops.
b) Cat does not hop.
c) It is not true that "Mouse jumps or cat hops".
d) Mouse does not jump and cat does not hop
e) It is not true that "Mouse jumps and cat hops".
f) Mouse does not jump or cat does not hop.

26. Let P =*"Mouse jumps"*, Q =*"Cat hops"* and R =*"Old bear plays the drum"*. Read the following propositions.

a) $P \Rightarrow Q$
b) $Q \Rightarrow P$
c) $\neg Q \Rightarrow \neg P$
d) $\neg(P \Rightarrow Q)$
e) $(P \wedge Q) \Rightarrow R$
f) $P \wedge (Q \Rightarrow R)$
g) $(R \vee Q) \Rightarrow P$
h) $R \vee (Q \Rightarrow P)$
i) $(\neg P \wedge \neg Q) \Rightarrow \neg R$

27. Let uppercase Latin letters denote the following sentences:
A ="It is Sunday",
B ="Kristjan goes to the theater",
C ="Kristjan goes to visit a friend",
D ="Kristjan goes to walk the dog",
E ="Kristjan sits at home".
Read the following propositions.

a) $A \Rightarrow B$
b) $D \wedge A \Rightarrow \neg B$
c) $\neg A \Rightarrow \neg(B \vee C)$
d) $\neg B \wedge \neg C \wedge \neg D \Rightarrow E$
e) $A \Rightarrow \neg E \wedge (B \vee C \vee D)$
f) $(A \Rightarrow \neg E) \wedge (\neg A \Rightarrow \neg E)$

28. Write the following statements in symbol form, where equalities and equations are unified by logical operators. Variables a, b and c denote real numbers.

a) $a \cdot b = 0$.
b) $a \cdot b \neq 0$.
c) $\frac{a}{b} = 0$.
d) $|a| = 3$.
e) The point (a, b) is in the first quadrant of the coordinate plane.
f) The point (a, b) is on the coordinate axis.
g) Real numbers a, b and c are the lengths of sides of some triangle.
h) Real numbers a, b and c are the lengths of sides of some isosceles triangle.

29. Write the following propositions in symbol form. Also give the ”key” of transformations, i.e., a table that shows which letters correspond to which sentences. For every sentence comment what gets lost in the transaction.

a) In every room there is a mom or a dad or both.
b) Anu loves dogs, but Mart loves cats.
c) If Mr. Kask is happy, then Mrs. Kask is sad and if Mr. Kask is sad, then Mrs. Kask is happy.
d) That person does not have neither compassion nor conscience.
e) In the winter ships get out of the marina if and only if an icebreaker has made them a path.

f) If accidents happen, then they happen all the time.
g) If I move the Rook, then I lose the queen, but if I do not move the Rook, then I get a checkmate.
h) TV is broken because of a bad emitter or the fuses have broke and the electricity does not reach TV.
i) Communism – that is the power of Soviet Union plus the electrification of the whole world.

30. Write the following sentences symbol form.
a) To conclude that the sun rises in the East it is necessary that it sets in the West.
b) To conclude that the sun rises in the East it is sufficient that it sets in the West.
c) A necessary and sufficient condition for a business to be successful is that its founders have enough initial capital.
d) For theaters to work it is sufficient that the government supports theaters.
e) Without the government support, theaters stop working.
f) A necessary and sufficient condition for series to converge is that all its rearrangements converge.

31. Find the truth values of following propositions.
a) If 20 is divisible by 10, then 20 is divisible by 5.
b) If 20 is divisible by 5, then 20 is divisible by 10.
c) If 20 is divisible by 3, then 20 is divisible by 6.
d) If 20 is divisible by 4, then 20 is divisible by 8.
e) 20 is divisible by 5 if and only if 20 is divisible by 4.
f) 10 is divisible by 5 if and only is 10 is divisible by 4.

32. Find the truth values of following statements. First of all, determine the truth values of components and then use that to calculate the truth value of the proposition.
a) Hydrogen is a gas and if mercury is a gas, then gold is also a gas.
b) It is not true that moon is not round.
c) Keskerakond won the election and did not win the election.
d) Narva is in northwestern Estonia if and only if Pärnu is in southeastern Estonia.

33. There are four cards on the table. Each card has a letter one side and an integer on the other side. We can see the cards as follows:

A K 7 4

We need to determine, if the statement "If there is a vowel on on the one side of the card, then there is an even number on the other side" is true. Which cards do we need to turn over to find out if the statement is true?

34. The truth values a and b of propositions A and B are either 1 (true) or 0 (false). Find the arithmetic expressions that contain variables a and b and correspond to the truth values of following propositions

a) $\neg A$, b) $A \wedge B$, c) $A \vee B$, d) $A \Rightarrow B$, e) $A \Leftrightarrow B$

35. The following propositions contain implication and equivalence. Find the truth values of these propositions by using the definitions of truth values of implication and equivalence.

a) If $1 = 1$, then $2 = 2$.
b) If $1 = 2$, then $2 = 3$.
c) If $1 = 1$ and $1 = 2$, then $1 = 3$.
d) If $1 = 2$ or $1 = 3$, then $1 = 1$.
e) $1 = 2$ if and only if $2 = 3$.
f) $1 = 1$ if and only if $2 = 2$ and $3 = 3$.
g) If 1 is not equal to 2, then $1 = 3$.
h) 1 is not equal to 3 if and only if 1 is not equal to 2 or 1 is not equal to 1.

36. Find the truth tables of the following formulas.

a) $(X \Rightarrow \neg Y) \vee (X \Rightarrow X \wedge Y)$
b) $\neg(X \Rightarrow \neg(Y \wedge X)) \Rightarrow (X \vee Z)$
c) $(X \wedge (Y \Rightarrow X)) \Rightarrow \neg X$
d) $((X \wedge \neg Y) \Rightarrow Y) \Rightarrow (X \Rightarrow Y)$
e) $(X \wedge (Y \vee \neg X)) \wedge ((\neg Y \Rightarrow X) \vee Y)$
f) $(\neg X \Rightarrow \neg Y) \Rightarrow ((Y \wedge Z) \Rightarrow (X \wedge Z))$
g) $(X \Rightarrow (Y \Rightarrow Z)) \Rightarrow ((X \Rightarrow Y) \Rightarrow (X \Rightarrow Z))$

37. Are the following propositions satisfiable? Justify the answer!

a) $\neg(X \Rightarrow \neg X)$
b) $(X \Rightarrow Y) \Rightarrow (Y \Rightarrow X)$

38. Prove that following formulas are tautologies.

a) $X \vee \neg X$
b) $X \Rightarrow (X \vee Y)$
c) $X \Rightarrow (Y \Rightarrow (X \wedge Y))$
d) $(X \Rightarrow Y) \Rightarrow ((Y \Rightarrow Z) \Rightarrow (X \Rightarrow Z))$

39. Determine the classification of following formulas (tautology, contradiction, satisfiable, invalid).

a) $P \Rightarrow \neg P$
b) $\neg(P \Rightarrow \neg P)$
c) $(P \Rightarrow Q) \vee (Q \Rightarrow P)$
d) $P \Leftrightarrow \neg P$

40. For writing propositional formulas there also exists a different notation – *Polish notation* – that was first used in 1928 by a Polish logician Jan Lukasiewicz. In that notation the operator is not between the components, but in front of them instead. An interesting feature of Polish notation is the fact that it is not necessary to use brackets in formulas. Formulas $\neg A$, $A \wedge B$, $A \vee B$, $A \Rightarrow B$, $A \Leftrightarrow B$ written in Polish notation are Na, Kab, Aab, Cab, Eab (propositional variables are denoted by lowercase letters). More complex formulas are composed analogically. For example the formula $A \wedge B \vee C$ is AKabc in Polish notation, but formula $A \wedge (B \vee C)$ is KaAbc. Transform formulas a)–c) to Polish notation and d)–f) to regular notation.

a) $((A \vee B) \wedge (B \vee C)) \wedge (A \vee C)$

b) $\neg A \wedge (\neg A \Rightarrow B) \Rightarrow B$

c) $(A \vee B) \Rightarrow (C \Leftrightarrow D) \wedge A$

d) EcNCbKad

e) AaCKNabc

f) KENA$abcd$

3. Transformation of formulas

Definition. We say that the formula $\mathcal{G}$ **concludes** from the formulas $\mathcal{F}_1, \ldots, \mathcal{F}_n$, if for all the valuations of the variables (that appear in the formulas), where formulas $\mathcal{F}_1, \ldots, \mathcal{F}_n$ are true, the formula $\mathcal{G}$ is also true.

Theorem. From formulas $\mathcal{F}_1, \ldots, \mathcal{F}_n$ we can conclude the formula $\mathcal{G}$ if and only if the formula $\mathcal{F}_1 \wedge \ldots \wedge \mathcal{F}_n \Rightarrow \mathcal{G}$ is a tautology.

Definition. Formulas $\mathcal{F}$ and $\mathcal{G}$ are called **logically equivalent**, if their truth values are equal for every possible valuation of the variables.

Let $\mathcal{F}$ and $\mathcal{G}$ be formulas.

1. **Identity laws**:
   a) $\mathcal{F} \wedge \mathcal{F} \equiv \mathcal{F}$,
   b) $\mathcal{F} \vee \mathcal{F} \equiv \mathcal{F}$.
2. **Commutative laws**:
   a) $\mathcal{F} \wedge \mathcal{G} \equiv \mathcal{G} \wedge \mathcal{F}$,
   b) $\mathcal{F} \vee \mathcal{G} \equiv \mathcal{G} \vee \mathcal{F}$.
3. **Associative laws**:
   a) $(\mathcal{F} \wedge \mathcal{G}) \wedge \mathcal{H} \equiv \mathcal{F} \wedge (\mathcal{G} \wedge \mathcal{H})$,
   b) $(\mathcal{F} \vee \mathcal{G}) \vee \mathcal{H} \equiv \mathcal{F} \vee (\mathcal{G} \vee \mathcal{H})$.
4. **Distributive laws**:
   a) $\mathcal{F} \wedge (\mathcal{G} \vee \mathcal{H}) \equiv \mathcal{F} \wedge \mathcal{G} \vee \mathcal{F} \wedge \mathcal{H}$,
   b) $\mathcal{F} \vee (\mathcal{G} \wedge \mathcal{H}) \equiv (\mathcal{F} \vee \mathcal{G}) \wedge (\mathcal{F} \vee \mathcal{H})$.
5. **Absorption laws**:
   a) $\mathcal{F} \wedge (\mathcal{F} \vee \mathcal{G}) \equiv \mathcal{F}$,
   b) $\mathcal{F} \vee \mathcal{F} \wedge \mathcal{G} \equiv \mathcal{F}$.
6. **De Morgan's laws**:
   a) $\neg(\mathcal{F} \wedge \mathcal{G}) \equiv \neg\mathcal{F} \vee \neg\mathcal{G}$,
   b) $\neg(\mathcal{F} \vee \mathcal{G}) \equiv \neg\mathcal{F} \wedge \neg\mathcal{G}$.
7. **Double negation law**: $\neg\neg\mathcal{F} \equiv \mathcal{F}$.
8. **Domination laws**, where t is a random tautology and f is a random contradiction:
   a) $\mathcal{F} \wedge t \equiv \mathcal{F}$,
   b) $\mathcal{F} \vee t \equiv t$,
   c) $\mathcal{F} \wedge v \equiv v$,
   d) $\mathcal{F} \vee v \equiv \mathcal{F}$.
9. **Implication expressed by conjunction and disjunction**:
   a) $\mathcal{F} \Rightarrow \mathcal{G} \equiv \neg(\mathcal{F} \wedge \neg\mathcal{G})$,
   b) $\mathcal{F} \Rightarrow \mathcal{G} \equiv \neg\mathcal{F} \vee \mathcal{G}$.
10. **Conjunction expressed by disjunction and implication**:
    a) $\mathcal{F} \wedge \mathcal{G} \equiv \neg(\mathcal{F} \Rightarrow \neg\mathcal{G})$,
    b) $\mathcal{F} \vee \mathcal{G} \equiv \neg\mathcal{F} \Rightarrow \mathcal{G}$.
11. **Equivalence expressed by other operators**:
    a) $\mathcal{F} \Leftrightarrow \mathcal{G} \equiv \mathcal{F} \wedge \mathcal{G} \vee \neg\mathcal{F} \wedge \neg\mathcal{G}$,
    b) $\mathcal{F} \Leftrightarrow \mathcal{G} \equiv (\mathcal{F} \Rightarrow \mathcal{G}) \wedge (\mathcal{G} \Rightarrow \mathcal{F})$.

Definition. The principal (full) disjunctive normal form of a propositional formula $\mathcal{F}$ is the formula that is equivalent to the formula $\mathcal{F}$ and that is a disjunction of principal basic conjunctions.

41. a) Can we conclude the formula $\neg(A \Rightarrow B)$ from formulas A and $\neg B$?
b) Can we conclude the formula C from formulas $A \Rightarrow B$, $B \Rightarrow C$ and A?

42. We know the following facts about three statements A, B and C.

a) If A is true, then both B and C are also true.
b) If B is true, then at least one of statements A and C is true.
c) If C is true, then A is true and B is false.

Which of the statements A, B, C are true?

43. Write the following discussion by propositional formulas and prove that the discussion is correct, i.e., prove that from the formulas corresponding to assumptions we can conclude the formula that corresponds to the conclusion.
Students are happy if and only if there is no test. If students are happy, then the teacher is happy. However, if the teacher is happy, then he/she does not want to teach a class and then a test takes place. Therefore, the students are not happy.

44. We know the following facts.

a) If Mihkel is coughing and is pale, then he is either ill or he has been smoking.
b) If Mihkel has not been smoking and still is coughing or is pale, then he is ill.
c) If Mihkel is ill, then he is coughing but is not pale.

After the recess Mihkel was pale. Can we conclude that he had been smoking?

45. We know the following facts.

a) If Jüri watches basketball and Estonia wins, then he is happy and drinks beer.
b) If Jüri does not watch basketball and he is still happy, then he drinks beer.
c) If Estonia wins, then Jüri is happy.

In the evening Jüri was happy. Can we say for sure that he had been drinking beer? Justify your answer.

46. A guru announced to his/her students: "If I am Buddha, then I am not Buddha." Students were astonished when they heard it. Express the statement by a formula, compile the truth table and find a simpler statement it is equivalent to.

47. Prove that if formulas A and $A \Rightarrow B$ are tautologies, then the formula B is also a tautology.

48. Someone said that he/she knows such propositional formulas F and G, that formula G concludes from formula F and formula $\neg G$ concludes from formula F. Is such situation possible?

49. Prove the following equivalences.

a) $(X \vee Y) \wedge Z \equiv X \wedge Z \vee Y \wedge Z$
b) $X \Rightarrow Y \equiv \neg X \vee Y$
c) $X \Leftrightarrow Y \equiv (X \Rightarrow Y) \wedge (Y \Rightarrow X)$

50. Transform the following formulas in such a way that they would contain only negation and conjunction.

a) $\neg P \wedge Q \Rightarrow \neg Q \wedge P$
b) $P \vee Q \Rightarrow (\neg P \Rightarrow Q)$
c) $(\neg A \Rightarrow B) \Rightarrow A \vee B$
d) $\neg(P \Rightarrow Q) \vee (\neg P \Rightarrow \neg Q)$
e) $(P \Leftrightarrow Q) \vee P$

51. Transform the following formulas in such way that they would contain only negation and disjunction.

a) $(P \Rightarrow R) \Rightarrow (Q \Rightarrow R)$
b) $\neg P \wedge Q \Rightarrow \neg Q \wedge P$
c) $(P \Rightarrow Q) \wedge P$
d) $(P \Leftrightarrow Q) \wedge P$
e) $((A \Rightarrow B) \wedge (B \Rightarrow \neg C)) \Rightarrow (C \Rightarrow \neg A)$

52. Simplify the following formulas.

a) $\neg(\neg A \vee B) \Rightarrow ((A \vee B) \wedge B)$
b) $\neg(\neg P \wedge \neg Q) \vee ((P \Rightarrow Q) \wedge P)$
c) $(A \Leftrightarrow B) \wedge (A \vee B)$
d) $((A \Rightarrow B) \wedge (B \Rightarrow C)) \Rightarrow (C \Rightarrow A)$
e) $(P \Rightarrow Q) \wedge (Q \Rightarrow P)$
f) $\neg((A \Rightarrow B) \wedge (B \Rightarrow \neg A))$
g) $(P \Rightarrow \neg Q) \vee \neg(P \vee Q)$
h) $\neg(P \wedge Q \wedge (P \Rightarrow \neg Q))$

53. Negate the following propositions.

a) If a, b and c are lengths of sides of a triangle, then the inequalities $a+b>c$, $a+c>b$ and $b+c>a$ are true.
b) If a, b and c are such integers that $a=bc$, then a is divisible by b and a is divisible by c.
c) If n is a positive integer, then n^2+n+41 is a prime number;
d) For all integers a and b it is true that if $a+b$ is an even number, then a and b are either both even or both odd.

54. Negate the following propositions.

a) Every student taking this subject has been to Finland or Sweden.
b) Every e-mail address that is bigger than one megabyte is compressed.
c) If a user is online, then there is at least one network connection available.
d) There exists a pig that can swim and catch fish.
e) There is no one taking this subject who can speak French or Russian.
f) Number x is positive, but number y is not positive.
g) If x is a prime number, then $\sqrt{x}$ is not a rational number.
h) If x is an odd number, then x^2 is an odd number.
i) If x is a rational number and $x \neq 0$, then $\tan x$ is not a rational number.
j) If $\sin x < 0$, then $0 \leqslant x \leqslant \pi$ is not true.

k) $\exists x \exists y\, (x+2y=2 \wedge 2x+4y=5)$.
l) $\forall x \exists y\, (x+y=2 \wedge 2x-y=1)$.
m) $\forall x \forall y\, (((x \geqslant 0) \wedge (y<0)) \Rightarrow (x-y>0))$.
n) $\exists x \exists y\, (((x \leqslant 0)) \wedge (y \leqslant 0) \wedge (x-y>0))$.
o) $\forall x \forall y\, (((x \neq 0)) \wedge (y \neq 0) \Leftrightarrow (xy \neq 0))$.

55. Transform the following formulas in such a way that the negation operator appears only in front of the variables.

a) $\neg(\neg P \vee Q)$
b) $\neg(P \wedge Q \vee \neg R)$
c) $\neg(P \wedge Q \vee R) \Rightarrow \neg(P \wedge Q)$
d) $\neg(P \wedge (\neg Q \vee \neg R) \wedge R)$

56. Find the principal disjunctive normal form of the following formulas.

a) $(\neg X \Rightarrow \neg Y) \Rightarrow ((Y \wedge Z) \Rightarrow (X \wedge Z))$
b) $((X \Rightarrow Y) \Rightarrow \neg X) \Rightarrow (X \Rightarrow (Y \wedge X))$
c) $\neg((X \wedge Y) \Rightarrow \neg X) \wedge \neg((X \wedge Y) \Rightarrow \neg Y)$
d) $(X \Rightarrow \neg Y) \vee (X \Rightarrow X \wedge Y)$
e) $\neg(X \Rightarrow \neg(Y \wedge X)) \Rightarrow (X \vee Z)$
f) $(X \wedge (Y \Rightarrow X)) \Rightarrow \neg X$
g) $((X \wedge \neg Y) \Rightarrow Y) \Rightarrow (X \Rightarrow Y)$
h) $(X \wedge (Y \vee \neg X)) \wedge ((\neg Y \Rightarrow X) \vee Y)$

57. Find a formula that satisfies the condition.

a) Formula is true if and only if X is true and Y is false.
b) Formula is true if and only if X and Y are both true.
c) Formula is true if and only if at least two of the propositional variables X, Y and Z are true.
d) Formula is true if and only if exactly two of the propositional variables X, Y and Z are true
e) Formula is true if and only if exactly one of the propositional variables X, Y and Z is true

58. Find a three variable formula that is true if and only if exactly two of the variables are false.

59. Find a three variable formula that has the same truth value as most of the variables.

60. A committee that has members A, B, C and D makes all the important decisions by voting. Every member has one vote, except the chairman of the committee A, whose vote is worth two votes. A decision is accepted if it has at least 3 votes. Find the formula that is true if and only if the decision is accepted.

61. Compile a formula that is true if and only if the sum of binary numbers $\overline{AB}$ and $\overline{CD}$ has at most two digits.

62. Confirm whether the following properties are true in propositional calculus.

a) associativity of implication;

b) associativity of equivalence;
c) distributivity between conjunction and implication.

63. Express every other propositional operator by
a) negation and disjunction;
b) negation and conjunction;
c) negation and implication;

64. Express disjunction by implication.

65*. Prove that
a) negation can not be expressed by conjunction, disjunction, implication and equivalence;
b) implication can not be expressed by disjunction and conjunction;
c) conjunction can not be expressed by disjunction and implication.

66*. *Duality principle.* Let F and G be equivalent formulas that contain only operators negation, disjunction and conjunction. Prove that if formulas F' and G' are obtained from formulas F and G by replacing all operators $\wedge$ with operators $\vee$ and vice versa and by replacing every propositional variable X with its negation $\neg X$, then formulas F' and G' are equivalent.

67*. Find a
a) propositional formula that can not be expressed by negation and equivalence.
b) propositional formula that can not be expressed by conjunction, disjunction and implication.
c) propositional formula that can not be expressed by conjunction, disjunction, implication and equivalence.

(Also give a proof for all cases.)

4. Concept of set

A collection of distinct objects that is considered an object itself is called a **set** if for any object we can determine whether it is in the set or not.

Sets are usually denoted by uppercase letters $A, B, C, X, Y, \dots$, however, elements of the set are denoted by lowercase letters $a, b, c, x, y, \dots$. Empty set $\varnothing = \{\}$ is a set that does not contain any elements.
The most significant sets of numbers are

- Natural numbers $\mathbb{N} = \{1, 2, 3, \dots\}$;
- Integers $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$;
- Rational numbers $\mathbb{Q} = \{q \mid q = \dfrac{m}{n}, m \in \mathbb{Z}, n \in \mathbb{N}\}$;
- Real numbers $\mathbb{R}$;
- Complex numbers $\mathbb{C} = \{z \mid z = x + iy, x, y \in \mathbb{R}, i^2 = -1\}$.

The intervals of numbers are:

- Closed interval $[a, b] = \{x \mid x \in \mathbb{R}, a \leqslant x \leqslant b\}$;
- Open interval $(a, b) = \{x \mid x \in \mathbb{R}, a < x < b\}$;
- Half open interval $[a, b) = \{x \mid x \in \mathbb{R}, a \leqslant x < b\}$;
- Half open interval $(a, b] = \{x \mid x \in \mathbb{R}, a < x \leqslant b\}$.

Two sets are considered equal if they consist of exactly the same elements

Definition. Set A is called a **subset** of set B if all the elements of the set A are also elements of the set B (that means every element in the set A belongs to the set B). That relation is denoted by $A \subset B$ or $B \supset A$.

Definition. Set A is called a **proper subset** of set B if set A is a subset of set B and $A \neq B$. In that case we write $A \subsetneq B$.

The power set of set A is the set of all subsets of A and is usually denoted by $\mathcal{P}(A) = \{X \mid X \subset A\}$.

68. Write the set A in the form $\{x \mid P(x)\}$:

a) $A=\{0,1,2,3,4,5,6,7,8,9\}$;

b) $A=\left\{1,\frac{1}{2},\frac{1}{3},\frac{1}{4},\dots\right\}$;

c) $A=\{0.5,1,1.5,2\}$;

d) $A=\{1,4,9,16,25,\dots\}$;

e) $A=\{3,6,9,12,\dots\}$;

f) $A=\{\frac{1}{3\cdot 6},\frac{1}{6\cdot 9},\frac{1}{9\cdot 12},\frac{1}{12\cdot 15},\dots\}$;

g) $A=\{1,\frac{1}{2},3,\frac{1}{4},5,\frac{1}{6},\dots\}$;

h) $A=[a,b)$;

i) A is a set of even numbers.

69. Write the following sets by listing their elements:

a) $\{(-1)^n \mid n\in\mathbb{N}\}$;

b) $\{n+(-1)^n : n\in\mathbb{N}\}$;

c) $\{n\in\mathbb{N} \mid n^2-13n+30\leqslant 0\}$;

d) $\{\cos n\pi : n\in\mathbb{N}\}$;

e) $\{x \mid x\in\mathbb{R}\wedge 7x^2-8x=0\}$;

f) $\{(x,y) : x,y\in\mathbb{R}\wedge x^2+y^2=1\wedge y-x=1\}$;

g) $\{x \mid x\in\mathbb{R}\wedge x^2+4=0\}$;

h) $\{x \mid x\in\mathbb{R}\wedge x^2-5x+6=0\}$.

70. Write all the elements and subsets of following sets:

a) $\{a,b\}$;

b) $\{1,\{1\}\}$;

c) $\{a,b,c\}$.

71. Decide if the following statements are true:

a) $3\in\{1,2,3\}$;

b) $\{2\}\in\{1,2,3\}$;

c) $\{a,b,c\}=\{a,c,b\}$;

d) $x\in\{x\}$;

e) $\{1\}\subset\{1,2,3\}$;

f) $\{2\}\in\{\{1\},\{2\},\{3\}\}$;

g) $\{1,2\}\subset\{1,2,\{3\}\}$;

h) $\{1,2\}\in\{\{1,2,3\},\{2,3\},1,2\}$;

i) $\{\{3\}\}\subset\{2,3,\{3\}\}$;

j) $\{a,b\}\subset\{a,b,c\}$;

k) $3\in\{\{1\},\{2\},\{3\}\}$;

l) $\{\varnothing\}\in\varnothing$;

m) $\varnothing\in\{\varnothing\}$;

n) $\varnothing\subset\varnothing$;

o) $\{\varnothing\}=\varnothing$.

72. Find $\mathcal{P}(\varnothing)$, $\mathcal{P}(\{\varnothing\})$ and $\mathcal{P}(\{\varnothing,\{\varnothing\}\})$.

73. Represent the following sets on the real line:

a) $\{x \mid x \in \mathbb{R} \wedge 3 < x \leqslant 7\}$;

b) $\{x \mid x \in \mathbb{R} \wedge |x| < c\}$;

c) $\{x \mid x \in \mathbb{R} \wedge |x-a| < \varepsilon\}$;

d) $\{x \in \mathbb{R} \mid 2x^2 + 9x + 7 \geqslant 0\}$;

e) $\{2x^2 - 9x + 7 \mid x \in \mathbb{R}\}$;

f) $\{x^2 + 2x + 1 \mid x \in (-2, \infty)\}$;

g) $\{x^2 + 2|x| + 1 \mid x \in (-1, 1)\}$;

h) $\{x \in \mathbb{R} \mid \sqrt{|1-2x|} \geqslant 1 + x\}$;

i) $\{x \in \mathbb{R} \mid \sqrt{|0.25 - x|} \geqslant x + 0.5\}$;

j) $\{x \in \mathbb{R} \mid |x| + 2x \geqslant 1\}$;

k) $\{x \in \mathbb{R} \mid |2x^2 - 9x + 6| = 2x^2 - 9x + 6\}$;

l) $\{1 - |x| \mid x \in [-2, 1]\}$;

m) $\{|1 + x| + 1 \mid x \in [1, 2]\}$;

n) $\{y \in \mathbb{R} \mid y = -x^2 + 6x - 2 \wedge x \in (0, \infty)\}$;

o) $\{y \in \mathbb{R} \mid y = x^2 - 4x + 3 \wedge x \in (0, \infty)\}$.

74. Represent the following sets on the coordinate plane:

a) $\{(x, y) \mid x, y \in \mathbb{R} \wedge 2 < x \leqslant 5\}$;

b) $\{(x, y) \mid x, y \in \mathbb{R} \wedge a < x \leqslant b \wedge c < y \leqslant d\}$;

c) $\{(x, y) \mid x, y \in \mathbb{R} \wedge x^2 + y^2 \leqslant 4\}$;

d) $\{(x, y) \mid x, y \in \mathbb{R} \wedge y \geqslant x^2\}$;

e) $\{(x, y) \mid x, y \in \mathbb{R} \wedge |x| + |y| = 1\}$;

75. Find all of the elements of the given set:

a) $\{A \mid \{1\} \subset A \subset \{1,2,3\}\}$;

b) $\{A \mid \{a, b\} \subset A \subset \{a, b, c, d\}\}$;

c) $\{A \mid A \subset \{a, b, c\} \wedge a \notin A \wedge b \in A\}$.

76. (Lewis Carroll exercise) In the old days lots of warriors got hurt in battles. From warriors who participated in battle 70% lost an eye, 75% lost an ear, 80% lost an arm and 85% lost a leg. Find the maximal and minimal percentage of warriors who lost all four.

77. Prove that if a set has n elements, then that set has 2^n subsets.

78. Let $A_1, A_2, \ldots, A_n$ be sets. Prove that $A_1 \subset A_2 \subset \ldots \subset A_n \subset A_1 \Leftrightarrow A_1 = A_2 = \ldots = A_n$.

79*. For every positive integer n find a set A_n such that A_n has n elements and for every $a, b \in A_n$, $a \neq b$ either $a \in b$ or $b \in a$.

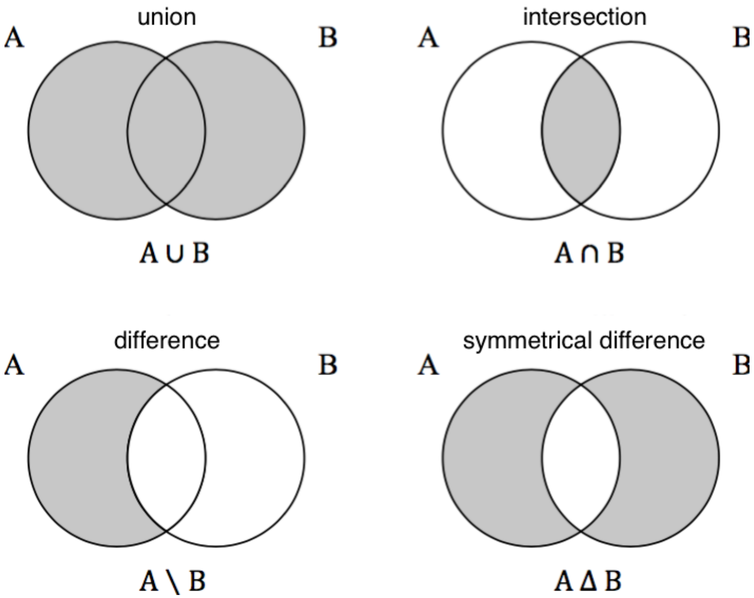
5. Set operations

union $A \cup B = \{x \mid x \in A \vee \text{(or) } x \in B\}$	$\bigcup_\alpha A_\alpha = \{x \mid \exists \text{ (exists) } \alpha \text{ such that } x \in A_\alpha\}$
intersection $A \cap B = \{x \mid x \in A \wedge \text{(and) } x \in B\}$	$\bigcap_\alpha A_\alpha = \{x \mid \forall \text{ (for every) } \alpha \text{ we have } x \in A_\alpha\}$
difference $A \setminus B = \{x \mid x \in A \wedge x \notin B\}$	symmetrical difference $A \triangle B = (A \setminus B) \cup (B \setminus A)$

Properties of union and intersection.

- idempotent laws: $A \cup A = A$, $A \cap A = A$
- commutative laws: $A \cup B = B \cup A$, $A \cap B = B \cap A$
- associative laws: $(A \cup B) \cup C = A \cup (B \cup C)$, $(A \cap B) \cap C = A \cap (B \cap C)$
- distributive laws: $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$, $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
- absorption laws: $A \cup (A \cap B) = A$, $A \cap (A \cup B) = A$

Venn diagrams.



Properties of symmetrical difference.

- commutative law: $A \triangle B = B \triangle A$
- associative law: $(A \triangle B) \triangle C = A \triangle (B \triangle C)$
- distributive law: $A \cap (B \triangle C) = (A \cap B) \triangle (A \cap C)$

Often we work with a fixed set U and all the sets we examine are subsets of that set. In that case, we call the set U an universal set. Let U be a fixed universal set.

Definition. The complement of a set A is the set A' that consists of all elements of the universal set U that do not belong to the set A, i.e.

$$A' = \{x \in U : x \notin A\} = U \setminus A.$$

De Morgan's laws: $(A \cup B)' = A' \cap B'$, $(A \cap B)' = A' \cup B'$.

Double complement law: $A'' = A$.

80. Let A be a random set. Describe the following sets

a) $A \cup \varnothing$; c) $A \setminus \varnothing$; e) $A \setminus A$; g) $A \cup A'$; i) $\varnothing'$.

b) $A \cap \varnothing$; d) $A \triangle \varnothing$; f) $\varnothing \setminus A$; h) $A \cap A'$;

81. Find sets $A \cup B$, $A \cap B$, $A \setminus B$, $B \setminus A$, $A \triangle B$, if

a) $A = \{-1, 0, 3, 4\}$, $B = \{0, 4, 6\}$;

b) $A = [0, 2]$, $B = [1, 5]$;

c) $A = [0, 2]$, $B = \{0, 4, 6\}$;

d) $A = (-\infty, 7]$, $B = [2, 4]$;

e) $A = \varnothing$, $B = \{1\}$;

f) $A = \{\varnothing\}$, $B = \{\varnothing, \{\varnothing\}\}$.

82. Find sets A and B, if $A \cup B = \{1, 2, 3, 4\}$, $A \cap B = \{2\}$, $A \setminus B = \{1\}$.

83. Represent the following sets on the real line:

a) $\{x \mid x \in \mathbb{R} \wedge -2 \leqslant x < 1\} \cup \{x \mid x \in \mathbb{R} \wedge 0 < x < 3\}$;
b) $\{x | x \in \mathbb{R} \wedge x \geqslant -1\} \cap \{x \mid x \in \mathbb{R} \wedge |x| < 3\}$;
c) $\{x \mid x \in \mathbb{R} \wedge x > 2\} \setminus \{x \mid x \in \mathbb{R} \wedge 3 < x \leqslant 5\}$;
d) $\{x \mid x \in \mathbb{R} \wedge 1 < x \leqslant 5\} \triangle \{x \mid x \in \mathbb{R} \wedge x < 0\}$.

84. Find the complement A' of the set A with respect to the universal set X:

a) $X = \{1,2,3,4,5,6,7,8\}$, $A = \{2,5,7\}$; b) $X = \mathbb{R}$, $A = \mathbb{Q}$.

85. Which conditions sets A and B must satisfy, so that the equality holds:

a) $A \cup B = A$; c) $A \cup B = A \cap B$; e) $A \setminus B = B$; g) $A \triangle B = \varnothing$.

b) $A \cap B = A$; d) $A \setminus B = A$; f) $A \triangle B = A$;

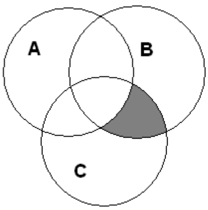
86. Draw the Venn diagrams for each of these combinations of the sets A, B, and C:

a) $A \setminus (B \cup C)$; d) $A \cap (B \setminus C)$; g) $(A \setminus B) \triangle C$;

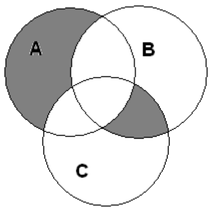
b) $A \setminus (B \cap C)$; e) $(A \triangle B) \triangle C$; h) $(A \cap B) \cap (A \triangle B)$;

c) $(A \setminus (B \cap C)) \cap B$; f) $(A \cup B) \cap C$; i) $(A \cap C) \cup (B \setminus (A \cup C))$.

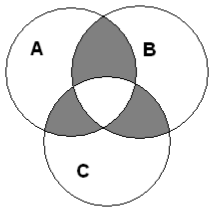
87. Write the sets presented in following Venn diagrams by using set-theoretic operations:



a)



b)



c)

88. In a group of 30 students, each student has at least one hobby. Students have a choice between three hobbies: chess, acting and cooking. Six students chose only cooking, five chose all three hobbies and two chose both chess and cooking but not acting. Altogether 15 students take cooking classes. Two students chose only chess and three chose only acting. How many students go to chess and acting classes but do not participate in cooking classes? How many students go to chess classes?

89. When inquiring 100 students, it was found out that 28 of them study English, 30 study German, 42 study French, 8 study both English and German, 10 study both English and French, 5 study German and French and 3 of them study all three languages. How many students study none of the mentioned languages? How many study only French, how many study only English and how many study only German?

90. In a military base out of 100 soldiers 80 play football, 60 play volleyball and 40 play basketball. We know that 40 of them play both football and volleyball, 30 play both football and basketball and 20 play both volleyball and basketball. How many of the soldiers play all three sports if every soldier plays at least one sport?

91. Find $(\{\{\varnothing\}\} \triangle \{\varnothing\}) \setminus (\{\varnothing\} \triangle \varnothing)$.

92. Sets A, B and $A \cup B$ contain m, n and p elements respectively. How many elements do sets $A \cap B$, $A \setminus B$ and $A \triangle B$ have?

93. Prove the following equalities.

a) $(A \cup B) \cup C = A \cup (B \cup C)$

b) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

c) $(A \cap B) \cap C = A \cap (B \cap C)$

d) $A \cup (A \cap B) = A$

e) $A \cap (A \cup B) = A$

f) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

g) $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$

h) $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$

i) $A \setminus (A \setminus B) = A \cap B$

j) $A \cup (B \setminus A) = A \cup B$

k) $A \setminus B = A \setminus (A \cap B)$

l) $(A \setminus B) \setminus C = A \setminus (B \cup C)$

m) $A \setminus (B \setminus C) = (A \setminus B) \cup (A \cap C)$

n) $(A \setminus B) \setminus C = (A \setminus C) \setminus (B \setminus C)$

o) $A \cap (B \setminus C) = (A \cap B) \setminus C$

p) $A \cap (B \setminus A) = \varnothing$

q) $A \setminus B = A \triangle (A \cap B)$

r) $(A \triangle B) \cap C = (A \cap C) \triangle (B \cap C)$

s) $A \triangle (A \triangle B) = B$

t) $A \triangle \varnothing = A$

u) $A \triangle B = (A \cup B) \setminus (A \cap B)$

v) $A \triangle B = (A \cup B) \setminus (A \cap B)$

94. Prove the following equalities.

a) $(A')' = A$

b) $(A' \cup B) \cap A = A \cap B$

c) $(A \cap B) \cup (A \cap B') = A$

d) $(A \cup B) \cap (A \cup B') = A$

95. Find sets.

a) $\bigcup_{n=1}^{\infty} (-n, n)$

b) $\bigcup_{x \in \mathbb{R}} \{x\}$

c) $\bigcup_{n \in \mathbb{N}} [0, 1 + 2^{-n}]$

d) $\bigcup_{n \in \mathbb{N}} [0, 1 + 2^{n}]$

e) $\bigcap_{n \in \mathbb{N}} [2^{-n}, 1]$

f) $\bigcap_{n \in \mathbb{N}} [0, 1 + 2^{-n})$

g) $\bigcap_{n \in \mathbb{N}} [0, 1 + 2^{n}]$

h) $\bigcap_{n=1}^{\infty} [-1, n^{-1}]$

i) $\bigcap_{n \in \mathbb{N}} (-2^{n}, 2^{n})$

j) $\bigcap\limits_{n\in\mathbb{N}}[-2^{-n},2^{-n}]$

k) $\bigcup\limits_{n\in\mathbb{N}}[n-1,n]$

l) $\bigcap\limits_{n\in\mathbb{N}}[n-1,n]$

m) $\bigcap\limits_{n=1}^{\infty}[2^{n},\infty)$

n) $\bigcap\limits_{n=1}^{\infty}[2^{-n},\infty)$

o) $\bigcup\limits_{n\in\mathbb{N}}[n,\infty)$

p) $\bigcap\limits_{n\in\mathbb{N}}[n,\infty)$

q) $\bigcup\limits_{n\in N}\{0,1,2,\ldots,2n\}$

r) $\bigcap\limits_{n\in N}\{0,1,2,\ldots,2n\}$

s) $\bigcup\limits_{n\in N}\{-n,n\}$

t) $\bigcap\limits_{n\in N}\{-n,n\}$

u) $\bigcup\limits_{j=k}^{n}[2j,2j+3]$

v) $\bigcup\limits_{j=k}^{n}\{-j,j\}$

w) $\bigcap\limits_{j=k}^{n}\{-j,j\}$

x) $\bigcap\limits_{n\in N}\{-2n,0,2n\}$

96. Prove that $A\subset B \Leftrightarrow A\cup B=B \Leftrightarrow A\cap B=A \Leftrightarrow A\setminus B=\varnothing$.

97. Prove that if $A\subset B$, then for every set C the following relations are true:

a) $A\cup C\subset B\cup C$;
b) $A\cap C\subset B\cap C$;
c) $(A\setminus C)\subset(B\setminus C)$;
d) $(C\setminus B)\subset(C\setminus A)$;
e) $B'\subset A'$.

98. Prove the following equivalences:

a) $A\cup B\subset C\Leftrightarrow A\subset C\wedge B\subset C$;
b) $C\subset A\cap B\Leftrightarrow C\subset A\wedge C\subset B$;
c) $C\subset A\vee C\subset B\Rightarrow C\subset A\cup B$.

99. Express

a) $A\cup B$ through operations $\cap$ and $\triangle$;
b) $A\setminus B$ through operations $\cap$ and $\triangle$;
c) $A\cap B$ through operations $\cup$ and $\triangle$;
d) $A\setminus B$ through operations $\cup$ and $\triangle$;
e) $A\cup B$ through operations $\setminus$ and $\triangle$;
f) $A\cap B$ through operation $\setminus$.

100. Let us have a sequence of sets $A_1,A_2,\ldots,A_n,\ldots$, where $A_1\subset A_2\subset\cdots\subset A_n\subset\ldots$. Prove that if we leave out a finite number of sets from the sequence, the union $\bigcup\limits_{n=1}^{\infty}A_n$ does not chance.

101. Let us have a sequence of sets $A_1,A_2,\ldots,A_n,\ldots$, where $A_1\supset A_2\supset\cdots\supset A_n\supset\ldots$. Prove that if we leave out a finite number of sets from the sequence, the intersection $\bigcap\limits_{n=1}^{\infty}A_n$ does not chance.

Definition. The **Cartesian product** of two sets A and B is a set of all pairs (a,b), where $a\in A$, $b\in B$ and where the order of the elements is important.

$$\boxed{\begin{gathered} A\times B=\{(a,b)\mid a\in A, b\in B\}\\ A_1\times\cdots\times A_n=\{(a_1,\ldots,a_n)\mid a_1\in A_1,\ldots,a_n\in A_n\}\\ \underbrace{A\times\cdots\times A}_{n}=A^n \end{gathered}}$$

Properties of the Cartesian product:

- Cartesian product with empty set: $A\times\varnothing=\varnothing$, $\varnothing\times A=\varnothing$;
- Distributive laws: $A\times(B\cup C)=(A\times B)\cup(A\times C)$,
  $A\times(B\cap C)=(A\times B)\cap(A\times C)$,
  $A\times(B\setminus C)=(A\times B)\setminus(A\times C)$.

102. Find the Cartesian product $A\times B$ of following sets:

a) $A=\{1,2,3\}$, $B=\{1,2\}$;

b) $A=\{1,2,3\}$, $B=\{1,2,3\}$;

c) $A=\{3,4\}$, $B=\{1,2,3\}$;

d) $A=\{1,2,3\}$, $B=\{3,4\}$;

e) $A=\{x\mid x\in\mathbb{N}, x=5\}$,
$B=\{x\mid x\in\mathbb{N}, 1<x\leqslant 3\}$.

103. Let us have $A=\{x\in\mathbb{R}:|x|<3\}$, $B=\{x\in\mathbb{R}:0\leqslant x\leqslant 4\}$ and $C=\{y\in\mathbb{N}:1\leqslant y\leqslant 5\}$. Draw each of these combinations the sets A, B, and C on the coordinate plane:

a) $A\times(B\cup C)$, c) $C\times(A\cup B)$, e) $(A\cup C)\times B$, g) $(B\cap A)\times C$, i) $(C\setminus A)\times B$,
b) $(A\triangle B)\times C$, d) $C\times(B\triangle A)$, f) $C\times(B\setminus A)$, h) $B\times(C\cup A)$, j) $A\times(B\cap C)$.

104. Sets A and B have m and n elements respectively. How many elements do sets $\mathcal{P}(A)$, $A\times B$, $\mathcal{P}(A)\times\mathcal{P}(B)$ and $\mathcal{P}(A\times B)$ have? Find those sets if:

a) $A=\{a,b\}$, $B=\{c\}$; b) $A=B=\{0,1\}$; c) $A=\{a,b,c\}$, $B=\varnothing$.

105. Prove the equalities

a) $A\times\varnothing=\varnothing$, $\varnothing\times A=\varnothing$

b) $A\times(B\cup C)=(A\times B)\cup(A\times C)$

c) $(A\setminus B)\times C=(A\times C)\setminus(B\times C)$

d) $(A\cap B)\times(C\cap D)=(A\times C)\cap(B\times D)$

106. Show that $(A\times B)\cup(C\times D)\subset(A\cup C)\times(B\cup D)$.

107. Prove that if $A\neq\varnothing$ and $B\neq\varnothing$, then $A\times B=B\times A$ if and only if $A=B$.

108. Prove that if $A_i\neq\varnothing, B_i\neq\varnothing, i=1,\ldots,n$, then

a) $A_1\times A_2\times\ldots\times A_n\subset B_1\times B_2\times\ldots\times B_n$ if and only if $A_1\subset B_1, A_2\subset B_2,\ldots,A_n\subset B_n$;

b) $A_1 \times A_2 \times \ldots \times A_n = B_1 \times B_2 \times \ldots \times B_n$ if and only if $A_1 = B_1, A_2 = B_2, \ldots, A_n = B_n$.

109*. Prove that

a) it is not possible to express $A \cup B$ through operations $\cap$ and $\smallsetminus$;

b) it is not possible to express $A \smallsetminus B$ through operations $\cap$ and $\cup$.

110*. Does there exist such sets $A, B \subset \mathbb{N}$ that $A \cap B = \varnothing$, $A \cup B = \mathbb{N}$ and

$$\begin{cases} \forall x, y \in A \quad x \neq y \Rightarrow x + y \in B, \\ \forall x, y \in B \quad x \neq y \Rightarrow x + y \in A? \end{cases}$$

111*. Let A, B, C, D and X be subsets of some universal set.
Prove that equation

$$(A \cap X) \cup (B \cap X') = (C \cap X) \cup (D \cap X')$$

has solutions (with respect to X) if and only if $B \vartriangle D \subset (A \vartriangle C)'$. In that case find all the solutions X.

112*. Prove that when given a set and n subsets of it, then by using operations $\cup$, $\cap$, $\smallsetminus$ it is possible to construct a maximum of 2^{2^n} different sets. (That means prove that it is not possible to get more sets and for each n give an example of how to get 2^{2^n} sets.)

113*. Let A, B, C, D be such sets that $A \neq \varnothing$, $B \neq \varnothing$ and $(A \times B) \cup (B \times A) = C \times D$. Prove that $A = B = C = D$.

6. Number theory and mathematical induction

Definition. We say that an integer a **divides** an integer b (denoted by $a \mid b$), if there exists an integer c, such that $ac = b$. Notation $a \mid b$ means the same as notation $b \vdots a$ i.e integer b is divisible by a.

Theorem. Let a be an integer and b be a natural number. Then there exist unique integers q (quotient) and r (remainder), such that

$$a = bq + r \quad \text{and} \quad 0 \leqslant r < b.$$

Definition. A natural number $p > 1$ that has exactly two positive dividers; 1 and p, is called a **prime number**. A natural number that is bigger than 1 and that is not a prime number is called a **composite number**.

Definitsioon. A natural number is called a perfect square if it is equal to the square of some integer.

114. Let a, b, c and d be integers. Prove following properties:

a) $a \mid a$ (reflexivity);
b) $1 \mid a$;
c) $0 \mid a$ if and only if $a = 0$;
d) if $a \mid b$ and $b \mid c$, then $a \mid c$ (transitivity);
e) if $a \mid b$ and $b \neq 0$, then $|a| \leq |b|$;
f) if $a \mid b$ and $a \mid c$, then $a \mid xb + yc$ for all integers x and y;
g) if $a \mid b$ and $a \mid (b \pm c)$, then $a \mid c$;
h) if $a \mid b$ and $b \mid a$, then $|a| = |b|$;
i) if $a \mid b$ and $b \neq 0$, then $(b/a) \mid b$;
j) if $c \neq 0$, then $a \mid b$ if and only if $ac \mid bc$.
k) if $a \mid b$, then $a \mid be$ for every integer e.
l) if $a \mid b$ and $c \mid d$, then $ac \mid bd$;

115. Show that if all digits of a three-digit number are equal then it is always divisible by 3.

116. Prove that for all integers a and b at least one of the numbers a, b, $a + b$ and $a - b$ is divisible by 3.

117. Five integers a, b, c, d, j satisfy the following conditions

a) $j \mid (ad - bc)$;
b) $j \mid (a - b)$, and
c) the only common divisors of b and j are ± 1.

Prove that $j \mid (c-d)$.

118. Prove that for all prime numbers p and q that are bigger than 3, the difference $p^2 - q^2$ is divisible by 24.

119. Explain why the square of an integer can not end with digits $2, 3, 7$ or 8.

120. Show that the sum of the squares of three consecutive natural numbers is not divisible by 3.

121. Prove that

a) Exactly one of three consecutive numbers is divisible by 3.
b) Exactly one of two consecutive even numbers is divisible by 4.
c) Exactly one of five consecutive numbers is divisible by 5.

122. Let n be a natural number. Prove that $6 \mid n(n+1)(n+2)$.

123. Let n be a natural number. Prove that $6 \mid n(n^2-1)$.

124. Let n be a natural number. Prove that $30 \mid n(n+1)(n+2)(n+3)(n+4)$.

125. Show that if n is a natural number, then $n^2 + 1$ is not divisible by 11.

126. Prove that if $(mn+pq) \vdots (m-p)$, then $(mq+np) \vdots (m-p)$.

127. Let a, b and c be such integers that $a+b+c$ is divisible by 6. Show that then $a^3+b^3+c^3$ is also divisible by 6.

128. Prove that a perfect square

a) has remainder 0 or 1 when divided by 3;
b) has remainder 0 or 1 when divided by 4;
c) has remainder 0, 1 or 4 when divided by 8;
d) has remainder 0, 1, 2 or 4 when divided by 7.

129. A natural number a has remainder 7 when divided by 8. What reminder a^3 has when divided by 8.

130. A natural number a has remainder 4 when divided by 5. Prove that $a^2 + a^3$ is divisible by 5.

131. Prove that for every natural number n the number $n^3 + 3n^2 + 2n$ is divisible by 6.

132. Prove that every prime number $p \geqslant 5$ has remainder 1 or 5 when divided by 6.

133. Prove that a square of a prime number $p \geqslant 5$ has remainder 1 when divided by 24.

134. Prove that when given any 12 natural numbers, we can choose two of them such that the difference of these two numbers is divisible by 11.

135. Prove that if p, $p+2$ and $p+4$ are all prime numbers, then $p=3$.
Hint. Examine separately the cases when number p has remainders 0, 1 or 2 when divided by 3.

Mathematical induction. Let us have a series of statements $S_1, S_2, \dots, S_n, \dots$. Every statement S_n of the given series is true if:

1. **Basis step.** S_1 is true, that means the first statement of the series is true;

2. **Inductive step.** $S_k \Rightarrow S_{k+1}$, that means if we assume that statement S_k is true, then we can conclude that the statement S_{k+1} is also true.

Strong (mathematical) induction. Let there be a series of statements $S_1, S_2, \dots, S_n, \dots$. In the given series every statement S_n is true, if

1. **Basis step.** S_1 is true, which means that the first statement of the series is true;

2. **Inductive step.** $S_1 \wedge S_2 \wedge \dots \wedge S_k \Rightarrow S_{k+1}$, which means that if all statements $S_1, \dots, S_k$ are true, then we can conclude that the statement S_{k+1} is also true.

Important! When using the strong induction, in the basis step it is sometimes not enough to prove just the first statement but it is necessary to also prove some of the following statements.

136. Prove the equalities by mathematical induction.

a) $\displaystyle\sum_{i=1}^{n} i^2 = \frac{1}{6}\cdot n(n+1)(2n+1) \quad \forall n \in \mathbb{N}$

b) $\displaystyle\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4} \quad \forall n \in \mathbb{N}$

c) $\displaystyle\sum_{i=1}^{n} \frac{1}{(2i-1)\cdot(2i+1)} = \frac{n}{2n+1} \quad \forall n \in \mathbb{N}$

d) $\displaystyle\sum_{i=1}^{n} (2i-1) = n^2 \quad \forall n \in \mathbb{N}$

e) $\displaystyle\sum_{i=0}^{n} i\cdot i! = (n+1)!-1 \quad \forall n \in \mathbb{N}\cup\{0\}$

f) $\displaystyle\sum_{i=1}^{n} (3i^2-i-2) = (n-1)n(n+2) \quad \forall n \in \mathbb{N}$

g) $\displaystyle\frac{1^2}{1\cdot 3} + \frac{2^2}{3\cdot 5} + \dots + \frac{n^2}{(2n-1)(2n+1)} = \frac{n(n+1)}{2(2n+1)} \quad \forall n \in \mathbb{N}$

h) $\displaystyle\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\cdots\left(1-\frac{1}{n^2}\right) = \frac{n+1}{2n} \quad \forall n \in \mathbb{N}$

i) $\displaystyle 1-2^2+3^2-\dots+(-1)^{n-1}n^2 = (-1)^{n-1}\frac{n(n+1)}{2} \quad \forall n \in \mathbb{N}$

j) $\displaystyle 1\cdot 2+2\cdot 3+\dots+n(n+1) = \frac{n(n+1)(n+2)}{3} \quad \forall n \geqslant 2$

k) $\displaystyle 1\cdot 3+2\cdot 4+\dots+n(n+2) = \frac{n(n+1)(2n+7)}{6} \quad \forall n \in \mathbb{N}$

l) $1^3+2^3+3^3+\dots+n^3 = (1+2+3+\dots+n)^2 \quad \forall n \in \mathbb{N}$

137. Prove the inequalities by mathematical induction.

a) $n^2 \geq 2n \;\; \forall n \in \mathbb{N} : n \geq 2$
b) $2^n \geq 2n \;\; \forall n \in \mathbb{N}$
c) $n^n \geq n! \;\; \forall n \in \mathbb{N}$
d) $(n+1)! \geq 3^n \;\; \forall n \in \mathbb{N} : n \geq 4$
e) $\frac{1}{2} \cdot \frac{3}{4} \cdot \dots \cdot \frac{2n-1}{2n} \leqslant \frac{1}{\sqrt{3n+1}} \;\; \forall n \in \mathbb{N}$
f) $1 + \frac{1}{4} + \frac{1}{9} + \dots + \frac{1}{n^2} \leqslant 2 - \frac{1}{n} \;\; \forall n \in \mathbb{N}$
g) $\frac{n}{2} < 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{2^n - 1} \leqslant n \;\; \forall n \in \mathbb{N}$
h) $\frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots + \frac{1}{n^2} < \frac{n-1}{n} \;\; \forall n \geqslant 2$
i) $\frac{1 \cdot 3 \cdot 5 \cdot \dots \cdot (2n-1)}{2 \cdot 4 \cdot 6 \cdot \dots \cdot 2n} \leqslant \frac{1}{\sqrt{2n+1}} \;\; \forall n \in \mathbb{N}$
j) $n! > 2^n \;\; \forall n \in \mathbb{N} : n \geq 4$
k) $\frac{n}{2} + 1 \geq \sum_{i=1}^{n} \frac{1}{i} \;\; \forall n \in \mathbb{N}$
l) $\sum_{l=1}^{n} \frac{1}{\sqrt{l}} \geq \sqrt{n} \;\; \forall n \in \mathbb{N}$

138. Prove that if $n \in \mathbb{N}$, then $(1+x)^n \geqslant 1 + nx$ holds for every $x > -1$, $x \in \mathbb{R}$.

139. The sum of the first n members of a geometric progression. Prove by mathematical induction that $1 + x + x^2 + x^3 + \dots + x^n = \frac{1 - x^{n+1}}{1 - x}$, if $n \geqslant 0$ and $x \neq 1$.

140. Prove by mathematical induction that the sum of the interior angles of an n–sided convex polygon is equal to $S_n = (n-2) \cdot 180^\circ$.

141. Let x be a real number such that $x + \frac{1}{x}$ is an integer. Prove that then $x^n + \frac{1}{x^n}$ is also an integer for all natural numbers n. *Hint: Use strong induction.*

142. There are n teams in a basketball league ($n \geq 2$). All teams play with each other exactly once and draws are not allowed (in every game someone will win and someone will lose). Prove that after the end of the season it is possible to line up the teams in such way that for every two teams that are next to each other, the winner of the game between the two teams, is left from the loser.

143. In a roundabout road with length 100 km there are n cars. All together they have enough fuel to drive 101 km. Prove that there exists a car that can cover the whole round by starting with its own fuel and gathering fuel from other cars on the way. (A car is allowed to take fuel from another car, if the cars are next to each other; pushing cars is not allowed).

144. There are $2n$ people in a party. All people who already know each other shake hands. It is known that there does not exist a group of three people, such that all three of them already know each other. Prove that the number of handshakes is at most n^2.

145. In a city we have n blabbermouths ($n \geqslant 4$). All of then have a phone. One day at the same time every one of them finds out a rumor. Prove that it is possible to organize phone calls in a way that after $2n - 4$ calls every blabbermouth knows all the rumors.

146. In a $2^n \times 2^n$ chessboard one square is colored red. Prove that it is possible to cover the board with L-shaped pieces in a way that the only visible square is the red square.

147. Prove the following statements ($n \in \mathbb{N}$).

a) $3 \mid n^3 + 2n$
b) $5 \mid n^5 - n$
c) $7 \mid n^7 - n$
d) $12 \mid 5 \cdot 9^n + 3$
e) $8 \mid 3^{2n} - 1$
f) $64 \mid 3^{2n} - 8n - 1$
g) $6 \mid n(n+1)(n+2)$
h) $6 \mid n(n^2 - 1)$
i) $30 \mid (n^5 - n)$
j) $30 \mid n(n+1)(n+2)(n+3)(n+4)$
k) $30 \mid mn(m^4 - n^4)$

148. Prove that the sum $\frac{a}{3} + \frac{a^2}{2} + \frac{a^3}{6}$ is an integer for every integer a.

149. Prove that $(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} \cdot a^{n-i} \cdot b^i$. $(a, b \in \mathbb{R}, n \in \mathbb{N})$

Hint: first make sure that $\binom{n}{i-1} + \binom{n}{i} = \binom{n+1}{i}$.

150. Let $a_0, a_1, a_2, \ldots$ be a sequence where the elements are defined as follows: $a_0 = \frac{1}{4}$ and $a_{n+1} = 2 \cdot a_n(1 - a_n)$ for $n \geqslant 0$. Show that for every $n \geqslant 0$ the term a_n is equal to $\frac{1 - 0{,}5^{2^n}}{2}$.

151. A recurrent sequence is defined as follows: $a_1 = 1$, $a_2 = 2$ and $a_{n+1} = a_n + 2a_{n-1}$ for every $n \geqslant 2$. Find a_{300}. *Hint: Use strong induction.*

Definition. Numbers $F_0, F_1, F_2, \ldots$, where $F_0 = 0$, $F_1 = 1$ and $F_{n+1} = F_n + F_{n-1}$ for every natural number n, are called Fibonacci numbers.

152. Prove the following statements for the Fibonacci sequence.

a) For every $n \geqslant 0$ the term F_{3n} is an even number.
b) For every $n \geqslant 0$ the equality $\sum_{i=0}^{n} (F_i)^2 = F_n \cdot F_{n+1}$ is true.
c) For every $n \geqslant 1$ the equality $\sum_{i=0}^{n-1} F_i = F_{n+1} - 1$ is true.
d) For every $n \geqslant 1$ the equality $F_1 + F_3 + F_5 + \cdots + F_{2n-1} = F_{2n}$ is true.
e) For every $n \geqslant 0$ and $m \geqslant 1$ the equality $F_{m+n} = F_m F_{n+1} + F_{m-1} F_n$ is true.
f) For every $n \geqslant 1$ the equality $F_0 - F_1 + F_2 - F_3 + \cdots - F_{2n-1} + F_{2n} = F_{2n-1} - 1$ is true.
g) For every $n \geqslant 1$ the equality $F_{n-1} F_{n+1} - F_n^2 = (-1)^n$ is true.
h) For every $n \geqslant 0$ the equality $F_n = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right]$ is true.

153. Find the mistake in the proof of the statement 'All birds are the same color'.
Basis step. If $n = 1$, we have exactly one bird and the statement is true.

Inductive step. Let us assume the statement is true for k birds. Let us now take a look at $k+1$ birds $L_1, \ldots, L_{k+1}$. By leaving out the last bird we get that according to the assumption the birds $L_1, \ldots, L_k$ are the same color. By leaving out the first bird, we also get that birds $L_2, \ldots,$ L_{k+1} are the same color. From that we can conclude that the bird L_{k+1} is the same color as birds $L_1, \ldots, L_k$ and therefore all $k+1$ birds are the same color.

154. Find the mistake in the proof of statement 'All positive integers are equal'.
Let $\max(x, y)$ denote the bigger number from numbers x and y. We will use induction to prove the statement.
Basis step. If $\max(x, y) = 1$, then $x = y = 1$, because we are examining positive integers.
Inductive step. Let us assume that the statement is true for all numbers that have a maximum of k. Let x and y be such numbers that $\max(x, y) = k+1$. The last equality is equal to the equality $\max(x-1, y-1) = k$. According to our assumption $x-1 = y-1$, from what we get $x = y$.

155. Let x_1 and x_2 be the solutions of the equation $x^2 + px - 1 = 0$, where p is an odd integer, and let $y_n = x_1^n + x_2^n$ for every $n = 0, 1, 2, \ldots$. Prove that for every natural number n numbers y_n and y_{n+1} are coprime.

156. A puzzle is put together step-by-step. A step can be either adding a single piece to the existing block or uniting two blocks. Prove by strong induction that no matter how to put together a puzzle, it will still take $n-1$ steps to put together a puzzle with n pieces.

157*. Prove that the sum of remainders that numbers $1^p, 2^p, 3^p, \ldots, (p-1)^p$ have when divided by p^2 is $\dfrac{p^3 - p^2}{2}$, if p is a prime number bigger than two.
Hint: Use the binomial formula.

158*. Prove that if n positive real numbers $x_1, x_2, \ldots, x_n$ are such that $x_1 + x_2 + \ldots + x_n \leqslant \dfrac{1}{2}$, then $(1-x_1)(1-x_2)\cdot\ldots\cdot(1-x_n) \geqslant \dfrac{1}{2}$.

159*. Let $S_j = 1 + \dfrac{1}{2} + \cdots + \dfrac{1}{j}$ for every $j \in \mathbb{N}$. Prove that $1 + \dfrac{n}{2} \leqslant S_{2^n} \leqslant 1 + n$ for every $n \in \mathbb{N}$.

160*. Prove that for every natural number $m \geqslant 2$ and for every natural number $N \geqslant m$ the equality

$$\sqrt{m\sqrt{(m+1)\sqrt{\ldots\sqrt{N}}}} < m+1$$

is true.

7. Different methods of proof

Direct proof

Definition. Integer n is said to be even if there exists a $k \in \mathbb{Z}$ such that $n = 2k$.

Definitsioon. Integer n is said to be odd if there exists a $k \in \mathbb{Z}$ such that $n = 2k + 1$.

161. The digits of a three-digit number are three successive natural numbers. Construct a new three-digit number by writing the digits of the first number in reverse. Prove that the difference between the larger number and the smaller number is 198.

162. Prove that a sum of two even numbers is even.

163. Prove that a sum of two rational numbers is rational.

164. Prove that if n is an odd integer then n^2 is also an odd integer.

165. Prove that if m and n are both perfect squares then mn is also a perfect square.

166. Prove that if a product of two natural numbers is odd then their sum is an even number.

167. Prove that the sum of the squares of two successive numbers is an odd number.

168. Let x and y be integers. Prove that if both xy and $x + y$ are even numbers then x and y are both even.
Hint. Instead of $A \Rightarrow B$ show that $\neg B \Rightarrow \neg A$. Assume that x and y both are not even numbers at the same time. Show that xy and $x + y$ both are not even numbers at the same time. Without loss of generality assume that x is even. Separately check cases where y is an odd number and where y is an even number.

169. Prove that fraction $\frac{12n+1}{30n+2}$ is irreducible for every positive integer n.

170. Prove that if $a^2 + b^2 = 1$ and $c^2 + d^2 = 1$ then $|ac + bd| \leqslant 1$. Use trigonometric substitutions.

171. Prove that if the interior angles of a triangle are α, β and γ, and $\sin\alpha : \sin\beta : \sin\gamma = 4 : 5 : 6$, then $\cos\alpha : \cos\beta : \cos\gamma = 12 : 9 : 2$.

Proof by contradiction

172. Prove by contradiction that if a product of two numbers is odd then their sum is even.

173. Prove that if n is an integer and n^2 is odd then n is an odd integer.

174. Prove that at least four of 22 successive days fall on the same weekday.

175. Prove that among 64 freely picked days there are at least ten which have the same weekday designation. (Hint: Use Dirichlet's principle.)

176. Prove that among 25 freely picked days there are at least 3 days from the same month.

177. Prove that if there are $n \geq 2$ people attending a part then at least two people at the party must have the same amount of friends.

178. Prove that if n is a natural number and $3n+2$ is an odd number then n is even.

179. Prove that if x is an odd number then $\sqrt{2x}$ is not an integer.

180. Show that if c is an odd number then equation $x^2 + x - c = 0$ has no integer solutions.

181. Prove that the set of prime numbers is infinite.

182. Prove that if n is a composite number (not a prime number) then it is divisible by a prime number which is less than or equal to $\sqrt{n}$.

183. Prove that the diagonal length of a unit square cannot be expressed as a rational number.

184. Prove that if for a real number x the number x^3 is irrational then x is also an irrational number.

185. Prove that if a positive real number x is irrational then $\sqrt{x}$ is an irrational number.

186. Prove that there does not exist a rational number r for which $r^3 + r + 1 = 0$. Hint. Proof by contradiction. If $r = a/b$ and a and b are integers such that their greatest common divisor is 1, then substitute $r = a/b$ in equation $r^3 + r + 1 = 0$. Multiply both sides of this equation by b^3 and analyze the result to see if either of a and b can be even or odd.

187. Let m and n be coprime integers. Show that $m+n$ and $m-n$ are also coprime numbers.

188. A blackboard has the numbers $1, 2, \ldots, 2000$ written on it. Two numbers are removed from the board and both of them are replaced with their arithmetic median. This process is repeated over and over again. Prove that the black board never has just the numbers 1000, 1000, ..., 1000 written on it.

189. Prove that every party has two participants who know the same amount of participants of the party. (Acquaintanceship is a symmetrical relation: if A knows B then B also knows A.)

190. Prove that a party of 51 people always has a person who knows an even number of other people at the party.

191. Prove that if line c crosses one of two parallel lines a and b then it also cuts the other.

192. Prove that if two lines a and b are perpendicular to one and same line c then lines a and b are parallel to each other.

193. Prove that if two angles of a triangle are unequal then the sides opposite to them also are unequal.

194. Prove that a natural number of form $4k+3$ has at least one prime factor with the same form.

195. Prove in two different ways that number $n^5 - n$ is divisible by 5.

196. Prove that any integer of form $p = 3k - 1$ is either a prime number or it has an odd number of prime factors of same form.

Counterexamples

197. Disprove the following assertions with a counterxample.

a) Let a and b be integers. If $a \mid b$ and $b \mid a$ then $a = b$.
b) If a, b and c are integers for which $a \mid (bc)$ then $a \mid b$ or $a \mid c$.
c) If n is a positive integer then $n^2 + n + 41$ is a prime number.
d) For every real number x holds $x^3 \geq x^2$.
e) For every positive real number x it holds that $2x^2 > x$.
f) $\forall x, y \in \mathbb{R},\ \sqrt{x+y} \leq \sqrt{x} + \sqrt{y}$.
g) $\forall x \in \mathbb{R},\ x^2 \neq x$.
h) $\forall x \in \mathbb{R},\ |x| > 0$.
i) $\forall x, y \in \mathbb{Z},\ [x^2 = y^2 \implies x = y]$.
j) $\forall x \in \mathbb{Z} \exists y \in \mathbb{Z},\ y^2 = x$.

198. Compare the two assertions and either prove them or disprove them with a counterexample.

a) number $2^n + 1$ is a prime only if n is a prime number;

b) number $2^n - 1$ is a prime only if n is a prime number;

Proof by equivalence

199. Let $x, y \in \mathbb{Z}$. Prove that $4 \mid (x^2 - y^2)$ if and only if both x and y are even numbers or if both x and y are odd numbers.

200. Prove that $3 \mid (2n^2 + 1)$ if and only if $3 \nmid n$.

201. Prove that a four-digit number $\overline{abcd}$ is divisible by 101 if and only if $\overline{ab} - \overline{cd} = 0$.

Constructive proof of existence

202. Prove that the difference between two consecutive prime numbers can be arbitrarily large (or for every natural number n there exist n consecutive natural numbers and none of them are prime numbers)

Existence and uniqueness

203. Prove that for every natural number n there exists exactly one natural number m for which $m^2 \leqslant n < (m+1)^2$.

204. Prove that if a and b are real numbers and $a \neq 0$ then equation $ax + b = c$ has exactly one real solution.

205. Let a and b be odd integers for which $a \neq b$. Prove that there exists exactly one integer c for which $|a - c| = |b - c|$.

206. Prove that if a is an irrational number then there exists an unique integer m for which $|a - m| < 1/2$.

207. Prove that if n is an odd integer then there exists a unique integer k for which n is the sum of $k - 2$ and $k + 3$.

208. Prove that if r is a real number then there exists a unique integer n and a unique real number ε for which $0 \leqslant \varepsilon < 1$ and $r = n + \varepsilon$.

Find mistake in proof.

209. Juku says $-1 = 1$ because $(-1)^2 = 1^2$. Where is the mistake?

210. Let m and n be arbitrary numbers and observe the equality

$$m^2 - 2mn + n^2 = n^2 - 2mn + m^2 \text{ or } (m-n)^2 = (n-m)^2.$$

From the last part we conclude that $m - n = n - m$ or $2m = 2n$ from which $m = n$. Therefore, every two numbers are equal! Where is the mistake?

211. Observe the equality $a^2 - a^2 = a^2 - a^2$ or in other words $a(a-a) = (a-a)(a+a)$. By dividing both sides of the last equality by $a-a$ we get $a = a+a$ or $a = 2a$. Thus every number is equal to its double. Where is the mistake?

Necessary and sufficient conditions

212. Prove that a four-digit number $\overline{abcd}$ divides by 101 if and only if $\overline{ab} - \overline{cd} = 0$.

213*. Seventeen mathematicians of different countries are in correspondence with each other. Every two mathematicians write in one of three languages: English, French or Russian. Prove that there are three mathematicians who write to each other in the same language.

214*. All points of a plane are colored using

a) two
b) three

different colors. Prove that there are always two points of same color which are only 1 unit away from each other.

8. Functions

Definition. Let X and Y be sets. If there is given a rule, which associates every element of X to a single element Y, then we say, that there is defined a **function** f, and we write $f\colon X \to Y$. If element $x \in X$ is associated with element $y \in Y$, then we use the notation $y = f(x)$ or $y = fx$ or $f\colon x \mapsto y$.

Definition. Functions $f\colon X \to Y$ and $g\colon Z \to W$ are said to be **equal**, if $X = Z$, $Y = W$ and $f(x) = g(x)$ for all $x \in X(= Z)$.

Definition. Let $f\colon X \to Y$ be a function. The set $G(f) = \{(x, f(x)) \mid x \in X\} \subset X \times Y$ is called the **graph** of function f.

Function $I_X : X \to X$ defined wit $I_X(x) = x\ \forall x \in X$ is called the identity transformation.

215. Sketch the graphs of given functions ($x \in \mathbb{R}$):

a) $f(x) = 1 - |x|$;

b) $f(x) = \begin{cases} 1 - x^2, & \text{if } x \leqslant 0, \\ 1/x, & \text{if } 0 < x < 2, \\ 1/2, & \text{if } x \geqslant 2. \end{cases}$

c) $f(x) = \begin{cases} x^3, & \text{if } x < 0, \\ 0, & \text{if } 0 \leqslant x \leqslant 1, \\ 2x, & \text{if } x > 1. \end{cases}$

216. Let $f_1 : X \to Y$ and $f_2 : X \to Y$. Prove that $f_1 = f_2$ if and only if $G(f_1) = G(f_2)$.

217. Let $f_1 : X \to Y$ and $f_2 : X \to Y$. Show that $G(f_1) \cup G(f_2)$ (also $G(f_1) \cap G(f_2)$) is a graph of a function with domain X if and only if $f_1 = f_2$.

Let X be a universal set.

Definition. The **characteristic function** of set $A \subset X$ is the function $\chi_A\colon U \to \{0, 1\}$, where

$$\chi_A(x) = \begin{cases} 1, & \text{if } x \in A \\ 0, & \text{if } x \in U \setminus A. \end{cases}$$

218. Prove the relations:

a) $\chi_\varnothing(x) \equiv 0$, $\chi_X(x) \equiv 1$;

b) $\chi_A(x)\chi_A(x) \equiv \chi_A(x)$;

c) $\chi_{A\cap B}(x) \equiv \chi_A(x)\chi_B(x)$;

d) $\chi_{A\cup B}(x) \equiv \chi_A(x) + \chi_B(x) - \chi_A\chi_B(x)$;

e) $\chi_{A\setminus B}(x) \equiv \chi_A(x) - \chi_A(x)\chi_B(x)$;

f) $\chi_{A'}(x) \equiv 1 - \chi_A(x)$;

g) $\chi_{A\times B}((x, y)) \equiv \chi_A(x)\chi_B(y)$.

219. Show that $\chi_{\cup_{i\in I} A_i}(x) = \max\limits_{i\in I} \chi_{A_i}(x)$, $\quad \chi_{\cap_{i\in I} A_i}(x) = \min\limits_{i\in I} \chi_{A_i}(x)$.

220. Express $\chi_{A \triangle B}$ through functions χ_A and χ_B.

221. Use the characteristic functions to prove the following equalities:

a) $(A \cap B) \cup (A \setminus B) = A$;
b) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$;
c) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$;
d) $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$;
e) $(A \setminus B) \setminus C = A \setminus (B \cup C)$;
f) $(A \cup B) \times C = (A \times C) \cup (B \times C)$.

222. Prove that $A \triangle (B \triangle C) = (A \triangle B) \triangle C$.

Definition. Let there be a function $f : X \to Y$ and elements $x \in X$ and $y \in Y$. If $y = f(x)$ then element y is called the **image** of element x and element x is called the **preimage** of element y.

Definition. If $A \subset X$ then

$$f(A) = \{y \in Y \mid \text{exists } x \in A \text{ such that } y = f(x)\} = \{f(x) \in Y \mid x \in A\}$$

is the image of set A.

Definition. If $B \subset Y$ then set

$$f^{-1}(B) = \{x \in X \mid f(x) \in B\}$$

is the preimage of set B.

223. Let there be given a function f and element x_0. Find the image $f(x_0)$ of element x_0.

a) $f : \mathbb{N} \to \mathbb{N}$, $f(x) = x^2 + x + 1$, $x_0 = 3$

b) $f :$ {Countries} $\to$ {People}, $f(x) =$ head of state x, $x_0 =$ Estonia

c) $f : \mathbb{N} \to \mathbb{N}$, $f(x) =$ total number of different prime factors of x, $x_0 = 360$

d) $f : \mathbb{Z} \to \mathbb{Z}$, $f(x) = \dfrac{x^2}{4}$, if x is even, $f(x) = \dfrac{x^2 - 1}{4}$, if x is odd, $x_0 = -5$

224. Let there be given a function $f : \mathbb{R} \to \mathbb{R}$ and element $y_0 \in \mathbb{R}$. Find all preimages of element y_0:

a) $f(x) = 4 - x^2$, $i)$ $y_0 = 0$; $ii)$ $y_0 = 4$; $iii)$ $y_0 = 2$;

b) $f(x) = 3^x$, $i)$ $y_0 = 3$; $ii)$ $y_0 = 1$; $iii)$ $y_0 = 5$.

225. Find all preimages of element y_0.

a) $f : \mathbb{N} \to \mathbb{N}$, $f(x) = |x|$, $y_0 = 2$

b) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x(x-1)$, $y_0 = 6$

c) $f: \mathbb{Z} \to \mathbb{Z}$, $f(x) = x \mod 5$ (that is to say $f(x)$ is the remainder of x after dividing it by 5), $y_0 = 3$

d) $f: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $f((r,s)) = (r+2)^2 + (s-3)^2$, $y_0 = 4$

226. Let there be given a function f and a set A. Find the image $f(A)$ of set A:

a) $f: \mathbb{R} \to \mathbb{R}, f(x) = x$, $A = [1,2]$;

b) $f: \mathbb{N} \to \mathbb{R}, f(x) = (-1)^x$, $A = \{2\}$;

c) $f: \mathbb{N} \to \mathbb{R}, f(x) = (-1)^x$, $A = \{2,4,6,8,\dots\}$;

d) $f: \mathbb{R} \to \mathbb{R}, f(x) = -3x$, $A = [0,3]$;

e) $f: \mathbb{R} \to \mathbb{R}, f(x) = |x-1|$, $A = [0,3]$;

f) $f: \mathbb{R} \to \mathbb{R}, f(x) = (x-2)^2$, $A = [1,4]$;

227. Let us consider the function $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x^2 + 2x + 2$. Find the following sets.

a) $f(\{-1,0,1\})$

b) $f([-1,1])$

c) $f([-2,2])$

d) $f([-5,-3] \cup [0,3])$

e) $f((-\infty,0])$

f) $f(\mathbb{R})$

228. Let there be a function f and a set B. Find the preimage $f^{-1}(B)$ of set B:

a) $f: \mathbb{R} \to \mathbb{R}, f(x) = \cos x$, $B = \{0\}$;

b) $f: \mathbb{N} \to \mathbb{R}, f(x) = (-1)^x$, $B = \{1\}$;

c) $f: \mathbb{N} \to \mathbb{R}, f(x) = (-1)^x$, $B = (0,\infty)$;

d) $f: \mathbb{R} \to \mathbb{R}, f(x) = -3x$, $B = [0,3]$.

229. Observe function $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x^2 + 2x + 2$. Find the following sets.

a) $f^{-1}(\{-1,0,1\})$ b) $f^{-1}([-2,2])$ c) $f^{-1}([10,50])$ d) $f^{-1}((-\infty,0])$ e) $f^{-1}(\mathbb{R})$

230. Let $f: \mathbb{R} \to \mathbb{R}$, $g: \mathbb{R} \to \mathbb{R}$, $f(x) = \lfloor x \rfloor := \max\{m \in \mathbb{Z} | m \leqslant x\}$. $g(x) = \lceil x \rceil := \min\{m \in \mathbb{Z} | m \geqslant x\}$. Find the following sets.

a) $f^{-1}(\{0\})$, $g^{-1}(\{0\})$

b) $f^{-1}(\{-1,0,1\})$, $g^{-1}(\{-1,0,1\})$

c) $f^{-1}(\{x \,|\, 0 < x < 1\})$, $g^{-1}(\{x \,|\, 0 < x < 1\})$

231. Let $m,k \in \mathbb{Z}$. Find $f^{-1}(\{m\})$, $f^{-1}(\{m,m+1\})$, $f([k,k+1])$, $f([k,k+2])$ for functions $f: \mathbb{R} \to \mathbb{R}$:

a) $f(x) = \lceil \lfloor x - \frac{1}{2} \rfloor + \frac{1}{2} \rceil$

b) $f(x) = \lfloor \frac{1}{2} - \lceil x + \frac{1}{2} \rceil \rfloor$

c) $f(x) = \lceil \frac{1}{2} - 2\lfloor \frac{x}{2} \rfloor \rceil$

d) $f(x) = \lfloor 2\lceil \frac{x}{2} \rceil + \frac{1}{2} \rfloor$

232. Let $f : X \to Y$, $A_1, A_2 \subset X$, $B_1, B_2 \subset Y$. Prove the following statements:

a) $A_1 \subset A_2 \Rightarrow f(A_1) \subset f(A_2)$;

b) $f(A_1 \cup A_2) = f(A_1) \cup f(A_2)$;

c) $B_1 \subset B_2 \Rightarrow f^{-1}(B_1) \subset f^{-1}(B_2)$;

d) $f^{-1}(B_1 \cup B_2) = f^{-1}(B_1) \cup f^{-1}(B_2)$;

e) $f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2)$;

f) $f^{-1}(B_1 \setminus B_2) = f^{-1}(B_1) \setminus f^{-1}(B_2)$.

233. Prove the following equalities:

a) $f(\bigcup_\alpha A_\alpha) = \bigcup_\alpha f(A_\alpha)$; b) $f^{-1}(\bigcup_\alpha B_\alpha) = \bigcup_\alpha f^{-1}(B_\alpha)$; c) $f^{-1}(\bigcap_\alpha B_\alpha) = \bigcap_\alpha f^{-1}(B_\alpha)$.

234. Let $f : X \to Y, B \subset Y$. Prove that $f^{-1}(Y \setminus B) = X \setminus f^{-1}(B)$.

235. Let $f : X \to Y, A \subset X$. Find an example where $f(X \setminus A) \nsubseteq \nsupseteq Y \setminus f(A)$.

236. Let $f : X \to Y, A \subset X, B \subset Y$. Prove the following inclusions and find examples where these inclusions are not equalities.

a) $A \subset f^{-1}(f(A))$

b) $f(f^{-1}(B)) \subset B$

237. Let $f : X \to Y, A \subset X, B \subset Y$. Prove that $f(A) \subset B \Leftrightarrow A \subset f^{-1}(B)$.

Definition. Function $f : X \to Y$ is injective or one-to-one if for all pairs $x_1, x_2 \in X$, $x_1 \neq x_2$ holds $f(x_1) \neq f(x_2)$.

Definition. Function $f : X \to Y$ is surjective or an onto map if $f(X) = Y$.

Definition. Function is bijective if it is injective and surjective.

Definition. The product or composition of functions $f : X \to Y$ and $g : Y \to Z$ is the function $gf : X \to Z$ which is defined as

$$(gf)(x) = g(f(x)),\ x \in X.$$

Definition. A substitution on set $M \neq \varnothing$ is any bijective map $f : M \to M$.

If function $f : X \to Y$ is bijective then we can define the inverse function $f^{-1} : Y \to X$ which assigns to every $y \in Y$ its preimage $x \in X$ by function f, that is to say

$$f^{-1}(y) = x \Leftrightarrow y = f(x).$$

238. Decide whether the following functions are surjective.

a) $q: \mathbb{N} \to \mathbb{N},\ q(x) = x + 2$

b) $p: \mathbb{Z} \to \mathbb{Z},\ p(x) = x + 2$

239. Determine if function f is injective, surjective or bijective. Explain your answer (denote $\mathbb{R}_+ = \{x \mid x \in \mathbb{R}, x \geqslant 0\}$):

a) $f: \mathbb{R} \to \mathbb{R},\ f(x) = 3x - 2$;

b) $f: \mathbb{R} \to \mathbb{R},\ f(x) = 2^x$;

c) $f: \mathbb{R} \to \mathbb{R}_+,\ f(x) = x^2$;

d) $f: \mathbb{R}_+ \to \mathbb{R}_+,\ f(x) = x^2$;

e) $f: \mathbb{R} \to \mathbb{R},\ f(x) = x^3$;

f) $f: \mathbb{R} \to \mathbb{R},\ f(x) = \sin x$;

g) $f: \mathbb{R} \to [-1, 1],\ f(x) = \sin x$;

h) $f: \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to [-1, 1],\ f(x) = \sin x$;

i) $f: \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to \mathbb{R},\ f(x) = \tan x$;

j) $f: \mathbb{R} \to \mathbb{R},\ f(x) = \cos x$;

k) $f: (0, \pi) \to \mathbb{R},\ f(x) = \cot x$;

240. Determine if function f is injective, surjective or bijective. Explain your answer.

a) $f: \mathbb{Z} \to \mathbb{N},\ f(n) = 1 + n + 3|n|$;

b) $f: \mathbb{N} \to \mathbb{Z},\ f(n) = (-1)^n n$.

c) $f: \mathbb{Z} \to \mathbb{Z},\ f(x) = (-1)^x x$

d) $f: \mathbb{N} \to \mathbb{N},\ f(x) = 1 + \dfrac{x + 3|x|}{2}$

241. Determine if function f is injective, surjective or transitive. Explain your answer.

a) $f: \mathbb{R}^2 \to \mathbb{R}^2,\ f(x, y) = (x + y, x - y)$

b) $g: \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z},\ g(m, n) = m^2 - n$

c) $g: \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z},\ g(m, n) = (2m, m - n)$

242. Let E denote the set of even numbers. For every function $f: \mathbb{Z} \to E$ determine if f is injective or surjective. Explain your answer.

a) $f(x) = \begin{cases} 2x, & \text{if } x \text{ is odd} \\ 4 - x, & \text{if } x \text{ is even} \end{cases}$

b) $f(x) = \begin{cases} 4 - 2x, & \text{if } x \text{ is odd} \\ 2x, & \text{if } x \text{ is even} \end{cases}$

243. Let function $f: [0, 1] \times \{1, 2, 3\} \to [1, 4]$ be defined as $f(x, y) = x + y$. Show that f is surjective but not injective.

244. Let function $g: [1, 4] \to [0, 1]$ be defined as $g(x) = \frac{x}{4}$. Show that g is injective but not surjective.

245. Let $X = \mathbb{N} \cup \{-1, 3.5\}$ and $Y = \mathbb{N} \cup \{0\}$. Find a function $f: X \to Y$ which is injective and surjective and $f(2) = 3$.

246. Let there be a two variable function $g: \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ for which $g(m,n) = m^2 - n$.

a) Is function g one-to-one? Is function g an onto mapping?

b) Find the values $g(0,3)$, $g(3,-2)$, $g(-3,-2)$ and $g(7,-1)$.

c) Find $g^{-1}(\{0\})$ and $g^{-1}(\{5\})$.

247. Let $\varphi: A \to B$ be a bijection. Prove that
a) φ^{-1} is a bijection,
b) $\varphi^{-1} \circ \varphi = I_A$,
c) $\varphi \circ \varphi^{-1} = I_B$.

248. Prove that any function $f: X \to Y$ satisfies the condition

$$f(A \cap B) \subset f(A) \cap f(B) \quad \forall A, B \subset X$$

and equality $f(A \cap B) = f(A) \cap f(B)$ holds if and only if f is injective.

249. Prove that $f(\bigcap\limits_{\alpha} A_\alpha) \subset \bigcap\limits_{\alpha} f(A_\alpha)$. Find an example of function f and sets A_α where $f(\bigcap\limits_{\alpha} A_\alpha) \neq \bigcap\limits_{\alpha} f(A_\alpha)$. Hint: consider the projections.

250. For every subset A of set X show that $f(X \setminus A) \subset Y \setminus f(A)$ if and only if f is injective.

251. For every subset A of set X show that $f(X \setminus A) \supset Y \setminus f(A)$ if and only if f is surjective.

252. Find an example of a function $f: \mathbb{Z} \to \mathbb{N}$ such that

a) f is injective but not surjective.

b) f is surjective but not injective.

c) f is both injective and surjective.

d) f is not injective nor surjective.

253. Grade the following proof with A, C or F, where grade A is for a correct proof, grade C is for a partially correct proof and grade F is for a wrong proof. Explain your grading.

Assertion: Function $f: \mathbb{R} \to \mathbb{R}$ where $f(x) = \begin{cases} -x, & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \\ 2x^2 - \sqrt{2}, & \text{if } x \in \mathbb{Q} \end{cases}$ is injective.

Proof: Let $x \neq y$. Then $2x^2 - \sqrt{2} \neq 2y^2 - \sqrt{2}$ and $-x \neq -y$. Thus $f(x) \neq f(y)$ which proves that f is injective.

254. Let A and B be nonempty sets. Define the function

$$p_1 : A \times B \to A, \; p_1(a,b) = a,$$

for all $(a,b) \in A \times B$. This is the **first projection function**.

a) Is function p_1 surjective? Explain your answer.

b) Let $B = \{b\}$, Is p_1 now an injection? Explain your answer.

c) Under which condition(s) function p_1 is not an injection? Construct an assertion to answer this qyestion and then prove it.

255. Let C be the set of all continuous functions on the closed interval $[0,1]$. Define the function $A: C \to \mathbb{R}$ for all $f \in C$ as

$$A(f) = \int_0^1 f(x)dx.$$

Is function A injective? Is it surjective? Explain your answers.

256. Let $A = \{(m,n) \mid m \in \mathbb{Z}, n \in \mathbb{Z} \text{ ja } n \neq 0\}$. Define the function $f: A \to \mathbb{Q}$ as follows

$$\text{for all } (m,n) \in A \text{ we have } f(m,n) = \frac{m+n}{n}.$$

a) Is function f injective? Explain your answer.

b) Is function f surjective? Explain your answer.

Denote $A^B = \{f\colon B \to A \mid f \text{ is a function}\}$.

257. Let sets A and A_1 but also B and B_1 have one-to-one correspondences (bijections). Prove that then the following sets are also in an one-to-one correspondence
a) $A \times B$ and $A_1 \times B_1$,
b) A^B and $A_1^{B_1}$,
c) $A \cup B$ and $A_1 \cup B_1$ if $A \cap B = \varnothing$ and $A_1 \cap B_1 = \varnothing$.

258. Construct an one-to-one correspondence between the following sets:
a) $A \times B$ and $B \times A$,
b) $A \times (B \times C)$ and $(A \times B) \times C$,
c) $(A \times B)^C$ ja $A^C \times B^C$,
d) $(A^B)^C$ and $A^{B \times C}$,
e) $A^{B \cup C}$ and $A^B \times A^C$ if $B \cap C = \varnothing$.

259. Let A and B be sets such that $B \neq \varnothing$. Define functions $f : \mathcal{P}(B) \to \mathcal{P}(B)$, $g : \mathcal{P}(B) \to \mathcal{P}(B)$ by formulae $f(X) = X \cup A$ and $g(X) = X \cap A$. Under which condition(s) is function f or g injective? Under which condition(s) is function f or g surjective?

260. Let $C \neq \varnothing$ and $A, B \subset C$ and $f, g : \mathcal{P}(C) \to \mathcal{P}(C) \times \mathcal{P}(C)$ be functions for which $f(X) = (X \cup A, X \cup B)$, $g(X) = (X \cap A, X \cap B)$ for all $X \in \mathcal{P}(C)$. Under which conditions is function f or g injective?

261. Let $f(x) = 3x$, $g(x) = x + 1$ and $h(x) = x^2 + 2$. Find

a) $(fg)(3)$;
b) $(fg)(-6)$;
c) $(gf)(x)$;
d) $(fh)(2)$;
e) $(fh)(x)$;
f) $(hf)(x)$;
g) $(gh)(x)$;
h) $(hg)(x)$.

262. Let $f(x) = x^2 - 1$ and $g(x) = 3 - x$. Find

a) $(gf)(1)$;
b) $(fg)(-4)$;
c) $(fg)(x+1)$;
d) $(fg)(x+2)$.

263. Let $f(x) = x^3 + 3$ and $g(x) = x - 4$. Find $(fg)(x)$ and $(gf)(x)$ and solve the equation $(fg)(x) = (gf)(x)$.

264. Find gf, fg, g^2, f^3 if

a) $f(x) = 2x + 1$ and $g(x) = 3x - 1$;
b) $f(x) = x^2$ and $g(x) = 2^x$.

265. Find f^{-1} if

a) $f(x) = 3x - 1$;
b) $f(x) = x^3 - 3$;
c) $f(x) = \frac{1}{4}x + 5$;
d) $f(x) = \sqrt{3}x - 3$;
e) $f(x) = 2x^3 + 3$;
f) $f(x) = \frac{2x}{5 - x}$, $x \neq 5$.

266. Find gf, fg, g^5, f^6 if f and g are substitution on set X:

a) $X = \{1,2,3,4,5\}$, $f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 2 & 5 & 4 \end{pmatrix}$, $g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 1 & 2 & 3 & 4 \end{pmatrix}$;

b) $X = \{1,2,3,4,5,6\}$, $f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 1 & 5 & 3 & 4 & 2 \end{pmatrix}$, $g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 5 & 1 & 4 & 3 & 2 \end{pmatrix}$;

c) $X = \{1,2,3,4,5,6\}$, $f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 1 & 2 & 6 & 5 & 4 \end{pmatrix}$, $g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 5 & 4 & 3 & 2 & 1 \end{pmatrix}$.

267. Find f^{-1}, g^{-1}, $g^{-2}f^3$, f^3g^2 if f and g are substitution on set X:

a) $X = \{1,2,3,4\}$, $f = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$, $g = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix}$;

b) $X = \{1,2,3,4,5\}$, $f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 1 & 5 & 4 \end{pmatrix}$, $g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 1 & 5 & 4 \end{pmatrix}$;

c) $X = \{1,2,3,4,5\}$, $f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 4 & 3 & 5 \end{pmatrix}$, $g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 5 & 3 & 2 & 1 \end{pmatrix}$;

268. Let $f : X \to Y$ and $g : Y \to X$ be functions. Prove that if $gf = I_X$, then f is injective and g is surjective.

269. Let $f_1 : X \to Y$, $f_2 : X \to Y$, $g : Y \to Z$ be functions. Prove that if $gf_1 = gf_2$ and g is injective then $f_1 = f_2$.

270. Let $f : X \to Y$, $g_1 : Y \to Z$, $g_2 : Y \to Z$ be functions. Prove that if $g_1 f = g_2 f$ and f is surjective then $g_1 = g_2$.

271. Let $f : X \to Y$ and $g : Y \to Z$. Let $h = g \circ f$. Prove or disprove the following assertions.

a) If f and g are injective then h is also injective.
b) If f and g are surjective then h is also surjective.
c) If f and g are bijective then h is also bijective.
d) If h is injective then f is also injective.
e) If h is injective then g is also injective.
f) If h is surjective then f is also surjective.
g) If h is surjective then g is also surjective.

272. Let X, Y and Z be nonempty sets and also $f : X \to Y$ and $g : Y \to Z$ be functions. Prove the assertions:

a) If f is surjective and $g \circ f$ is injective then g is injective.
b) If g is injective and $g \circ f$ is surjective then f is surjective.

273*. Find all functions $f : \mathbb{R} \to \mathbb{R}$ which for every real number x satisfy the condition $f(2015x + f(0)) = 2015x^2$.

274*. Find all functions $f : \mathbb{R} \to \mathbb{R}$ which for all $x, y \in \mathbb{R}$ satisfy the condition

$$f(x \cdot f(y)) = x \cdot y.$$

275*. Let a be a real number for which $|a| \neq 1$. Find all functions $f : (0, \infty) \to \mathbb{R}$ which for all $x \in (0, \infty)$ satisfy condition

$$ax^2 f\left(\frac{1}{x}\right) + f(x) = \frac{x}{x+1}.$$

276*. 1) Find all surjective functions $f : \mathbb{R} \to \mathbb{R}$ which for all $x, y \in \mathbb{R}$ satisfy

$$f(x + f(y)) = f(x + y) + 1.$$

2) Find all injective functions $f : \mathbb{R} \to \mathbb{R}$ which for any $x, y \in \mathbb{R}$ satisfy

$$f(x + f(y)) = f(x + y) + 1.$$

277*. Prove that for all $f : X \to Y$ exists a set Z, an injective mapping $g : X \to Z$ and a surjective mapping $h : Z \to Y$ such that $f = h \circ g$.

9. Cardinality of sets

Definition.

1. Sets X and Y are said to have the same cardinality of are equivalent if there exists a bijective function $f : X \to Y$. We use the notation $|X| = |Y|$ or $X \sim Y$. If X and Y do not have the same cardinality then we use the notation $|X| \neq |Y|$ or $X \nsim Y$.
2. The cardinality of X does not exceed cardinality of Y if there exists an injective function from X to Y and we denote $|X| \leqslant |Y|$.
3. The cardinality of X is less than the cardinality of Y if $|X| \leqslant |Y|$ and $|X| \neq |Y|$. We use the notation $|X| < |Y|$.

Definition.

1. Set X is finite if $X = \varnothing$ or there exists an $n \in \mathbb{N}$ such that $|X| = |\{1, \ldots, n\}|$.
2. Set X is infinite if it is not finite.
3. Set X is countable if $X \sim \mathbb{N}$.
4. Set X is at most countable if $|X| \leqslant |\mathbb{N}|$.
5. Set X has the cardinality of continuum if $|X| = |\mathbb{R}|$.

Theorem. If the cardinality of set A does not exceed the cardinality of set B and the cardinality of B does not exceed the cardinality of A, then sets A and B have the same cardinality.

278. Determine whether the given set is finite or infinite. If the set is infinite then determine whether it is countable or not. Explain your answer.

a) $X = \{-10, -9, -8, \ldots, -1, 0, 1, 2, \ldots, 5, 6\}$;
b) $X = \{10^x : x \in \mathbb{N}\}$;
c) $X = \varnothing$;
d) $X = (1, 2)$.

279. Construct bijections $f : \mathbb{N} \to M$ and $g : M \to \mathbb{N}$ for the following sets M (here $k \in \mathbb{N}$)

a) $M = \mathbb{Z}$;
b) M is the set of even numbers;
c) M is the set of odd numbers;
d) $M = \mathbb{N} \cup \{-k, 1-k, \ldots, 0\}$;
e) $M = \mathbb{N} \cup \{a_1, a_2, \ldots, a_k\}$;
f) $M = \mathbb{N} \times \mathbb{N}$.

280. Hilbert's hotel has a countably infinite number of rooms. How to accommodate new guests if:

a) arrives a train with a million guests;
b) arrives a train with a countably infinite number of guests;
c) arrives a countably infinite number of trains each with a countably infinite number of guests.

281. Is the set $\mathbb{N}$ equivalent to a proper subset of itself?

282. Prove that every infinite set has a countable subset.

283. Prove that a set is infinite if and only if it is equivalent to one of its proper subsets.

284. Let A and B be countable set. Prove that $A \cup B$ is countable.

285. Let A be a countable set and B be a finite set. Prove that $A \cup B$ and $A \setminus B$ are countable and $A \cap B$ is finite.

286. Prove that the set of all finite subsets of the set of natural numbers is countable.

287. Present the set of all natural numbers $\mathbb{N}$ in the form $\bigcup\limits_{n=1}^{\infty} A_n$ where all sets $A_1, A_2, \ldots$ are infinite and pairwise disjoint sets.

288. Let A be a nonempty set. We call it alphabet, its elements letters and every finite writing of consecutive letters a word. If $a, b, c \in A$ then words for example are a and $bacaa$. Prove that the set of all words is countable.

289. Prove that the set of all finite binary sequences is countable. Prove that the set of all binary sequences has the cardinality of continuum.

290. Prove that the set of all intervals with rational end points is countable.

291. Prove that for sets A, B and C the following holds: if $A \sim B$ and $B \sim C$ then $A \sim C$.

292. Let A and B be sets. Prove that if $A \setminus B \sim B \setminus A$ then $A \sim B$. Give an example of sets A and B for which the opposite implication does not hold.

293. Give an example of sets A, B, C and D for which $A \sim C$ and $B \sim D$ but $A \cup B \nsim C \cup D$ and $A \cap B \nsim C \cap D$.

294. Let X and Y be sets.

a) Prove that if $X \subset Y$ then $|X| \leqslant |Y|$.

b) Prove that if $X \subsetneqq Y$ and Y is finite then $|X| < |Y|$.

295. Let X, Y and Z be sets. Prove that if $|X| \leqslant |Y|$ and $|Y| \leqslant |Z|$ then $|X| \leqslant |Z|$.

296. Let A and B be finite sets. Prove that if $|B| < |A|$ then not a single mapping $A \to B$ is injective.

297. Prove that the interval of real numbers $(0, 1)$ is not countable.

298. Construct concrete bijective mappings $f : A \to B$, $g : B \to A$ between sets A and B (here $a, b, c, d \in \mathbb{R}$, $a < b$, $c < d$):

a) $A = (a,b),\ B = (c,d)$

b) $A = (a,b),\ B = (-\infty,d)$

c) $A = (a,b),\ B = (c,\infty)$

d) $A = (a,b),\ B = \mathbb{R}$

e) $A = (-\infty,b),\ B = (c,\infty)$

f) $A = (-\infty,b),\ B = \mathbb{R}$

g) $A = (a,\infty),\ B = \mathbb{R}$

h) $A = [a,b),\ B = (c,d)$

i) $A = [a,b],\ B = [c,d)$

j) $A = [a,b],\ B = (c,d)$

k) $A = [a,\infty),\ B = (c,\infty)$.

299. Let X be an infinite set and let Y be at most a countable set.

a) Prove that $X \cup Y \sim X$.

b) Prove that if $X \setminus Y$ is infinite then $X \setminus Y \sim X$.

300. What is the cardinality of the set of all irrational numbers?

301. Find the cardinality of the set consisting of all sequences of natural numbers.

302. Find the cardinality of the set of all increasing sequences of natural numbers.

303. Let A_n, $n = 1, 2, \ldots$, be sets. Prove that if every set A_n, $n = 1, 2, \ldots$, has the cardinality of continuum then the set $\bigcup\limits_{n=1}^{\infty} A_n$ also has the cardinality of continuum.

304. Bring forth an example of sets A and B that both have the cardinality of continuum and a) $A \cap B$ is (i) finite, (ii) countable, (iii) has the cardinality of continuum. Problem b) contains the same sub-problems (i)-(iii) like a) but instead of set $A \cap B$ use the set $A \setminus B$.

305*. Let X be a countable subset of the set of real numbers. Does there exist an $a \in \mathbb{R}$ such that

$$\{x + a : x \in X\} \cap X = \varnothing?$$

306*. Prove that there exists a set X such that $X \subset \mathcal{P}(\mathbb{N})$ and $|X| = c$ but for every $A, B \in X$ the set $A \cap B$ is finite.

10. Relations

Definition. Let A and B be sets. A **relation** or **connection** between sets A and B is any subset of $A \times B$.

If $R \subset X \times X$ then we talk of a relation on set X. For the pair $(x, y) \in R$ we say that elements x and y are in relation R and denote xRy.

Definition. Let X and Y be sets. Relation $R \subset X \times Y$ is a function if the following conditions are satisfied:

a) for all $x \in X$ there exist a $y \in Y$ such that $(x, y) \in R$

b) if $x \in X$ and $y, z \in Y$ are such that $(x, y) \in R$ and $(x, z) \in R$, then $y = z$.

307. Let $A = \{1,2,3,4\}$. Which ordered pairs are in the relation $R = \{(a, b) : a|b\}$ defined on set A?

308. Let $A = \{1,2,3,4,5,6\}$. Draw a directed graph of the following relation

$$R = \{(x, y) \in A \times A \mid x + y \text{ is an even number}\}.$$

309. Present the following relations on set $X = \{1,2,3,4\}$ in matrix form and as a graph.

a) $R = \{(1,3), (2,1), (2,4), (3,2), (4,1)\}$;

b) $R = \varnothing$;

c) $R = \{(x, y) \mid x > y\}$;

d) $R = \{(x, y) \mid \neg(x = y)\}$;

310. Let $X = \{1,2,3,4\}$ and let relations R, S, T, P be defined on set X. Write the relations R, S, T, P as sets of ordered pairs based on the matrix forms of these relations given below.

a) $R = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$;

b) $S = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$;

c) $T = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$;

d) $P = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$.

311. Let $X = \{1,2,3,4\}$ ja $Y = \{a, b, c\}$. Present the relation R as a subset of set $X \times Y$ if its matrix form is

$$\text{(a)} \quad \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}, \qquad \text{(b)} \quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}.$$

312. Present the following relations visually as a collection of points on a plane. Assume that $(x,y) \in \mathbb{R}\times\mathbb{R}$.

a) $R = \{(x,y) \mid x^2+y^2=1\}$;
b) $R = \{(x,y) \mid x^2+y^2 \leqslant 1\}$;
c) $R = \{(x,y) \mid x^2+y^2 > 1\}$;
d) $R = \{(x,y) \mid x,y \in \mathbb{N}\}$;
e) $R = \{(x,y) \mid 4x^2+y^2=16\}$;
f) $R = \{(x,y) \mid |x| = |y|\}$;
g) $R = \{(x,y) \mid |x|+|y|=1\}$.

313. Check if the given relation holds for the given elements. Elaborate on every answer.

a) On the set $\mathbb{Z}$ we have the relation $R = \{(m,n) \mid m$ and n divide by same prime number$\}$. Does $22\ R\ 45$ hold?
b) On the set $\mathcal{P}(\{1,2,3,4,5\})$ we have the relation $R = \{(A,B) \mid A$ and B have the same number of elements$\}$. Does $\{2,3,4\}\ R\ \{3,4,5\}$ hold?
c) On the set of all finite words containing only letters a and b we have the relation $R = \{(s,t) \mid s$ and t start or end with the same letter$\}$. Does $abbab\ R\ bbaba$ hold?
d) On the set of all formulae of propositional calculus we have the relation $R = \{(F,G) \mid F$ and G are both true on a valuation of variables$\}$. Does $\neg C \wedge (B \Rightarrow A)\ R\ B \vee C \Rightarrow \neg A \wedge B$ hold?

314. Let $R = \{(a,b) \mid |a-b| \leqslant 2\}$ be a relation on the set of integers $\mathbb{Z}$.

a) Give examples of ordered pairs that belong to R.
b) Find all integers x such that $xR5$ and find all integers x such that $5Rx$.
c) If possible, find all the numbers x and y for which $xR8$ and $8Ry$ but x and y are not in relation R.
d) If $a \in \mathbb{Z}$, find all the integers x for which xRa.

315. Find the ordered pairs which belong to relation

a) $R = \{(x,y) \in \mathbb{Z}\times\mathbb{Z} \mid x+y = xy\}$;
b) $R = \{(x,y) \in \mathbb{Z}\times\mathbb{Z} \mid \frac{y}{x} + 2x = 0\}$.

316. Present graphically the relation $\{(x,y) \in \mathbb{R}\times\mathbb{R} \mid \lfloor x \rfloor = \lfloor y \rfloor\}$, where $\lfloor x \rfloor \in \mathbb{Z}$ is such that $\lfloor x \rfloor \leqslant x < \lfloor x \rfloor + 1$.

317. Decide whether the following relations $R \subset \{1,2,3,4,5\}\times\mathbb{N}$ are functions:

a) $R = \{(1,1),(3,2),(5,3),(3,4),(1,5)\}$,
b) $R = \{(1,1),(2,1),(3,1),(4,1),(5,1)\}$,
c) $R = \{(3,4),(5,9),(1,9),(2,3)\}$,
d) $R = \{(1,2),(2,3),(3,4),(4,5),(4,1)\}$.

318. Decide if the following relations R in matrix form are functions:

a) $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, b) $\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$, c) $\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$,

d) $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$, e) $\begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{pmatrix}$, f) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}$, g) $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$, h) $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$.

319. Decide if the following relations $R \subset \mathbb{R} \times \mathbb{R}$ are functions:

a) $R = \mathbb{R} \times \mathbb{R}$,
b) $R = \mathbb{R} \times [0, \infty)$,
c) $R = \mathbb{R} \times \{1\}$,
d) $R = \{1\} \times \mathbb{R}$,
e) $R = \{(t^3, t) : t \in \mathbb{R}\}$,
f) $R = \{(t, 2^t) : t \in [0, \infty)\} \cup \{(t, -t) : t \leqslant 0\}$,
g) $R = \{(t, 2^t) : t \in [0, \infty)\} \cup \{(t, -t) : t < 0\}$,
h) $R = \left\{(\tan t, t), t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\right\}$.

Definition. Relation $R \subset X \times X$ is called

- reflexive if $xRx \ \forall x \in X$;
- irreflexive if $x \not R x \ \forall x \in X$;
- symmetric if $xRy \Rightarrow yRx \ \forall x, y \in X$;
- antisymmetric if $xRy \wedge yRx \Rightarrow x = y \ \forall x, y \in X$;
- transitive if $xRy \wedge yRz \Rightarrow xRz \ \forall x, y, z \in X$.

Definition. The **inverse relation** of relation $R \subset A \times B$ is the relation $R^{-1} \subset B \times A$, which is determined by the equivalence $bR^{-1}a \Leftrightarrow aRb$ or $R^{-1} = \{(b, a) : (a, b) \in R\}$.

Definition. The relation $\triangle X = \{(x, x) \mid x \in X\}$ is called the diagonal of set X.

320. Find all the relations on set $\{a, b\}$ which are:

a) reflexive; b) symmetric; c) transitive; d) antisymmetric.

321. Let $A = \{1, 2, 3, 4\}$. Which properties (reflexivity, irreflexivity, symmetry, antisymmetry, transitivity) does the relation $R = \{(a, b) : a | b\}$ defined on the set A have?

322. Let $A = \{1, 2, 3\}$ and observe three relations on A.

a) $R = \{(1,1), (1,2), (2,1), (2,2), (3,3)\}$;
b) $R = \{(1,1), (1,3), (2,2), (3,2)\}$;
c) $R = \{(1,2), (1,3), (2,3)\}$.

Which properties do each of them satisfy?

323. Let $A = \{1, 2, 3, 4\}$. Give an example of a relation R on A which

a) is reflexive but it is not antisymmetric or transitive;

b) is not reflexive, symmetric or transitive;
c) is symmetric and transitive;
d) is not symmetric or antisymmetric;
e) is both symmetric and antisymmetric.

324. Let $X = \{1,2,3,4\}$. Explain if relation R on set X is reflexive, symmetric or antisymmetric and then present the relation R as a subset of $X \times X$ (as a list of pairs) if its matrix form is

$$\text{(a)} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}, \quad \text{(b)} \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \text{(c)} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}.$$

325. Explain whether the relation is transitive if its matrix form is

$$\text{(a)} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix}, \quad \text{(b)} \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \text{(c)} \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix}.$$

326. Is relation $xRy \Leftrightarrow x + y = 0$ on the set of real numbers reflexive? Irreflexive?

327. Check what properties (reflexivity, symmetry, antisymmetry, transitivity) the following relations R on set $\mathbb{Z}$ have:

a) $xRy \Leftrightarrow x = y$;
b) $xRy \Leftrightarrow x < y$;
c) $xRy \Leftrightarrow x \leq y$;
d) $xRy \Leftrightarrow |x - y| > 1$;
e) $xRy \Leftrightarrow x - y \in \mathbb{N}$;
f) $xRy \Leftrightarrow x^2 - y^2 = 4$.

328. Check what properties (reflexivity, symmetry, antisymmetry, transitivity) the following relations R on set $\mathbb{R}$ have:

a) $xRy \Leftrightarrow x = y^2$;
b) $xRy \Leftrightarrow |x| = |y|$;
c) $xRy \Leftrightarrow |x - y| \geqslant 2$;
d) $xRy \Leftrightarrow x = -y$;
e) $xRy \Leftrightarrow x - y \in \mathbb{Q}$;
f) $xRy \Leftrightarrow y \leqslant x^2$.

329. Let $X \neq \varnothing$. Check what properties (reflexivity, symmetry, antisymmetry, transitivity) the following relations on $\mathcal{P}(X)$ have:
a) $(A,B) \in R \Leftrightarrow$ sets A and B are finite and they have the same amount of elements;
b) $(A,B) \in R \Leftrightarrow A \cap B = \varnothing$;
c) $(A,B) \in R \Leftrightarrow A \cap B \neq \varnothing$;
d) $(A,B) \in R \Leftrightarrow$ sets A and B have exactly two elements in common.

330. Let $X \neq \varnothing$. Observe the relation $\subset$ on the set of all subsets of X, $\mathcal{P}(X) = \{Y \mid Y \subset X\}$. What properties does the relation $\subset$ on set $\mathcal{P}(X)$ satisfy?

331. Observe the relation $R = \{(a,d),(b,b),(d,a),(d,c)\}$ on set $A = \{a,b,c,d\}$. Find the smallest relation on the set A which contains R and is

a) reflexive and symmetric;
b) symmetric and transitive;
c) reflexive, symmetric and transitive.

332. Check what properties (reflexivity, symmetry, antisymmetry, transitivity) apply to the following relations R on set $\mathbb{R}\times\mathbb{R}$:

a) $(m,n)R(k,l) \Leftrightarrow (m \leqslant k \wedge n \leqslant l)$;
b) $(m,n)R(k,l) \Leftrightarrow (m \leqslant k \vee n \leqslant l)$;
c) $(m,n)R(k,l) \Leftrightarrow m \leqslant k$;
d) $(m,n)R(k,l) \Leftrightarrow m+n \geqslant k+l$;
e) $(m,n)R(k,l) \Leftrightarrow m+l = n+k$.

333. Prove that if a relation is symmetric and antisymmetric then it is also transitive.

334. Prove that a relation is reflexive, symmetric and antisymmetric if and only if it is a diagonal of some set X.

335. Find R^{-1} and check which properties the relation R and its inverse R^{-1} have if

a) $R = \{(a,b),(b,a),(b,b)\}$;
b) $R = \{(5,4),(1,2),(3,1),(2,1),(4,3)\}$;
c) $R = \{(x,y)\in\mathbb{R}\times\mathbb{R} \mid x=-y\}$;
d) $R = \{(x,y)\in\mathbb{Z}\times\mathbb{Z} \mid y-x=2\}$.

336. Let $R \subset X \times X$ be a relation. Prove that
a) R is reflexive if and only if $\triangle X \subset R$;
b) R is symmetric if and only if $R = R^{-1}$.

337. Find the inverses of the following relations.

a) $(\{0,1,2\},\leqslant)$
b) $(\mathbb{N},|)$
c) $(\mathcal{P}(\mathbb{Z}),\subset)$

Definition. A relation is said to be an equivalence relation if it is reflexive, symmetric and transitive.

338. Which of the following relations are equivalence relations?
a) $R = \{(a,b)\in\mathbb{Z}\times\mathbb{Z} \mid a+b \text{ is an even number }\}$
b) $R = \{(x,y)\in\mathbb{R}\times\mathbb{R} \mid |x|=|y|\}$
c) $R = \{(x,y)\in\mathbb{R}\times\mathbb{R} \mid |x-y|\geqslant 2\}$
d) Set is $\mathbb{R}^2$ and $(m,n)R(k,l) \Leftrightarrow (m\leqslant k)\wedge(n\leqslant l)$.
e) Set is $\mathbb{R}^2$ and $(m,n)R(k,l) \Leftrightarrow (m\leqslant k)\vee(n\leqslant l)$.
f) Set is $\mathbb{R}^2$ and $(m,n)R(k,l) \Leftrightarrow |m-l|=|k-n|$.
g) Set is $\mathbb{R}^2$ and $(m,n)R(k,l) \Leftrightarrow (m\leqslant k)\wedge(n\leqslant l)$.
h) X is the set of all people living in Tartu and xRy means that people x and y know each other.

339. Which relations R on set $\mathbb{R}$ are equivalence relations?

a) $xRy \Leftrightarrow |x| = |y|$;
b) $xRy \Leftrightarrow x - y \in \mathbb{Q}$;
c) $xRy \Leftrightarrow x - y \in \mathbb{Z}$;
d) $xRy \Leftrightarrow [x - y] = 0$.

340. Which relations R on set $\mathbb{R}^2$ are equivalence relations?

a) $(m,n)R(k,l) \Leftrightarrow m \leqslant k$;
b) $(m,n)R(k,l) \Leftrightarrow m + l = n + k$;
c) $(m,n)R(k,l) \Leftrightarrow |m - l| = |k - n|$.

341. Prove that R is an equivalence relation if and only if R^{-1} is an equivalence relation.

342. Find an example of a relation which is reflexive and symmetric but not transitive.

343. Show that the logical equivalence in propositional calculus is an equivalence relation, that is to say reflexive, symmetric and transitive.

Definition. A partition of a nonempty set X is a system of sets $\{X_\alpha, \alpha \in A\}$ which satisfies the following conditions:

- $X_\alpha \neq \varnothing$ for all $\alpha \in A$;
- $\bigcup\limits_{\alpha \in A} X_\alpha = X$;
- $X_\alpha \neq X_\beta \Rightarrow X_\alpha \cap X_\beta = \varnothing$.

If R is an equivalence relation on X then $X_x = \{y \in X \mid (y,x) \in R\}$, $x \in X$ are called the equivalence classes.

Definition. The quotient set of set X by equivalence relation R is the set of all equivalence classes $X/R = \{X_x \mid x \in X\}$.

344. Write down every possible partition of set $A = \{a, b, c\}$.

345. From the sets $A_1, \ldots, A_8$ construct systems of sets which would form a partition of $\mathbb{Z}$ if $A_1 = \{0\}$, $A_2 = \{1\}$, $A_3 = \{1,2,3,\ldots\}$, $A_4 = \{-1,-2,-3,\ldots\}$, $A_5 = \{2,4,6,\ldots\}$, $A_6 = \{1,3,5,7,\ldots\}$, A_7 is the set of all prime numbers and A_8 is the set of all composite natural numbers.

346. Is a system of intervals $[n, n+1)$, $n \in \mathbb{N}$, a partition of $\mathbb{R}$?

347. Find the corresponding equivalence relation to partition $\{[k, k+1), k \in \mathbb{Z}\}$ of set $\mathbb{R}$.

348. Find the equivalence classes of equivalence relation R on $X = \{1,2,3,4,5\}$ if $R = \{(1,1),(2,2),(3,3),(4,4),(5,5),(1,3),(3,1),(3,5),(5,3),(1,5),(5,1)\}$.

349. Find the equivalence relation R on set $X = \{1,2,3\}$ if the corresponding equivalence classes are $\{1,3\}$ and $\{2\}$.

350. Find the equivalence relation R on set $X = \{1,2,3,4,5,6\}$ if the corresponding equivalence classes are $\{1\}$, $\{2,3\}$ and $\{4,5,6\}$.

351. Let $A = \{a,b,c,d,e\}$ and let R be an equivalence relation on A. Assume that relation R has two equivalence classes. It is known that aRd, bRc and eRd. Find the relation R.

352. Let R be a relation on $\mathbb{N}$ where mRn if and only if m and n give exactly the same remainder if divided by 3. Find the quotient set $\mathbb{N}/R$.

353. Let $X = \{a,b,c,d\}$. Check that relation $R = \{(a,a),(b,b),(c,c),(d,d),(a,c),(c,a)\}$ on set X is an equivalence relation and find X/R.

354. Find $\mathbb{R}/R$, if relation R on set $\mathbb{R}$ is defined as follows:

a) $(x,y) \in R \Leftrightarrow |x| = |y|$;

b) $(x,y) \in R \Leftrightarrow x - y \in \mathbb{Z}$.

Definition. Relation R on set A is called an **ordering relation**, if it is reflexive, antisymmetric and transitive.

Definition. A set which has been given an ordering relation is called a **partially ordered set (poset)**.

Definition. A partially ordered set is said to be a **linearly ordered set**, if for every pair of elements a and b either $a \leqslant b$ or $b \leqslant a$, that is to say that two arbitrary elements are comparable.

355. Let us consider the relation R on the set $\mathbb{N}$ for which $aRb \Leftrightarrow a \mid b$ (that is to say aRb if and only a divides number b). What properties does the relation R on the set $\mathbb{N}$ satisfy? Is it an ordering relation on $\mathbb{N}$?

356. Which of the following relations on the set $\{0,1,2,3\}$ are ordering relations?

a) $\{(0,0),(1,1),(2,2),(3,3)\}$;
b) $\{(0,0),(1,1),(2,0),(2,2),(2,3),(3,2),(3,3)\}$;
c) $\{(0,0),(1,1),(1,2),(2,2),(3,3)\}$;
d) $\{(0,0),(1,1),(1,2),(1,3),(2,2),(2,3),(3,3)\}$;
e) $\{(0,0),(2,2),(3,3)\}$;
f) $\{(0,0),(1,1),(2,0),(2,2),(2,3),(3,3)\}$;
g) $\{(0,0),(1,1),(1,2),(1,3),(2,0),(2,2),(2,3),(3,0),(3,3)\}$.

357. Is the set (H,R) partially ordered if H is the set of all people living on Earth and people a and b are in relation R if

a) a is taller than b;
b) a is not taller than b;
c) a is not shorter than b;
d) a and b have a common friend;
e) a and b have no friends in common;
f) a weighs more than b;
g) $a = b$ or a is the parent of b;
h) $a = b$ or a is the child of b.

358. Which of the following sets are partially ordered?

a) $(\mathbb{Z},=)$; b) $(\mathbb{Z},\neq)$; c) $(\mathbb{Z},\geqslant)$; d) $(\mathbb{Z},\nmid)$; e) $(\mathbb{R},<)$; f) $(\mathbb{R},\leqslant)$.

359. Find out which of the following relations presented in matrix form are ordering relations:

a) $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, c) $\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, e) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$,

b) $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, d) $\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, f) $\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{pmatrix}$.

360. Give examples of elements which are comparable and which are incomparable in the partially ordered set $(\mathbb{N},|)$.

361. Find two incomparable elements in the following partially ordered sets:

a) $(\mathcal{P}(\{0,1,2\}),\subset)$; b) $(\{1,2,4,6,8\},|)$.

362. Let $\prec^X$ and $\prec^Y$ be ordering relations respectively on sets X and Y. Define relation $\prec$ on set $X\times Y$ by $(x_1,y_1)\prec(x_2,y_2)\Leftrightarrow x_1 \prec^X x_2 \wedge y_1 \prec^Y y_2$. Prove that $\prec$ is an ordering relation on $X\times Y$.

363. Prove that if R is an ordering relation then $xRy\wedge yRx \Leftrightarrow x=y$.

364. Prove that relation $(m,n)\in R \Leftrightarrow \dfrac{m}{n}\in\mathbb{N}$ is an ordering relation on $\mathbb{N}$.

365. Prove that R is an ordering relation if and only if R^{-1} is an ordering relation.

Definition. Element a_0 of a partially ordered set A is said to be **smallest**, if $a_0 \leqslant a$ for all $a\in A$. Analogically, element $a_0\in A$ is said to be **largest**, if $a\leqslant a_0$ for all $a\in A$.

Definition. Element a_0 of a partially ordered set A is called **minimal**, if from $a\leqslant a_0$ and $a\in A$ follows $a=a_0$. Analogically, element a_0 is called **maximal**, if from $a_0\leqslant a$ and $a\in A$ follows $a=a_0$.

366. Let $H=\{1,2,3,4\}$ and consider the lexicographic ordering on this set. Find

a) all pairs in set $H\times H$ which are smaller than $(2,3)$;
b) all pairs in set $H\times H$ which are larger than $(3,1)$.

367. For the partially ordered set $(\{3,5,9,15,24,45\},|)$ find

a) all minimal elements;
b) all maximal elements;
c) smallest element if it exists;
d) largest element if it exists.

368. For the partially ordered set $(\{2,4,6,9,12,18,27,36,48,60,72\},|)$ find
a) all minimal elements;
b) all maximal elements;
c) smallest element if it exists;
d) largest element if it exists.

369. For the partially ordered set $(\{\{1\},\{2\},\{4\},\{1,2\},\{1,4\},\{2,4\},\{3,4\},$ $\{1,3,4\},\{2,3,4\}\},\subset)$ find
a) all minimal elements;
b) all maximal elements;
c) smallest element if it exists;
d) largest element if it exists.

370. For the partially ordered set $(\{\{a,b\},\{a,b,c\},\{c,b,d,a\},\{b,d,e\}\},\subset)$ find
a) all minimal elements;
b) all maximal elements;
c) smallest element if it exists;
d) largest element if it exists.

371. Find an example of a partially ordered set which
a) has a minimal element but lacks a maximal element;
b) has a maximal element but lacks a minimal element;
c) has neither a maximal or a minimal element.

372*. Let $p(n)$ be the number of different equivalence relations (so also partitions) on a set with n elements. Prove that

$$p(n)=\sum_{j=0}^{n-1}\binom{n-1}{j}p(n-j-1).$$