§ 1. The notion of a metric space

1.1. The notion of a metric space

Definition 1.1. Let X be a set.

A function $\rho\colon X \times X \to \mathbb{R}$ is called a *distance* (or a *metric*) if, for every $x, y, z \in X$,

1° $\rho(x, y) = 0 \Leftrightarrow x = y$;

2° $\rho(x, y) = \rho(y, x)$;

3° $\rho(x, z) \leqslant \rho(x, y) + \rho(y, z)$.

In this case, one says that (X, ρ) is a *metric space.* If the metric ρ is clear from the context, one just calls X a metric space. The number $\rho(x, y)$ is called the distance between x and y. The conditions 1°, 2°, and 3°—the *axioms of metric*—are referred to as, respectively, the *axiom of identity,* the *axiom of symmetry,* and the *triangle inequality.*

Simpler properties of metric are collected in the following

Proposition 1.1. *Let (X, ρ) be a metric space, and let $x, y, z, u, v \in X$. Then*

(a) $\rho(x, y) \geqslant 0$*;*

(b) (the quadrangle inequality) $\big|\rho(x, y) - \rho(u, v)\big| \leqslant \rho(x, u) + \rho(y, v)$*;*

(c) (the reverse triangle inequality) $\big|\rho(x, y) - \rho(y, z)\big| \leqslant \rho(x, z)$*;*

(d) *every subset of X is again a metric space with respect to the distance ρ.*

Proof. (a). One has (by taking $z = x$ in the triangle inequality 3°)

$$2\rho(x, y) = \rho(x, y) + \rho(y, x) \geqslant \rho(x, x) = 0,$$

hence $\rho(x, y) \geqslant 0$.

(b). We must show that

$$-\rho(x, u) - \rho(y, v) \leqslant \rho(x, y) - \rho(u, v) \leqslant \rho(x, u) + \rho(y, v),$$

i.e.,

$$\rho(u, v) \leqslant \rho(x, y) + \rho(x, u) + \rho(y, v) \quad \text{and} \quad \rho(x, y) \leqslant \rho(x, u) + \rho(y, v) + \rho(u, v).$$

Both these inequalities follow from the triangle inequality:

$$\rho(u, v) \leqslant \rho(u, x) + \rho(x, v) \leqslant \rho(x, u) + \rho(x, y) + \rho(y, v) = \rho(x, y) + \rho(x, u) + \rho(y, v),$$
$$\rho(x, y) \leqslant \rho(x, u) + \rho(u, y) \leqslant \rho(x, u) + \rho(u, v) + \rho(v, y) = \rho(x, u) + \rho(y, v) + \rho(u, v).$$

(c). The claim follows by taking, in the quadrangle inequality (b), $u = z$ and $v = y$.

(d) is more than obvious. □

1.2. Simpler examples of metric spaces

Example 1.1. The set $\mathbb{R}$ of real numbers is a metric space with respect to the natural distance

$$d(x,y) = |x-y|, \quad x,y \in \mathbb{R}.$$

Example 1.2. Let $n \in \mathbb{N}$. The *Euclidean metric* d in

$$\mathbb{R}^n := \left\{(\xi_j)_{j=1}^n := (\xi_1,\dots,\xi_n) :\ \xi_1,\dots,\xi_n \in \mathbb{R}\right\}$$

is defined by

$$d(x,y) = \sqrt{\sum_{j=1}^n |\xi_j - \eta_j|^2}, \qquad x = (\xi_j)_{j=1}^n,\ y = (\eta_j)_{j=1}^n \in \mathbb{R}^n.$$

(The axioms of metric for d will be veryfied in Section 5.) In particular, in $\mathbb{R}^2$,

$$d(x,y) = \sqrt{|\xi_1-\eta_1|^2 + |\xi_2-\eta_2|^2}, \qquad x = (\xi_1,\xi_2),\ y = (\eta_1,\eta_2) \in \mathbb{R}^2,$$

and, in $\mathbb{R}^3$,

$$d(x,y) = \sqrt{|\xi_1-\eta_1|^2 + |\xi_2-\eta_2|^2 + |\xi_3-\eta_3|^2}, \qquad x = (\xi_1,\xi_2,\xi_3),\ y = (\eta_1,\eta_2,\eta_3) \in \mathbb{R}^3,$$

i.e., $d(x,y)$ is the "natural" distance between x and y.

Example 1.3. Let X be an arbitrary set. Define, for $x,y \in X$,

$$\rho(x,y) = \begin{cases} 0, & \text{if } x = y, \\ 1, & \text{if } x \neq y. \end{cases}$$

Exercise 1.1. Prove that ρ is a distance.

The metric ρ is called the *discrete metric.* The metric space (X,ρ) is called the *discrete metric space.*

1.3. Balls and bounded sets in a metric space

Definition 1.2. Let X be a metric space, and let $a \in X$ and $r > 0$. The sets

$$\begin{aligned} B(a,r) &:= \left\{x \in X :\ \rho(x,a) < r\right\}, \\ \overline{B}(a,r) &:= \left\{x \in X :\ \rho(x,a) \leqslant r\right\}, \\ S(a,r) &:= \left\{x \in X :\ \rho(x,a) = r\right\} \end{aligned}$$

are called, respectively, the the *open ball, closed ball,* and the *sphere* of radius r centered at a.

Example 1.4. The open and closed balls, and the sphere in $\mathbb{R}^2$ centered at $a = (a_1, a_2) \in \mathbb{R}^2$ of radius $r > 0$ with respect to the Euclidean distance d

$$\begin{aligned}
B(a,r) &= \left\{x = (\xi_1, \xi_2) \in \mathbb{R}^2 \colon\ d(x,a) < r\right\} \\
&= \left\{(\xi_1, \xi_2) \in \mathbb{R}^2 \colon\ \sqrt{|\xi_1 - a_1|^2 + |\xi_2 - a_2|^2} < r\right\} \\
&= \left\{(\xi_1, \xi_2) \in \mathbb{R}^2 \colon\ |\xi_1 - a_1|^2 + |\xi_2 - a_2|^2 < r^2\right\}, \\
\overline{B}(a,r) &= \left\{x = (\xi_1, \xi_2) \in \mathbb{R}^2 \colon\ d(x,a) \leqslant r\right\} \\
&= \left\{(\xi_1, \xi_2) \in \mathbb{R}^2 \colon\ \sqrt{|\xi_1 - a_1|^2 + |\xi_2 - a_2|^2} \leqslant r\right\} \\
&= \left\{(\xi_1, \xi_2) \in \mathbb{R}^2 \colon\ |\xi_1 - a_1|^2 + |\xi_2 - a_2|^2 \leqslant r^2\right\}, \\
S(a,r) &= \left\{x = (\xi_1, \xi_2) \in \mathbb{R}^2 \colon\ d(x,a) = r\right\} \\
&= \left\{(\xi_1, \xi_2) \in \mathbb{R}^2 \colon\ \sqrt{|\xi_1 - a_1|^2 + |\xi_2 - a_2|^2} = r\right\} \\
&= \left\{(\xi_1, \xi_2) \in \mathbb{R}^2 \colon\ |\xi_1 - a_1|^2 + |\xi_2 - a_2|^2 = r^2\right\}
\end{aligned}$$

are, respectively, the open disk, the closed disk, and the circle centered at a of radius r.

Exercise 1.2. Describe the balls $B(a,r)$ and $\overline{B}(a,r)$ and the sphere $S(a,r)$ in the discrete metric space.

Remark 1.1. The behaviour of balls in a metric space may be very “unballish”. For instance,

- there exist metric spaces X such that for some $a \in X$ and $r_2 > r_1 > 0$, one has $B(a, r_2) \subset B(a, r_1)$ (think about the discrete metric space);

and, moreover,

- there exist metric spaces X such that for some $a_1, a_2 \in X$ and $r_2 > r_1 > 0$, one has $B(a_2, r_2) \subsetneq B(a_1, r_1)$ (look at the metric space $(\mathbb{N}, \rho)$ in [HOP, exercise 7]).

Exercise 1.3. Let X be a metric space and let $a_1, a_2 \in X$ and $r_1, r_2 > 0$ satisfy $r_1 \leqslant r_2 - \rho(a_1, a_2)$. Prove that $B(a_1, r_1) \subset B(a_2, r_2)$ and $\overline{B}(a_1, r_1) \subset \overline{B}(a_2, r_2)$.

Definition 1.3. Let X be a metric space and let $x \in X$. Any set containing a ball centered at x is called a *neighbourhood* of x.

Definition 1.4. Let X be a metric space and let $A \subset X$. The subset A is said to be *bounded,* if it is contained in some ball.

Exercise 1.4. Prove that, in a metric space (X, ρ),

(a) every closed ball is contained in some open ball, and every open ball is contained in some closed ball;

(b) for every ball $\overline{B}(a,r)$ and every $b \in X$, there exists $R > 0$ such that $\overline{B}(a,r) \subset \overline{B}(b,R)$.

1.4. Convergence in a metric space

Definition 1.5. Let (X, ρ) be a metric space, and let $x_n, x \in X$, $n \in \mathbb{N}$.

One says that the sequence $(x_n)_{n=1}^{\infty}$ *converges* to the element x in the space X, and writes

$$x_n \xrightarrow[n\to\infty]{} x \qquad \text{or simply} \qquad x_n \to x \qquad \text{or} \qquad \lim_{n\to\infty} x_n = x$$

if

$$\rho(x_n, x) \xrightarrow[n\to\infty]{} 0.$$

The element x is called the *limit* of the sequence $(x_n)_{n=1}^{\infty}$.

In other words, $x_n \xrightarrow[n\to\infty]{} x$ if and only if

- for every $\varepsilon > 0$, there is an index $N \in \mathbb{N}$ such that

$$\rho(x_n, x) < \varepsilon \quad \text{for every } n \geqslant N$$

or, equivalently,

- for every $\varepsilon > 0$, there is an index $N \in \mathbb{N}$ such that

$$x_n \in B(x, \varepsilon) \quad \text{for every } n \geqslant N$$

or, equivalently,

- for every neighbourhood U of x, there is an index $N \in \mathbb{N}$ such that

$$x_n \in U \quad \text{for every } n \geqslant N.$$

A sequence in a metric space which is not convergent to any element of this space is said to *diverge*.

Example 1.5. Let X be a metric space and let $(x_n)_{n=1}^{\infty}$ be a sequence in X such that, for some index $N \in \mathbb{N}$,

$$x_n = x_N \quad \text{for all } n \geqslant N.$$

Then $x_n \xrightarrow[n\to\infty]{} x_N$. Indeed, whenever $\varepsilon > 0$, one has, for $n \geqslant N$,

$$\rho(x_n, x_N) = \rho(x_N, x_N) = 0 < \varepsilon.$$

Exercise 1.5. Prove that, in a discrete metric space, the only convergent sequences are those having the property described in the previous example.

Example 1.6. Let X be a metric space and let $x, y \in X$, $x \neq y$. Then the sequence

$$x,\ y,\ x,\ y,\ x. y,\ \ldots,$$

i.e., the sequence $(x_n)_{n=1}^{\infty}$ in X given by

$$\begin{cases} x_n = x, & \text{if } n \text{ is odd;} \\ x_n = y, & \text{if } n \text{ is even,} \end{cases}$$

is divergent.

Exercise 1.6. Prove that the sequence $(x_n)_{n=1}^{\infty}$ is divergent.

Simpler properties of convergent sequences are collected in the following

Proposition 1.2. (a) *A convergent sequence in a metric space may have at most one limit.*

(b) *A convergent sequence in a metric space is bounded (i.e., the set of its elements is bounded).*

(c) *Every subsequence of a convergent sequence in a metric space converges to the same limit.*

(d) *Distance ρ in a metric space X is continuous in the following sense: If x_n, x, $y_n, y \in X$, $n \in \mathbb{N}$, and*

$$x_n \xrightarrow[n\to\infty]{} x \qquad \text{and} \qquad y_n \xrightarrow[n\to\infty]{} y, \tag{1.1}$$

then

$$\rho(x_n, y_n) \xrightarrow[n\to\infty]{} \rho(x, y).$$

PROOF. Let X be a metric space, and let $x_n, x \in X$, $n \in \mathbb{N}$, be such that $x_n \xrightarrow[n\to\infty]{} x$.

(a). Suppose that $x_n \xrightarrow[n\to\infty]{} y$ for some $y \in X$. We have to show that $x = y$ or, equivalently, $\rho(x, y) = 0$.

For every $n \in \mathbb{N}$,

$$0 \leqslant \rho(x, y) \leqslant \rho(x, x_n) + \rho(x_n, y) = \rho(x_n, x) + \rho(x_n, y). \tag{1.2}$$

Since, $x_n \xrightarrow[n\to\infty]{} x$ and $x_n \xrightarrow[n\to\infty]{} y$, one has $\rho(x_n, x) \xrightarrow[n\to\infty]{} 0$ and $\rho(x_n, y) \xrightarrow[n\to\infty]{} 0$, thus (1.2) implies that $0 \leqslant \rho(x, y) \leqslant 0$, i.e., $\rho(x, y) = 0$, as desired.

(b). It suffices to show that there is an $r > 0$ such that $x_n \in \overline{B}(x, r)$ for every $n \in \mathbb{N}$. Since $\rho(x_n, x) \xrightarrow[n\to\infty]{} 0$, there is an $N \in \mathbb{N}$ such that

$$\rho(x_n, x) < 1 \qquad \text{for all } n > N.$$

Putting $r := \max\{\rho(x_1, x), \dots, \rho(x_N, x), 1\}$, one has $x_n \in \overline{B}(x, r)$ for every $n \in \mathbb{N}$.

(c). Let $(x_{k_n})_{n=1}^{\infty}$ be any subsequence of $(x_n)_{n=1}^{\infty}$. Since $x_n \xrightarrow[n\to\infty]{} x$, one has $\rho(x_n, x) \xrightarrow[n\to\infty]{} 0$. Since any subsequence of a convergent sequence of numbers converges to the same limit, the latter implies that also $\rho(x_{k_n}, x) \xrightarrow[n\to\infty]{} 0$, but this means that $x_{k_n} \xrightarrow[n\to\infty]{} x$ in X.

(d). Suppose that $x_n, x, y_n, y \in X$, $n \in \mathbb{N}$, satisfy (1.1). Then, by the quadrangle inequality,

$$\big|\rho(x_n, y_n) - \rho(x, y)\big| \leqslant \rho(x_n, x) + \rho(y_n, y) \xrightarrow[n\to\infty]{} 0.$$

□

§ 2. Normed spaces

Throughout in what follows, either $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$.

2.1. The notion of a linear space

Definition 2.1. A non-empty set X is called a *linear space* (or a *vector space*) (over the field $\mathbb{K}$) if two operations—*adddition*

$$X \times X \ni (x, y) \mapsto x + y \in X$$

and *multiplication by a scalar*

$$\mathbb{K} \times X \ni (\alpha, x) \mapsto \alpha x \in X$$

—have been defined in X satisfying the following axioms:

1° $x + y = y + x$ for all $x, y \in X$;

2° $(x + y) + z = x + (y + z)$ for all $x, y, z \in X$;

3° there is an element $0 \in X$ (called the *zero element* of X) satisfying $x + 0 = x$ for all $x \in X$;

4° for every $x \in X$, there exists an element $-x \in X$ (called the *additive inverse* of x) satisfying $x + (-x) = 0$;

5° $1x = x$ for all $x \in X$;

6° $\alpha(x + y) = \alpha x + \alpha y$ for all $\alpha \in \mathbb{K}$ and all $x, y \in X$;

7° $(\alpha + \beta)\, x = \alpha x + \beta\, x$ for all $\alpha, \beta \in \mathbb{K}$ and all $x \in X$;

8° $(\alpha\beta)\, x = \alpha(\beta\, x)$ for all $\alpha, \beta \in \mathbb{K}$ and all $x \in X$.

Remark 2.1. For $x, y \in X$ and $\alpha \in \mathbb{K}\backslash\{0\}$, it is customary to denote

$$x - y := x + (-y) \qquad \text{and} \qquad \frac{x}{\alpha} := \frac{1}{\alpha}\, x.$$

A non-empty subset Y of X is called a *linear subspace* of X if its closed with respect to the linear space operations, i.e., whenever $x, y \in Y$ and $\alpha \in \mathbb{K}$, also $x + y \in Y$ and $\alpha x \in Y$.

Example 2.1. A prototypical example of a linear space is, for $n \in \mathbb{N}$,

$$\mathbb{K}^n := \big\{ (\xi_j)_{j=1}^n := (\xi_1, \ldots, \xi_n)\colon\ \xi_1, \ldots, \xi_n \in \mathbb{K} \big\}$$

with respect to the operations

$$\begin{aligned} x + y &:= (\xi_j + \eta_j)_{j=1}^n, & x &= (\xi_j)_{j=1}^n, y = (\eta_j)_{j=1}^n \in \mathbb{K}^n, \\ \alpha x &:= (\alpha \xi_j)_{j=1}^n, & x &= (\xi_j)_{j=1}^n \in \mathbb{K}^n,\ \alpha \in \mathbb{K}. \end{aligned}$$

2.2. The notion of a normed space

Definition 2.2. Let X be a linear space over the scalar field $\mathbb{K}$.

A function $\|\cdot\|\colon X \to \mathbb{R}$ is called a *norm* if, for all $x, y \in X$ and $\alpha \in \mathbb{K}$,

1° $\|x\| = 0 \Leftrightarrow x = 0$;

2° $\|\alpha x\| = |\alpha|\,\|x\|$;

3° $\|x + y\| \leqslant \|x\| + \|y\|$.

In this case, one says that $(X, \|\cdot\|)$ is a *normed space.* If the norm $\|\cdot\|$ is clear from the context, one just calls X a normed space. The number $\|x\|$ is called the norm of x. The conditions 1°, 2°, and 3°—the *axioms of norm*—are referred to as, respectively, the *axiom of identity,* the *axiom of homogeneity,* and the *triangle inequality.*

Notice that a linear subspace of a normed space is again a normed space with respect to the same norm.

The following proposition says that every normed space can be viewed, in a natural way, as a metric space.

Proposition 2.1. *Let $(X, \|\cdot\|)$ be a normed space. Then X is a metric space with respect to the distance*

$$\rho(x, y) = \|x - y\|, \quad x, y \in X.$$

Proof.

Exercise 2.1. Prove that ρ satisfies the axioms of metric.

□

Thus a normed space shares all the properties of a metric space. In particular, *whenever $x, y \in X$, one has*

- $\|x\| \geqslant 0$;
- (the reverse triangle inequality for norm) $\big|\|x\| - \|y\|\big| \geqslant \|x - y\|$.

Exercise 2.2. Prove the above assertions.

Example 2.2. A prototypical example of a normed space is $\mathbb{K}^n$ ($n \in \mathbb{N}$) with respect to the *Euclidean norm*

$$\|x\| = \sqrt{\sum_{j=1}^{n} |\xi_j|^2}, \qquad x = (\xi_j)_{j=1}^{n} \in \mathbb{K}^n.$$

The axioms of norm for the Euclidean norm will be verified in Section 5. Notice that the Euclidean norm induces the Euclidean metric (in the sense of Proposition 2.1).

Exercise 2.3. Let $X \neq \{0\}$ be a normed space. Prove that

(a) there exists an $x \in X$ satisfying $\|x\| = 1$;

(b) for every $c \in [0, \infty)$ there exists an $x \in X$ satisfying $\|x\| = c$.

2.3. Balls and boundedness in a normed space

Remark 2.2. One may observe that, contrary to the possible "unballish" behaviour of balls in a general metric space, the behavior of balls in a normed space is always very ballish: the pathologies described in Remark 1.1 never occur in a normed space.

*__Exercise 2.4__ (cf. Exercise 1.3). Let X be a normed space, and let $a_1, a_2 \in X$ and $r_1, r_2 > 0$. Prove that

(a) if $B(a_1, r_1) \subset B(a_2, r_2)$, then $r_1 \leqslant r_2 - \|a_1 - a_2\|$;

(b) if $\overline{B}(a_1, r_1) \subset \overline{B}(a_2, r_2)$, then $r_1 \leqslant r_2 - \|a_1 - a_2\|$.

Definition 2.3. The two balls and the sphere

$$\begin{aligned} B_X &:= \overline{B}(0,1) = \{x \in X\colon\ \|x\| \leqslant 1\}, \\ B_X^\circ &:= B(0,1) = \{x \in X\colon\ \|x\| < 1\}, \\ S_X &:= S(0,1) = \{x \in X\colon\ \|x\| = 1\} \end{aligned}$$

in a normed space X are called, respectively, the *closed unit ball,* the *open unit ball,* and the *unit sphere* of X.

Proposition 2.2. *Let $(X, \|\cdot\|)$ be a normed space, and let $A \subset X$. Then A is bounded if and only if there exists an $M \geqslant 0$ such that*

$$\|x\| \leqslant M \qquad \text{for all } x \in A. \tag{2.1}$$

Proof. *Necessity.* If A is bounded, then it is contained in some closed ball centered at the origin, say $\overline{B}(0, M)$, i.e., $A \subset \overline{B}(0, M)$, but this is equivalent to (2.1).

Sufficiency. If (2.1) holds for some $M \geqslant 0$, then it also holds for some $M > 0$, but this means that A is contained in the ball $\overline{B}(0, M)$, thus A is bounded. □

Exercise 2.5. For a set A in a linear space X, and $z \in X$ and $\alpha \in \mathbb{K}$, the *translate* $A + z$ and *dilation* αA are defined by

$$A + z := \{x + z\colon\ x \in A\} \qquad \text{and} \qquad \alpha A := \{\alpha x\colon\ x \in A\}.$$

Prove that, if X is a normed space and A is bounded, then also $A + z$ and αA are bounded.

Exercise 2.6. Let X be a normed space, and let $a, b \in X$ and $\alpha, r > 0$. Prove that

(a) $\overline{B}(0, r) = r\overline{B}(0, 1)$;

(b) $\overline{B}(a, r) = \overline{B}(0, r) + a$;

(c) $\alpha\overline{B}(a, r) = \overline{B}(\alpha a, \alpha r)$;

(d) $\overline{B}(a, r) + b = \overline{B}(a + b, r)$.

2.4. Convergence in a normed space

Convergence $x_n \to x$ in a normed space, of course, means that

$$\|x_n - x\| \xrightarrow[n\to\infty]{} 0.$$

Proposition 2.3. *The norm and the algebraic operations in a normed space are continuous in the following sense: if $x_n, x, y_n, y \in X$ and $\alpha_n, \alpha \in \mathbb{K}$, $n \in \mathbb{N}$, are such that*

$$x_n \xrightarrow[n\to\infty]{} x, \qquad y_n \xrightarrow[n\to\infty]{} y, \qquad \text{and} \qquad \alpha_n \xrightarrow[n\to\infty]{} \alpha, \tag{2.2}$$

then

$$\|x_n\| \xrightarrow[n\to\infty]{} \|x\|, \qquad x_n + y_n \xrightarrow[n\to\infty]{} x + y, \qquad \text{and} \qquad \alpha_n x_n \xrightarrow[n\to\infty]{} \alpha x.$$

PROOF. Suppose that $x_n, x, y_n, y \in X$ and $\alpha_n, \alpha \in \mathbb{K}$, $n \in \mathbb{N}$, satisfy (2.2). By the reverse triangle inequality

$$\big|\|x_n\| - \|x\|\big| \leqslant \|x_n - x\| \xrightarrow[n\to\infty]{} 0.$$

By the triangle inequality

$$\|(x_n + y_n) - (x + y)\| = \|(x_n - x) + (y_n - y)\| \leqslant \|x_n - x\| + \|y_n - y\| \xrightarrow[n\to\infty]{} 0,$$

and

$$\begin{aligned}\|\alpha_n x_n - \alpha x\| &= \|\alpha_n x_n - \alpha_n x + \alpha_n x - \alpha x\| \\ &\leqslant \|\alpha_n (x_n - x)\| + \|(\alpha_n - \alpha)x\| = |\alpha_n|\|x_n - x\| + |\alpha_n - \alpha|\|x\| \xrightarrow[n\to\infty]{} 0.\end{aligned}$$

□

2.5. When a metric is induced by a norm

In the light of Proposition 2.1, it is natural to ask: if a metric space (X, ρ) is such that X is simultaneously a normed space, does there exist a norm $\|\cdot\|$ on X which induces the distance ρ, i.e.,

$$\rho(x, y) = \|x - y\| \qquad \text{for all } x, y \in X? \tag{2.3}$$

The answer, in general, is negative: endow a linear space $X \neq \{0\}$ with the discrete metric ρ; if one would have (2.3), then, for all $x \in X$,

$$\|x\| = \|x - 0\| = \rho(x, 0) \leqslant 1,$$

which can not be the case (because, by Exercise 2.3, any normed space $X \neq \{0\}$ admits elements of arbitrarily large norm).

Exercise 2.7. Prove that

$$\rho(x, y) = \sqrt{|x - y|}, \qquad x, y \in \mathbb{R},$$

is a metric in $\mathbb{R}$. Is this metric induced by some norm in $\mathbb{R}$?

The following Proposition gives necessary and sufficient conditions for a metric to be induced by a norm.

Proposition 2.4. *Suppose that (X,ρ) is a metric space, and that X is simultaneously a linear space. The following assertions are equivalent.*

(i) *The metric ρ is induced by a norm (i.e., there exists a norm $\|\cdot\|$ in X satisfying* (2.3)*).*

(ii) *For all $x, y, z \in X$ and $\alpha \in \mathbb{K}$,*

4° $\rho(\alpha x, \alpha y) = |\alpha| \rho(x,y)$;

5° $\rho(x+z, y+z) = \rho(x,y)$.

(iii) *For all $x, z \in X$ and $\alpha \in \mathbb{K}$,*

$4^{\circ\circ}$ $\rho(\alpha x, 0) = |\alpha| \rho(x,0)$;

$5^{\circ\circ}$ $\rho(x+z, z) = \rho(x,0)$.

If any of the equivalent conditions (i)–(iii) *holds, then the norm in condition* (i) *is defined by*

$$\|x\| := \rho(x,0), \quad x \in X.$$

PROOF.

Exercise 2.8. Prove Proposition 2.4.

□

2.6. Convex sets in a linear space

Definition 2.4. Let X be a linear space and let $A \subset X$.

The set A is said to be *convex,* if, whenever $x, y \in A$ and $\lambda \in [0,1]$, one has

$$(1-\lambda)x + \lambda y \in A.$$

Given $x, y \in X$, the set

$$[x,y] := \{x + \lambda(y-x) \colon \lambda \in [0,1]\} = \{(1-\lambda)x + \lambda y \colon \lambda \in [0,1]\}$$

is called the *(straight) line segment* connecting x and y.

Thus convexity of a set A in a normed space means that, together with any two points $x, y \in A$, the set A contains also the straight line segment $[x,y]$ connecting these points.

Exercise 2.9. Prove that balls in a normed space are convex sets.

Exercise 2.10. Let X be a normed space and let $x, y \in X$. Prove that

(a) $\|z\| \leqslant \max\{\|x\|, \|y\|\}$ for every $z \in [x,y]$;

(b) $\|x\| \leqslant \max\{\|x-y\|, \|x+y\|\}$.

§ 3. Some useful inequalities

Proposition 3.1 (Minkowski's inquality). *Let* $p \in (1, \infty)$ *and* $n \in \mathbb{N}$. *Whenever* $a_1, \ldots, a_n, b_1, \ldots, b_n \geqslant 0$, *one has*

$$\sum_{j=1}^{n} (a_j + b_j)^p \leqslant \left(\sum_{j=1}^{n} a_j^p\right)^{\frac{1}{p}} + \left(\sum_{j=1}^{n} b_j^p\right)^{\frac{1}{p}}. \tag{3.1}$$

Minkowski's inequality follows from

Proposition 3.2 (The Rogers–Hölder inequality). *Let* $p, q \in (1, \infty)$ *be conjugate exponents, i.e.,* $\frac{1}{p} + \frac{1}{q} = 1$, *and let* $n \in \mathbb{N}$. *Whenever* $a_1, \ldots, a_n, b_1, \ldots, b_n \geqslant 0$ *one has*

$$\sum_{j=1}^{n} a_j b_j \leqslant \left(\sum_{j=1}^{n} a_j^p\right)^{\frac{1}{p}} \left(\sum_{j=1}^{n} b_j^q\right)^{\frac{1}{q}}. \tag{3.2}$$

The Rogers–Hölder inequality, in turn, follows from

Proposition 3.3 (Young's inequality). *Let* $p, q \in (1, \infty)$ *be conjugate exponents, i.e.,* $\frac{1}{p} + \frac{1}{q} = 1$. *Whenever* $a, b \geqslant 0$, *one has*

$$a^{\frac{1}{p}} b^{\frac{1}{q}} \leqslant \frac{a}{p} + \frac{b}{q}. \tag{3.3}$$

Remark 3.1. Young's inequality is often formulated in the following (equivalent to Proposition 3.3) form: *if* $p, q \in (1, \infty)$ *are conjugate exponents, then, whenever* $a, b \geqslant 0$,

$$ab \leqslant \frac{a^p}{p} + \frac{b^q}{q}.$$

The proof of Minkowski's inequality. Let $a_1, \ldots, a_n, b_1, \ldots, b_n \geqslant 0$. Since, for each $j \in \{1, \ldots, n\}$,

$$(a_j + b_j)^p = (a_j + b_j)(a_j + b_j)^{p-1} = a_j(a_j + b_j)^{p-1} + b_j(a_j + b_j)^{p-1},$$

by the Rogers–Hölder inequality, letting $q \in (1, \infty)$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$ (then $\frac{1}{q} = \frac{p-1}{p}$ and $q(p-1) = p$),

$$\begin{aligned}
\sum_{j=1}^{n} (a_j + b_j)^p &= \sum_{j=1}^{n} a_j(a_j + b_j)^{p-1} + \sum_{j=1}^{n} b_j(a_j + b_j)^{p-1} \\
&\leqslant \left(\sum_{j=1}^{n} a_j^p\right)^{\frac{1}{p}} \left(\sum_{j=1}^{n} (a_j + b_j)^{(p-1)q}\right)^{\frac{1}{q}} + \left(\sum_{j=1}^{n} b_j^p\right)^{\frac{1}{p}} \left(\sum_{j=1}^{n} (a_j + b_j)^{(p-1)q}\right)^{\frac{1}{q}} \\
&= \left(\left(\sum_{j=1}^{n} a_j^p\right)^{\frac{1}{p}} + \left(\sum_{j=1}^{n} b_j^p\right)^{\frac{1}{p}}\right) \left(\sum_{j=1}^{n} (a_j + b_j)^p\right)^{\frac{1}{q}}.
\end{aligned}$$

If $a_1 = \cdots = a_n = 0$, then the inequality (3.1) is obvious. Suppose that $a_k > 0$ for some $k \in \{1, \ldots, n\}$. In this case, from the previous chain, we obtain

$$\left(\sum_{j=1}^{n}(a_j + b_j)^p\right)^{1-\frac{1}{q}} \leqslant \left(\sum_{j=1}^{n} a_j^p\right)^{\frac{1}{p}} + \left(\sum_{j=1}^{n} b_j^p\right)^{\frac{1}{p}},$$

which, in view of the equality $1 - \frac{1}{q} = \frac{1}{p}$, is equivalent to (3.1). □

Proof of the Rogers–Hölder inequality. If $a_1 = \cdots = a_n = 0$ or $b_1 = \cdots = b_n = 0$, then the inequality (3.2) clearly holds. Suppose that $a_k \neq 0$ and $b_l \neq 0$ for some $k, l \in \{1, \ldots, n\}$. Then, for each $j \in \{1, \ldots, n\}$, taking in Young's inequality (3.3)

$$a = \frac{a_j^p}{\sum_{i=1}^n a_i^p} \quad \text{and} \quad b = \frac{b_j^q}{\sum_{i=1}^n b_i^q},$$

one obtains

$$\frac{a_j b_j}{\left(\sum_{i=1}^n a_i^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^n b_i^q\right)^{\frac{1}{q}}} \leqslant \frac{a_j^p}{p\sum_{i=1}^n a_i^p} + \frac{b_j^q}{q\sum_{i=1}^n b_i^q}.$$

Thus

$$\sum_{j=1}^{n} \frac{a_j b_j}{\left(\sum_{i=1}^n a_i^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^n b_i^q\right)^{\frac{1}{q}}} \leqslant \frac{\sum_{j=1}^n a_j^p}{p\sum_{i=1}^n a_i^p} + \frac{\sum_{j=1}^n b_j^q}{q\sum_{i=1}^n b_i^q} = \frac{1}{p} + \frac{1}{q} = 1,$$

and the inequality (3.2) follows. □

The proof of Young's inequality. Let $a, b \geqslant 0$. If $b = 0$, then the inequality (3.3) clearly holds. Thus we may assume that $b > 0$. Putting $\lambda = \frac{1}{p}$, the inequality (3.3) is equivalent to

$$a^\lambda b^{1-\lambda} \leqslant \lambda a + (1-\lambda)b$$

or (dividing by b)

$$\left(\frac{a}{b}\right)^\lambda \leqslant \lambda\frac{a}{b} + 1 - \lambda.$$

Putting $t = \frac{a}{b}$, it thus suffices to show that, for every $t \in (0, \infty)$, one has

$$t^\lambda - \lambda t \leqslant 1 - \lambda.$$

To this end, consider the function $\phi(t) = t^\lambda - \lambda t$. Since

$$\phi'(t) = \lambda t^{\lambda-1} - \lambda = \lambda(t^{\lambda-1} - 1),$$

one has $\phi'(t) > 0$ if $t \in (0, 1)$, and $\phi'(t) < 0$ if $t \in (1, \infty)$. It follows that $\phi(1)$ is the maximal value of ϕ in $(0, \infty)$; thus, for all $t \in (0, \infty)$,

$$t^\lambda - \lambda t \leqslant \phi(1) = 1 - \lambda.$$

□

Proposition 3.4. *Let* $1 \leqslant p \leqslant q < \infty$*, and let* $n \in \mathbb{N}$ *and* $a_1, \ldots, a_n \geqslant 0$*. Then*

$$\left(\sum_{j=1}^{n} a_j^q\right)^{\frac{1}{q}} \leqslant \left(\sum_{j=1}^{n} a_j^p\right)^{\frac{1}{p}}.$$

Proof.

***Exercise 3.1.** Prove Proposition 3.4.

□

§ 4. Classical metric spaces

4.1. Finite-dimensional spaces

Throughout this subsection, $n \in \mathbb{N}$ will be a fixed natural number, and $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$.

The classical finite-dimensional normed spaces are:

- for $1 \leqslant p < \infty$, $\ell_p^n := (\mathbb{K}^n, \|\cdot\|_p)$ where

$$\|x\|_p := \left(\sum_{j=1}^{n} |\xi_j|^p\right)^{\frac{1}{p}}, \qquad x = (\xi_j)_{j=1}^n \in \mathbb{K}^n.$$

  The norm $\|\cdot\|_p$ is referred to as the p-norm.

- $\ell_\infty^n := m_n := (\mathbb{K}^n, \|\cdot\|_\infty)$ where

$$\|x\|_\infty := \max_{1 \leqslant j \leqslant n} |\xi_j|, \qquad x = (\xi_j)_{j=1}^n \in \mathbb{K}^n.$$

  The norm $\|\cdot\|_\infty$ is referred to as the *maximum norm.*

For $1 \leqslant p < \infty$, the most interesting among the p-norms are, perhaps, the 1-norm—also known as the *sum norm*—

$$\|x\|_1 := \sum_{j=1}^{n} |\xi_j|, \qquad x = (\xi_j)_{j=1}^n \in \mathbb{K}^n,$$

and the 2-norm—also known as the *Euclidean norm*—

$$\|x\|_2 := \sqrt{\sum_{j=1}^{n} |\xi_j|^2}, \qquad x = (\xi_j)_{j=1}^n \in \mathbb{K}^n.$$

Exercise 4.1. (a) Verify the axioms of norm for $\|\cdot\|_\infty$.

(b) Let $1 \leqslant p < \infty$. Verify the axioms of norm for $\|\cdot\|_p$ (use Minkowski's inequality for the triangle inequality in the case $1 < p < \infty$).

Exercise 4.2. Draw the closed unit balls in the spaces ℓ_1^2, ℓ_2^2, and ℓ_∞^2.

Remark 4.1. The notation ℓ_∞^n is justified by

Proposition 4.1. *For every* $x = (\xi_j)_{j=1}^n \in \mathbb{K}^n$,

$$\|x\|_p \xrightarrow[p \to \infty]{} \|x\|_\infty.$$

PROOF.

***Exercise 4.3.** Prove Proposition 4.1

□

Remark 4.2. Proposition 3.4 may be equivalently reformulated in terms of p-norms as follows: *Whenever* $1 \leqslant p \leqslant q < \infty$, *one has, for every* $x \in \mathbb{K}^n$,

$$\|x\|_\infty \leqslant \|x\|_q \leqslant \|x\|_p \leqslant \|x\|_1.$$

Proposition 4.2. *Convergence in the spaces* ℓ_p^n, $1 \leqslant p \leqslant \infty$, *is equivalent to coordinatewise convergence, i.e., for* $x_k = (\xi_j^k)_{j=1}^n, x = (\xi_j)_{j=1}^n \in \ell_p^n$, $k \in \mathbb{N}$,

$$x_k \underset{k\to\infty}{\longrightarrow} x \quad \text{in } \ell_p^n \qquad \Longleftrightarrow \qquad \xi_j^k \underset{k\to\infty}{\longrightarrow} \xi_j \quad \text{for all } j \in \{1, \ldots, n\}.$$

PROOF. By Remark 4.2, for every $j \in \{1, \ldots, n\}$ and all $p \in [1, \infty]$,

$$|\xi_j^k - \xi_j| \leqslant \max_{1\leqslant i\leqslant n} |\xi_i^k - \xi_i| = \|x_k - x\|_\infty \leqslant \|x_k - x\|_p \leqslant \|x_k - x\|_1 = \sum_{i=1}^n |\xi_i^k - \xi_i|.$$

It follows that, on one hand, if $x_k \underset{k\to\infty}{\longrightarrow} x$ in ℓ_p^n for some $p \in [1, \infty]$, i.e., $\|x_k - x\|_p \underset{k\to\infty}{\longrightarrow} 0$, then

$$|\xi_j^k - \xi_j| \underset{k\to\infty}{\longrightarrow} 0, \quad \text{i.e.,} \quad \xi_j^k \underset{k\to\infty}{\longrightarrow} \xi_j \qquad \text{for every } j \in \{1, \ldots, n\}. \tag{4.1}$$

On the other hand, if (4.1) holds, then also $\|x_k - x\|_1 \underset{k\to\infty}{\longrightarrow} 0$ and thus, for all $p \in [1, \infty]$, one has $\|x_k - x\|_p \underset{k\to\infty}{\longrightarrow} 0$, i.e., $x_k \underset{k\to\infty}{\longrightarrow} x$ in ℓ_p^n. □

4.2. Sequence spaces

Throughout this subsection, $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. By a sequence $(\xi_j) = (\xi_j)_{j=1}^\infty$, we shall mean a sequence of numbers, i.e., $\xi_j \in \mathbb{K}$, $j \in \mathbb{N}$.

In all of the linear spaces of sequences below, for sequences $(\xi_j)_{j=1}^\infty$ and $(\eta_j)_{j=1}^\infty$, and a number $\alpha \in \mathbb{K}$, the linear space operations are defined coordinatewise:

$$(\xi_j)_{j=1}^\infty + (\eta_j)_{j=1}^\infty := (\xi_j + \eta_j)_{j=1}^\infty,$$
$$\alpha(\xi_j)_{j=1}^\infty := (\alpha\xi_j)_{j=1}^\infty.$$

- The linear space of all sequences

  $$s := \left\{(\xi_j)_{j=1}^\infty : \ \xi_j \in \mathbb{K}, \ j \in \mathbb{N}\right\}$$

  is a metric space with respect to the distance

  $$\rho(x, y) := \sum_{j=1}^\infty \frac{1}{2^j} \frac{|\xi_j - \eta_j|}{1 + |\xi_j - \eta_j|}, \qquad x = (\xi_j)_{j=1}^\infty, \ y = (\eta_j)_{j=1}^\infty \in s.$$

Exercise 4.4. Verify the axioms of metric for the distance ρ.

HINT. For the triangle inequality, it suffices to show that, whenever $0 \leqslant \alpha \leqslant \beta$, on has $\frac{\alpha}{1+\alpha} \leqslant \frac{\beta}{1+\beta}$, because, in this case, for all $j \in \mathbb{N}$,

$$\frac{|\xi_j - \zeta_j|}{1 + |\xi_j - \zeta_j|} = \frac{|\xi_j - \eta_j + \eta_j - \zeta_j|}{1 + |\xi_j - \eta_j + \eta_j - \zeta_j|} \leqslant \frac{|\xi_j - \eta_j| + |\eta_j - \zeta_j|}{1 + |\xi_j - \eta_j| + |\eta_j - \zeta_j|}$$
$$= \frac{|\xi_j - \eta_j|}{1 + |\xi_j - \eta_j| + |\eta_j - \zeta_j|} + \frac{|\eta_j - \zeta_j|}{1 + |\xi_j - \eta_j| + |\eta_j - \zeta_j|} \leqslant \frac{|\xi_j - \eta_j|}{1 + |\xi_j - \eta_j|} + \frac{|\eta_j - \zeta_j|}{1 + |\eta_j - \zeta_j|}.$$

Notice that the distance in s is not induced by any norm.

Exercise 4.5. Prove that there is no norm in s satisfying $\rho(x,y) = \|x-y\|$, $x,y \in s$.

Proposition 4.3. *Convergence in the space s is equivalent to coordinatewise convergence, i.e., for $x_k = (\xi_j^k)_{j=1}^\infty, x = (\xi_j)_{j=1}^\infty \in s$, $k \in \mathbb{N}$,*

$$x_k \xrightarrow[k\to\infty]{} x \quad \text{in } s \qquad \iff \qquad \xi_j^k \xrightarrow[k\to\infty]{} \xi_j \quad \text{for all } j \in \mathbb{N}.$$

Proof. Let $x_k = (\xi_j^k)_{j=1}^\infty, x = (\xi_j)_{j=1}^\infty \in s$, $k \in \mathbb{N}$.

"$\Rightarrow$". Assume that $x_k \xrightarrow[k\to\infty]{} x$ in the space s, i.e

$$\rho(x_k, x) = \sum_{j=1}^\infty \frac{1}{2^j} \frac{|\xi_j^k - \xi_j|}{1+|\xi_j^k - \xi_j|} \xrightarrow[k\to\infty]{} 0.$$

For every $k \in \mathbb{N}$,

$$\frac{1}{2^j} \frac{|\xi_j^k - \xi_j|}{1+|\xi_j^k - \xi_j|} \leqslant \rho(x_k, x) \quad \text{for every } j \in \mathbb{N},$$

thus

$$t_j^k := \frac{|\xi_j^k - \xi_j|}{1+|\xi_j^k - \xi_j|} \leqslant 2^j \rho(x_k, x) \quad \text{for every } j \in \mathbb{N}.$$

Since, for every $j \in \mathbb{N}$, one has $2^j\, \rho(x_k, x) \xrightarrow[k\to\infty]{} 0$, it follows that

$$t_j^k \xrightarrow[k\to\infty]{} 0 \quad \text{for every } j \in \mathbb{N};$$

therefore, observing that, for all $k, j \in \mathbb{N}$, one has $t_j^k + t_j^k|\xi_j^k - \xi_j| = |\xi_j^k - \xi_j|$ and thus $|\xi_j^k - \xi_j| = \dfrac{t_j^k}{1-t_j^k}$ (note that $1 - t_j^k \neq 0$, because $t_j^k < 1$),

$$|\xi_j^k - \xi_j| = \frac{t_j^k}{1-t_j^k} \xrightarrow[k\to\infty]{} 0 \quad \text{for every } j \in \mathbb{N}.$$

"$\Leftarrow$". Assume that $\xi_j^k \xrightarrow[k\to\infty]{} \xi_j$ for every $j \in \mathbb{N}$. We must show that $x_k \xrightarrow[k\to\infty]{} x$ in the space s, i.e., $\rho(x_k, x) \xrightarrow[k\to\infty]{} 0$. To this end, letting $\varepsilon > 0$ be arbitrary, it suffices to find an $N \in \mathbb{N}$ so that, for $k \geqslant N$,

$$\rho(x_k, x) = \sum_{j=1}^\infty \frac{1}{2^j} \frac{|\xi_j^k - \xi_j|}{1+|\xi_j^k - \xi_j|} < \varepsilon.$$

To this end, first choose an index $n \in \mathbb{N}$ so that

$$\sum_{j=n+1}^\infty \frac{1}{2^j} < \frac{\varepsilon}{2}$$

(such an $n \in \mathbb{N}$ exists because the series $\sum_{j=1}^{\infty} \frac{1}{2^j}$ converges and the remainder term of a convergent series converges to zero). Since $|\xi_j^k - \xi_j| \xrightarrow[k\to\infty]{} 0$ for every $j \in \mathbb{N}$, also, for every $j \in \{1, \ldots, n\}$,

$$\frac{|\xi_j^k - \xi_j|}{1 + |\xi_j^k - \xi_j|} \xrightarrow[k\to\infty]{} 0,$$

thus, for every $j \in \{1, \ldots, n\}$, there exists an $N_j \in \mathbb{N}$ such that, for $k \geqslant N_j$, one has

$$\frac{|\xi_j^k - \xi_j|}{1 + |\xi_j^k - \xi_j|} < \frac{\varepsilon}{2}. \tag{4.2}$$

Therefore, defining $N := \max\{N_1, \ldots, N_n\}$, the inequality (4.2) holds for all $j \in \{1, \ldots, n\}$ and $k \geqslant N$. Thus, whenever $k \geqslant N$, one has

$$\begin{aligned}\rho(x_k, x) &= \sum_{j=1}^{\infty} \frac{1}{2^j} \frac{|\xi_j^k - \xi_j|}{1 + |\xi_j^k - \xi_j|} = \sum_{j=1}^{n} \frac{1}{2^j} \frac{|\xi_j^k - \xi_j|}{1 + |\xi_j^k - \xi_j|} + \sum_{j=n+1}^{\infty} \frac{1}{2^j} \frac{|\xi_j^k - \xi_j|}{1 + |\xi_j^k - \xi_j|} \\ &< \sum_{j=1}^{n} \frac{\varepsilon}{2^{j+1}} + \sum_{j=n+1}^{\infty} \frac{1}{2^j} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

□

The most important *classical normed sequence spaces* are:

- for $1 \leqslant p < \infty$, the linear space of *p-summable sequences*

$$\ell_p := \left\{ (\xi_j)_{j=1}^{\infty} : \sum_{j=1}^{\infty} |\xi_j|^p < \infty \right\}$$

  with respect to the norm

$$\|x\| = \|x\|_p := \left(\sum_{j=1}^{\infty} |\xi_j|^p \right)^{\frac{1}{p}}, \qquad x = (\xi_j)_{j=1}^{\infty} \in \ell_p;$$

- the linear space of *bounded sequences*

$$\ell_\infty := m := \left\{ (\xi_j)_{j=1}^{\infty} : \text{ there exists } M \geqslant 0 \text{ such that } |\xi_j| \leqslant M \text{ for all } j \in \mathbb{N} \right\}$$

  and its linear subspaces of *convergent sequences*

$$c := \left\{ (\xi_j)_{j=1}^{\infty} : \text{ the limit } \lim_{j\to\infty} \xi_j \in \mathbb{K} \text{ exists} \right\} \subset \ell_\infty$$

  and

$$c_0 := \left\{ (\xi_j)_{j=1}^{\infty} : \lim_{j\to\infty} \xi_j = 0 \right\} \subset c \subset \ell_\infty$$

  with respect to the norm

$$\|x\| = \|x\|_\infty := \sup_{j\in\mathbb{N}} |\xi_j|, \qquad x = (\xi_j)_{j=1}^{\infty} \in \ell_\infty.$$

Exercise 4.6. Prove that, in the space c_0, the norm can be computed by

$$\|x\| = \|x\|_\infty := \max_{j\in\mathbb{N}} |\xi_j|, \qquad x = (\xi_j)_{j=1}^\infty \in c_0.$$

The most interesting among these spaces ℓ_p, $1 \leqslant p < \infty$, are the space of *summable sequences*

$$\ell_1 := \left\{ (\xi_j)_{j=1}^\infty \colon \ \sum_{j=1}^\infty |\xi_j| < \infty \right\}$$

where the norm is given by

$$\|x\| = \|x\|_1 := \sum_{j=1}^\infty |\xi_j|, \qquad x = (\xi_j)_{j=1}^\infty \in \ell_1,$$

and the space

$$\ell_2 := \left\{ (\xi_j)_{j=1}^\infty \colon \ \sum_{j=1}^\infty |\xi_j|^2 < \infty \right\}$$

where the norm is given by

$$\|x\| = \|x\|_2 := \sqrt{\sum_{j=1}^\infty |\xi_j|^2}, \qquad x = (\xi_j)_{j=1}^\infty \in \ell_2.$$

Proposition 4.4. (a) *For $1 \leqslant p < \infty$, convergence in the space ℓ_p implies coordinatewise convergence, i.e., for $x_k = (\xi_j^k)_{j=1}^\infty$, $x = (\xi_j)_{j=1}^\infty \in \ell_p$, $k \in \mathbb{N}$,*

$$x_k \xrightarrow[k\to\infty]{} x \quad \text{in } \ell_p \qquad \Longrightarrow \qquad \xi_j^k \xrightarrow[k\to\infty]{} \xi_j \quad \text{for all } j \in \mathbb{N}.$$

(b) *Convergence in the space ℓ_∞ is equivalent to uniform coordinatewise convergence, i.e., for $x_k = (\xi_j^k)_{j=1}^\infty$, $x = (\xi_j)_{j=1}^\infty \in \ell_\infty$, $k \in \mathbb{N}$,*

$$x_k \xrightarrow[k\to\infty]{} x \quad \text{in } \ell_\infty \qquad \Longleftrightarrow \qquad \xi_j^k \xrightarrow[k\to\infty]{} \xi_j \quad \text{uniformly in } j \in \mathbb{N}$$
$$\left(\text{i.e.,} \quad \sup_{j\in\mathbb{N}} |\xi_j^k - \xi_j| \xrightarrow[k\to\infty]{} 0 \right).$$

Proof.

Exercise 4.7. Prove Proposition 4.4.

□

Lause 4.1. *Let $1 \leqslant p < q \leqslant \infty$. Then $\ell_p \subsetneqq \ell_q$ and*

$$\|x\|_q \leqslant \|x\|_p \quad \text{for every } x \in \ell_p.$$

Proof.

Exercise 4.8. Prove Proposition 4.1

Hint. Use Proposition 3.4.

□

4.3. Function spaces

If $T \subset \mathbb{K}$, in all of the linear spaces of functions $T \to \mathbb{K}$ below, for functions $x, y\colon T \to \mathbb{K}$ and a number $\alpha \in \mathbb{K}$, the linear space operations are defined pointwise:

$$\begin{aligned} (x+y)(t) &:= x(t) + y(t), \\ (\alpha x)(t) &:= \alpha x(t), \end{aligned} \qquad t \in T.$$

The most important *classical function spaces* are the following.

- $M[a, b]$ is the normed space of bounded functions $x\colon [a, b] \to \mathbb{K}$ and $C[a, b]$ is its linear subspace of continuous functions. The norm in thes spaces is defined by
$$\|x\| = \sup_{t \in [a,b]} |x(t)|, \quad x \in M[a, b].$$
In the space $C[a, b]$, the norm can be computed by
$$\|x\| = \max_{t \in [a,b]} |x(t)|, \quad x \in C[a, b].$$
Notice that convergence in these spaces is the uniform convergence on $[a, b]$: for $x_n, x \in M[a, b]$,
$$x_n \xrightarrow[n \to \infty]{} x \quad \text{in } M[a, b] \qquad \iff \qquad x_n(t) \xrightarrow[n \to \infty]{} x(t) \quad \text{uniformly in } t \in [a, b].$$

- $C^n[a, b]$ $(n \in \mathbb{N})$ is the normed space of n times continuously differentiable on $[a, b]$ functions, where the norm is defined by
$$\|x\| = \max_{t \in [a,b]} |x(t)| + \sum_{j=1}^{n} \max_{t \in [a,b]} |x^{(j)}(t)|, \quad x \in C^n[a, b].$$

- $L_p(a, b)$ $(1 \leqslant p < \infty)$ is the normed space of *p-integrable* (in the sense of Lebesgue) functions $(a, b) \to \mathbb{K}$ i.e., measurable (in the sense of Lebesgue) functions $x\colon (a, b) \to \mathbb{K}$ for which
$$\int_a^b |x(t)|^p \, dt < \infty.$$
The norm in $L_p(a, b)$ is defined by
$$\|x\| = \|x\|_p := \left(\int_a^b |x(t)|^p \, dt \right)^{\frac{1}{p}}, \quad x \in L_p(a, b).$$
Equality of functions $x, y \in L_p(a, b)$ is understood as almost everywhere (a.e.) equality:
$$x = y \quad \text{in } L_p(a, b) \qquad \iff \qquad x(t) = y(t) \quad \text{for almost every in } t \in (a, b).$$

Equivalently, $L_p(a,b)$ can be interpreted as the normed space of equivalence classes of a.e. equal on (a,b) p-integrable functions, where the algebraic operations and norm are defined as above via representatives of equivalence classes.

The most interesting among the spaces $L_p(a,b)$, $1 \leqslant p < \infty$, are the space of integrable in the sense of Lebesgue functions $L_1(a,b)$ where the norm is given by

$$\|x\| = \|x\|_1 := \int_a^b |x(t)|\, dt, \quad x \in L_1(a,b)$$

and the space of square-integrable functions $L_2(a,b)$ where the norm is given by

$$\|x\| = \|x\|_2 := \sqrt{\int_a^b |x(t)|^2\, dt,}, \quad x \in L_2(a,b).$$

- $L_\infty(a,b)$ is the normed space of *essentially bounded* (in the sense of Lebesgue) functions $(a,b) \to \mathbb{K}$ (i.e., Lebesgue measurable functions $(a,b) \to \mathbb{K}$ which are bounded outside a set of Lebesgue measure 0). The norm for $x \in L_\infty(a,b)$ is defined by

$$\begin{aligned}\|x\| = \|x\|_\infty &:= \operatorname*{ess\,sup}_{t\in(a,b)} |x(t)| := \operatorname*{vrai\,sup}_{t\in(a,b)} |x(t)| \\ &:= \inf\{M \geqslant 0\colon\ |x(t)| \leqslant M \text{ a.e.}\} \\ &= \inf\Big\{M \geqslant 0\colon\ m\big(\{t \in (a,b)\colon\ |x(t)| > M\}\big) = 0\Big\},\end{aligned}$$

where m is the Lebesgue measure. Notice that $|x(t)| \leqslant \|x\|_\infty$ a.e.

As in $L_p(a,b)$ for $1 \leqslant p < \infty$, equality of functions in $L_\infty(a,b)$ is understood as their equality a.e. Equivalently, $L_\infty(a,b)$ is often interpreted as the normed space of equivalence classes of a.e. equal on (a,b) essentially bounded functions.

Remark 4.3. In general, convergence in the space $L_p(a,b)$ where $1 \leqslant p < \infty$ does not imply convergence a.e. (let alone pointwise convergence): if $x_n, x \in L_p(a,b)$ $(1 \leqslant p < \infty)$, $n \in \mathbb{N}$, are such that $x_n \xrightarrow[n\to\infty]{} x$ in $L_p(a,b)$, one does not necessarily have that

$$x_n \xrightarrow[n\to\infty]{} x \quad \text{a.e. in } (a,b),$$

let alone

$$x_n(t) \xrightarrow[n\to\infty]{} x(t) \quad \text{for every } t \in (a,b).$$

On the other hand, if $x_n, x \in L_p(a,b)$ $(1 \leqslant p < \infty)$, $n \in \mathbb{N}$, are such that

(1) $x_n \xrightarrow[n\to\infty]{} x$ in $L_p(a,b)$;

(2) there exist a subsequence $(x_{k_n})_{n=1}^\infty$ and a function $z\colon (a,b) \to \mathbb{K}$ such that $x_{k_n} \xrightarrow[n\to\infty]{} z$ a.e. in (a,b),

then $x = z$ a.e. (thus also $z \in L_p(a,b)$, and $x = z$ in the space $L_p(a,b)$).

Convergence in the space $L_\infty(a,b)$ implies convergence a.e.

Remark 4.4. In the set-theoretical sense, the spaces $L_p(a,b)$, $p \in [1,\infty]$, are related as follows: if $1 < p < q < \infty$, one has

$$L_1(a,b) \supsetneqq L_p(a,b) \supsetneqq L_q(a,b) \supsetneqq L_\infty(a,b).$$

***Exercise 4.9.** Prove the assertion above.

§ 5. Open and closed sets in metric spaces. Interior, boundary, and closure

Throughout this section, X will be a metric space.

5.1. Open sets in metric spaces. Interior points

Definition 5.1. A subset $A \subset X$ is said to be an *open* set, if every point of A has a neighbourhood which is contained in A.

Definition 5.2. Let $A \subset X$.

A point $a \in A$ is called an *interior point* of A, if a has a neighbourhood which is contained in A.

Equivalently, a point $a \in A$ is called an interior point of A, if there exists a ball centered at a which is contained in A, i.e., there is an $\varepsilon > 0$ such that $B(x, \varepsilon) \subset A$.

The following corollary is straightforward from the preceding definitions.

Corollary 5.1. *Let $A \subset X$. The following assertions are equivalent:*

(i) *the set A is open;*

(ii) *every point of A is an interior point of A;*

(iii) *for every $a \in A$, there exists an $\varepsilon > 0$ such that $B(a, \varepsilon) \subset A$.*

Proposition 5.2. *An open ball is an open set.*

PROOF. Let $a \in X$ and $r > 0$, and let $b \in B(a, r)$. In order to see that the open ball $B(a, r)$ is an open set, it suffices to find an $\varepsilon > 0$ such that $B(b, \varepsilon) \subset B(a, r)$. The latter inclusion clearly holds for $\varepsilon := r - \rho(b, a)$ (here $\varepsilon > 0$ because, since $b \in B(a, r)$, one has $\rho(b, a) < r$), because whenever $x \in B(b, \varepsilon)$, one has

$$\rho(x, a) \leqslant \rho(x, b) + \rho(b, a) < \varepsilon + \rho(b, a) < r - \rho(b, a) + \rho(b, a) = r$$

and thus $x \in B(a, r)$. □

Example 5.1. The half-open interval $[0, 1)$ in $\mathbb{R}$ is not an open set (with respect to the metric $d(x, y) = |x - y|$, $x, y \in \mathbb{R}$). Indeed, the point $0 \in [0, 1)$ is not an interior point of the interval $[0, 1)$, because no open ball $B(0, \varepsilon) = (-\varepsilon, \varepsilon)$ (here $\varepsilon > 0$) is contained in $[0, 1)$.

The following proposition collects some basic properties of open sets.

Proposition 5.3. (a) *$\varnothing$ and X are open sets;*

(b) *any finite intersection of open sets is an open set, i.e., whenever $n \in \mathbb{N}$ and $A_1, \ldots, A_n \subset X$ are open sets, also their intersection $\bigcap_{j=1}^{n} A_j$ is an open set;*

(c) *any union of open sets is an open set, i.e., whenever I is a set of indices and A_j, $j \in I$, are open sets, also their union $\bigcup_{j \in I} A_j$ is an open set.*

Proof. (a). Every point of $\varnothing$ is an interior point of $\varnothing$ (because there are no points in $\varnothing$), thus $\varnothing$ is open. Since X contains every ball in it, every point in X is an interior point of X and thus X is open.

(b). Let $n \in \mathbb{N}$ and let $A_1, \ldots, A_n \subset X$ be open sets. In order to show that the intersection $\bigcap_{j=1}^{n} A_j =: A$ is open, letting $x \in A$ be arbitrary, it suffices to find an $\varepsilon > 0$ such that $B(x, \varepsilon) \subset A$. For every $j \in \{1, \ldots, n\}$, since the set A_j is open and $x \in A_j$, there is an $\varepsilon_j > 0$ such that $B(x, \varepsilon_j) \subset A_j$. Putting $\varepsilon := \min\{\varepsilon_1, \ldots, \varepsilon_n\}$, one has $B(x, \varepsilon) \subset B(x, \varepsilon_j) \subset A_j$ for every $j \in \{1, \ldots, n\}$, and thus $B(x, \varepsilon) \subset \bigcap_{j=1}^{n} A_j = A$, as desired.

(c). Let I be a set of indices and let A_j, $j \in I$, be open sets. In order to show that the union $\bigcup_{j \in I} A_j =: A$ is open, letting $x \in A$ be arbitrary, it suffices to find an $\varepsilon > 0$ such that $B(x, \varepsilon) \subset A$. Let $i \in I$ be such that $x \in A_i$. Since the set A_i is open, there is an $\varepsilon > 0$ such that $B(x, \varepsilon) \subset A_i \subset \bigcup_{j \in I} A_j = A$, as desired. □

5.2. Closed sets in metric spaces. Boundary points

Definition 5.3. Let $A \subset X$.

A point $a \in X$ is called a *boundary point* of A, if every neighbourhood of a contains both points in A and points not belonging to A.

Equivalently, a point $a \in A$ is called a boundary point of A, if, for every $\varepsilon > 0$,

$$B(a, \varepsilon) \cap A \neq \varnothing \quad \text{and} \quad B(a, \varepsilon) \cap (X \backslash A) \neq \varnothing.$$

Since $A = X \backslash (X \backslash A)$, the following corollary is straightforward from the preceding definition.

Corollary 5.4. *Let $A \subset X$. The set A and its complement $X \backslash A$ have the same boundary points.*

Corollary 5.5. *Let $A \subset X$. Then any point of A is either an interior point or a boundary point of A. No point of A can be simultaneously an interior point and a boundary point of A.*

Proof. Let $a \in A$. There are two (mutually excluding each other) alternatives:

(I) there exists a neighbourhood of a which is contained in A;

(II) every neighbourhood of a contains a point not belonging to A.

In the case (I), the point a is an interior point of A by definition.

In the case (II), since every neighbourhood of a also contains points in A (note that every neighbourhood of a contains the point $a \in A$ itself), the point a is a boundary point.

It is clear from the corresponding definitions that no point of A can be simultaneously an interior point and a boundary point of A. □

From Corollary 5.5 it follows that, for a subset A of X,

$$A \text{ is open} \quad \Longleftrightarrow \quad A \text{ contains none of its boundary points.}$$

Definition 5.4. Let $A \subset X$. The set A is said to be a *closed* set, if it contains all its boundary points.

Remark 5.1. A set $A \subset X$ may be both open and closed simultaneously. This happens precisely when A has no boundary points. Examples of such a phenomenon are, e.g., $A = \varnothing$ and $A = X$. Also, every set in a discrete metric space is both open and closed (see exercise 5.6).

Remark 5.2. A set in a metric space may be neither open nor closed. E.g., the half-open interval $[0, 1)$ in $\mathbb{R}$ is not an open set (with respect to the metric $d(x, y) = |x - y|$, $x, y \in \mathbb{R}$) (see example 5.1); nor is it closed, because its boundary point $1 \in \mathbb{R}$ is not in $[0, 1)$.

The following proposition shows the duality between open and closed sets in metric spaces.

Proposition 5.6. *Let $A \subset X$.*

(a) *The set A is closed if and only if its complement $X\backslash A$ is open.*

(b) *The set A is open if and only if its complement $X\backslash A$ is closed.*

Proof. (a). Since by Corollary 5.4, the set A and its complement $X\backslash A$ have the same boundary points,

$$\begin{aligned} A \text{ is closed} \quad &\Longleftrightarrow \quad A \text{ contains all of its boundary points} \\ &\Longleftrightarrow \quad X\backslash A \text{ contains none of its boundary points} \\ &\Longleftrightarrow \quad X\backslash A \text{ is open.} \end{aligned}$$

(b). Since $A = X\backslash(X\backslash A)$, one has, by (a),

$$A \text{ is open} \quad \Longleftrightarrow \quad X\backslash(X\backslash A) \text{ is open} \quad \Longleftrightarrow \quad X\backslash A \text{ is closed.}$$

□

By courtesy of Proposition 5.6, the following proposition becomes a corollary from Proposition 5.3.

Proposition 5.7. (a) *$\varnothing$ and X are closed sets;*

(b) *any finite union of closed sets is a closed set, i.e., whenever $n \in \mathbb{N}$ and $A_1, \ldots, A_n \subset X$ are closed sets, also their union $\bigcup_{j=1}^{n} A_j$ is a closed set;*

(c) *any intersection of closed sets is a closed set, i.e., whenever I is a set of indices and A_j, $j \in I$, are closed sets, also their intersection $\bigcap_{j \in I} A_j$ is a closed set.*

Proof. (a) Since $\varnothing = X\backslash X$ and $X = X\backslash\varnothing$, and, by Proposition 5.3, X and $\varnothing$ are open sets, the sets $\varnothing$ and X are closed by Proposition 5.6, (a).

(b). Let $n \in \mathbb{N}$ and let $A_1, \dots, A_n \subset X$ be closed sets. In order to show that the union $\bigcup_{j=1}^{n} A_j =: A$ is closed, by Proposition 5.6, (a), it suffices to show that its complement $X\backslash A$ is open. By De Morgan's law,

$$X\backslash A = X\backslash \bigcup_{j=1}^{n} A_j = \bigcap_{j=1}^{n} X\backslash A_j.$$

Since the sets $A_1, \dots, A_n$ are closed, their complements $X\backslash A_1, \dots, X\backslash A_n$ are open by Proposition 5.6, (a), and thus also their intersection $\bigcap_{j=1}^{n} X\backslash A_j$ is open by Proposition 5.3, (b), i.e., the complement $X\backslash A$ is open, as desired.

(c). Let I be a set of indices and let A_j, $j \in I$, be closed sets. In order to show that the intersection $\bigcap_{j\in I} A_j =: A$ is closed, by Proposition 5.6, (a), it suffices to show that its complement $X\backslash A$ is open. By De Morgan's law,

$$X\backslash A = X\backslash \bigcap_{j\in I} A_j = \bigcup_{j\in I} X\backslash A_j.$$

Since, for every $j \in I$, the set A_j is closed, its complement $X\backslash A_j$ is open by Proposition 5.6, (a), thus also the union $\bigcup_{j\in I} X\backslash A_j$ is open by Proposition 5.3, (c), i.e., the complement $X\backslash A$ is open, as desired.

□

Proposition 5.8. *Let $A \subset X$. The following assertions are equivalent:*

(i) *A is closed;*

(ii) *whenever a sequence $(x_n)_{n=1}^{\infty}$ of elements of A converges to some $x \in X$, also $x \in A$.*

Proof. (i)⇒(ii). Let A be closed, and let $(x_n)_{n=1}^{\infty}$ be a sequence of elements of A converging to an element $x \in X$. We must show that $x \in A$. There are two (mutually excluding each other) alternatives:

(1) there exists a neighbourhood of x which is contained in A;

(2) every neighbourhood of x contains a point not belonging to A.

In the case (1), clearly $x \in A$ (in fact, x is an interior point of A). In the case (2), observing that, since $x_n \xrightarrow[n\to\infty]{} x$,

- every neighbourhood of x contains some x_n (and thus every neighbourhood of x contains a point belonging to A),

the point x is a boundary point of A and thus $x \in A$ by the closedness of A.

(ii)⇒(i). Assume that (ii) holds and let $x \in X$ be a boundary point of A. In order for A to be closed, it suffices to show that $x \in A$. Since x is a boundary point of A, for every $n \in \mathbb{N}$, there is some

$$x_n \in B\Big(x, \frac{1}{n}\Big) \cap A,$$

For every $n \in \mathbb{N}$, since $x_n \in B\Big(x, \frac{1}{n}\Big)$, one has

$$0 \leqslant \rho(x_n, x) < \frac{1}{n}$$

and thus $\rho(x_n, x) \xrightarrow[n\to\infty]{} 0$, i.e., $x_n \xrightarrow[n\to\infty]{} x$ in X. By (ii), $x \in A$, as desired. □

Proposition 5.9. *Closed balls and spheres in metric spaces are closed sets.*

Proof. We only show that closed balls are closed. (The closedness of spheres can be shown analogously.)

Let $a \in X$ and $r > 0$, and let $x_n \in \overline{B}(a, r)$, $n \in \mathbb{N}$, and $x \in X$ be such that $x_n \xrightarrow[n\to\infty]{} x$ in X. For the closedness of the closed ball $\overline{B}(a, r)$, by Proposition 5.8, it suffices to show that $x \in \overline{B}(a, r)$, i.e., $\rho(x, a) \leqslant r$.

By the continuity of the metric ρ (see Proposition 1.2, (d)),

$$\rho(x_n, a) \xrightarrow[n\to\infty]{} \rho(x, a).$$

For every $n \in \mathbb{N}$, since $x_n \in B(a, r)$, one has $\rho(x_n, a) \leqslant r$, thus also $\rho(x, a) \leqslant r$, as desired. □

Exercise 5.1. Prove Proposition 5.9 by showing that the complement of a closed ball and the complement of a sphere are open sets (and then applying Proposition 5.6).

5.3. Interior, boundary, and closure

Definition 5.5. Let $A \subset X$.

- The set of all interior points of A is called the *interior* of A and denoted by A° or $\operatorname{int} A$.
- The set of all boundary points of A is called the *boundary* of A and denoted by ∂A or $\operatorname{fr} A$.
- The union of the set A and its boundary is is called the *closure* of A and denoted by $\overline{A}$ or $\operatorname{cl} A$. The points in $\overline{A}$ are called *closure points* of A.

Thus, by definition,

$$\overline{A} := \operatorname{cl} A := A \cup \partial A = A^\circ \cup \partial A.$$

The following proposition just a restatement of Corollaries 5.4 and 5.5 (which were immediate corollaries from the corresponding definitions).

Proposition 5.10. *Let $A \subset X$. Then*

(a) $\partial A = \partial(X \backslash A)$;

(b) *for every $x \in A$, either $x \in A^\circ$ or $x \in \partial A$;*

(c) $A^\circ \cap \partial A = \varnothing$.

Clearly,

- A is open $\iff$ $A = A^\circ$ $\iff$ $A \cap \partial A = \varnothing$;
- A is closed $\iff$ $A = \overline{A}$.

Exercise 5.2. Let $A \subset X$. Prove that

(a) $A^\circ = X \backslash \overline{X \backslash A}$; (b) $\overline{A} = X \backslash (X \backslash A)^\circ$; (c) $(X \backslash A)^\circ = X \backslash \overline{A}$; (d) $\overline{X \backslash A} = X \backslash A^\circ$.

Closure points of a set in a metric space are described by

Proposition 5.11. *Let $A \subset X$ and $x \in X$. The following assertions are equivalent:*

(i) $x \in \overline{A}$;

(ii) *for every $\varepsilon > 0$, one has $B(x, \varepsilon) \cap A \neq \varnothing$;*

(iii) *there is a sequence (x_n) of elements of A such that $x_n \xrightarrow[n\to\infty]{} x$.*

Proof. (i)⇒(ii). Let $x \in \overline{A}$, and let $\varepsilon > 0$. If $x \in A^\circ$, then $x \in B(x, \varepsilon) \cap A$, hence $B(x, \varepsilon) \cap A \neq \varnothing$. If $x \in \partial A$, then the latter holds by definition.

(ii)⇒(iii). Suppose that (ii) holds. Then, for every $n \in \mathbb{N}$, there exists some $x_n \in B\left(x, \frac{1}{n}\right) \cap A$. Now $x_n \in A$ for all $n \in \mathbb{N}$ and $x_n \xrightarrow[n\to\infty]{} x$ because

$$\rho(x_n, x) \leqslant \frac{1}{n} \xrightarrow[n\to\infty]{} 0.$$

(iii)⇒(i). Suppose that there are $x_n \in A$, $n \in \mathbb{N}$, such that $x_n \xrightarrow[n\to\infty]{} x$. If $x \in A$, then clearly $x \in \overline{A}$, i.e., (i) holds. Suppose that $x \notin A$. In this case, since every neighbourhood of x contains x, every neighbourhood of x contains points not belonging to A, and since every neighbourhood of x contains some x_n (because $x_n \xrightarrow[n\to\infty]{} x$), every neighbourhood of x contains points in A; thus x is a boundary point of A and therefore $x \in \overline{A}$. □

Simpler properties of closure are collected in

Proposition 5.12. *Let $A, B \subset X$. Then*

(a) $\overline{A}$ *is a closed set;*

(b) *if* $A \subset B$, *then* $\overline{A} \subset \overline{B}$*;*

(c) $\overline{\overline{A}} = \overline{A}$*;*

(d) $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

PROOF. (a). Let (x_n) be a sequence of elements of $\overline{A}$ converging to some x in X. In order to prove that $\overline{A}$ is closed, by Proposition 5.8, it suffices to show that $x \in \overline{A}$.

For every $n \in \mathbb{N}$, since $x \in \overline{A}$, by Proposition 5.11, there exists some $z_n \in A \cap B\big(x_n, \frac{1}{n}\big)$. Now (z_n) is a sequence of elements of A satisfying $z_n \xrightarrow[n\to\infty]{} x$ in X, because

$$\rho(z_n, x) \leqslant \rho(z_n, x_n) + \rho(x_n, x) < \frac{1}{n} + \rho(x_n, x) \xrightarrow[n\to\infty]{} 0;$$

thus, by Propostion 5.11, $x \in \overline{A}$, as desired.

(b). Assume that $A \subset B$, and let $x \in \overline{A}$. It suffices to show that $x \in \overline{B}$. To this end, letting $\varepsilon > 0$ be arbitrary, by Proposition 5.11, it suffices to show that $B(x, \varepsilon) \cap B \neq \varnothing$. The latter holds because $A \subset B$ and $B(x, \varepsilon) \cap A \neq \varnothing$ by Proposition 5.11 (because $x \in \overline{A}$).

(c) follows immediately from (a).

(d). On the one hand, since $A \subset \overline{A}$ and $B \subset \overline{B}$, also $A \cup B \subset \overline{A} \cup \overline{B}$, thus $\overline{A \cup B} \subset \overline{\overline{A} \cup \overline{B}}$ by (b). Since $\overline{A}$ and $\overline{B}$ are closed sets (by (a)) and finite unions of closed sets are closed (by Proposition 5.7, (b)), also the union $\overline{A} \cup \overline{B}$ closed and therefore $\overline{\overline{A} \cup \overline{B}} = \overline{A} \cup \overline{B}$. Thus $\overline{A \cup B} \subset \overline{\overline{A} \cup \overline{B}} = \overline{A} \cup \overline{B}$.

On the other hand, since $A \subset A \cup B$ and $B \subset A \cup B$, by (b), one has $\overline{A} \subset \overline{A \cup B}$ and $\overline{B} \subset \overline{A \cup B}$, and thus also $\overline{A} \cup \overline{B} \subset \overline{A \cup B}$.

□

Simpler properties of interior are collected in

Proposition 5.13. *Let* $A, B \subset X$. *Then*

(a) A° *is an open set;*

(b) *if* $A \subset B$, *then* $A^\circ \subset B^\circ$*;*

(c) $(A^\circ)^\circ = A^\circ$*;*

(d) $(A \cap B)^\circ = A^\circ \cap B^\circ$.

PROOF.

Exercise 5.3. Prove Proposition 5.13

HINT. Use Exercise 5.2 together with Proposition 5.12.

□

Proposition 5.14. *Let $A \subset X$. Then*

$$\text{(a)}\ \ A^\circ = \bigcup_{\substack{G \subset A \\ G \text{ is open}}} G\,; \qquad\qquad \text{(b)}\ \ \overline{A} = \bigcap_{\substack{A \subset F \subset X \\ F \text{ is closed}}} F\,.$$

Proof.

Exercise 5.4. Prove Proposition 5.14

□

Corollary 5.15. *Let $A \subset X$.*

(a) *The interior A° is the biggest open set contained in A.*

(b) *The closure $\overline{A}$ is the smallest closed set containing A.*

Proof. (a). The interior A° is an open set by Proposition 5.13, (a). The interior A° is the biggest open set contained in A because, again by Proposition 5.14, (a), every open set contained in A is contained in A°.

(b). The closure $\overline{A}$ is a closed set by Proposition 5.12, (a). The closure $\overline{A}$ is the smallest closed set containing A because, again by Proposition 5.14, (b), every closed set containing A also contains $\overline{A}$. □

5.4. Additional exercises

Exercise 5.5. Let $a \in X$ and $r > 0$. Prove that

(a) $B(a,r)^\circ = B(a,r)$;

(b) $\overline{\overline{B}(a,r)} = \overline{B}(a,r)$;

(c) $\overline{B(a,r)} \subset \overline{B}(a,r)$;

(d) $\overline{B}(a,r)^\circ \supset B(a,r)$;

(e) $\partial B(a,r) \subset S(a,r)$;

(f) $\partial \overline{B}(a,r) \subset S(a,r)$.

Hint. For (a) and (b), use the facts that open balls are open sets and closed balls are closed sets.

Solution. (c). Since $B(a,r) \subset \overline{B}(a,r)$, one has, by Proposition 5.12, (a), and the closedness of $\overline{B}(a,r)$,

$$\overline{B(a,r)} \subset \overline{\overline{B}(a,r)} = \overline{B}(a,r).$$

(d). Since $\overline{B}(a,r) \supset B(a,r)$, one has, by Proposition 5.13, (a), and the openness of $B(a,r)$,

$$\overline{B}(a,r)^\circ \supset B(a,r)^\circ = B(a,r).$$

(e). It suffices to show that

$$\partial B(a,r) \cap B(a,r) = \varnothing \quad \text{and} \quad \partial B(a,r) \cap \big(X \backslash \overline{B}(a,r)\big) = \varnothing.$$

The first equality follows from the openness of $B(a,r)$. The second equality holds because, by (c),

$$\partial B(a,r) \subset \overline{B(a,r)} \subset \overline{B}(a,r).$$

(f). By the closedness of $\overline{B}(a,r)$,

$$\partial \overline{B}(a,r) \subset \overline{B}(a,r) = B(a,r) \cup S(a,r).$$

Since, by (d), $B(a,r) \subset \overline{B}(a,r)^\circ$, it follows that $\partial \overline{B}(a,r) \subset S(a,r)$.

Remark 5.3. The inclusions (c)–(f) in Exercise 5.5 are, in general, strict. In order to give the corresponding examples, it is, first, useful to solve the following Exercise 5.6.

Exercise 5.6. Prove that

(a) every set in a discrete metric space is open;

(b) every set in a discrete metric space is closed.

(c) the boundary of every set in a discrete metric space is empty.

Example 5.2. We give some examples of cases where the inclusions (c)–(f) in Exercise 5.5 are strict.

Let X be a discrete metric space containing at least two elements, and let $a \in X$. Then

- $\overline{B(a,1)} = B(a,1) = \{a\} \subsetneqq X = \overline{B}(a,1)$;
- $\overline{B}(a,1)^\circ = \overline{B}(a,1) = X \supsetneqq \{a\} = B(a,1)$;
- $\partial B(a,1) = \varnothing \subsetneqq X \backslash \{a\} = S(a,1)$;
- $\partial \overline{B}(a,1) = \varnothing \subsetneqq X \backslash \{a\} = S(a,1)$.

Remark 5.4. The following Exercise 5.7 shows that if $X \neq \{0\}$ is a normed space, then the inclusions (c)–(f) in Exercise 5.5 are actually equalities.

Exercise 5.7. Let $X \neq \{0\}$ be a normed space. Prove that

(a) $\partial B(a,r) = S(a,r)$;

(b) $\partial \overline{B}(a,r) = S(a,r)$;

(c) $\overline{B(a,r)} = \overline{B}(a,r)$;

(d) $\overline{B}(a,r)^\circ = B(a,r)$.

Exercise 5.8. Prove that, in a normed space,

(a) the closure of a subspace is a subspace;

(b) the closure of a convex set is convex.

Exercise 5.9. Prove that $\partial(\partial A) \subset \partial A$. It follows that *the boundary ∂A is a closed set.*

Exercise 5.10. Let $A \subset X$ and let $x \in X$. The distance of x from A is defined by

$$d(x,A) := \inf_{a \in A} \rho(x,a).$$

Prove that

(a) $d(x,\overline{A}) = d(x,A)$;

(b) $x \in \overline{A}$ if and only if $d(x,a) = 0$.

§ 6. Complete metric spaces

6.1. The notion of completeness

Definition 6.1. A sequence (x_n) in a metric space X is called a *Cauchy sequence* (or a *fundamental sequence*) if

$$\rho(x_n, x_m) \xrightarrow[n,m\to\infty]{} 0,$$

i.e., for every $\varepsilon > 0$, there is an index $N \in \mathbb{N}$ such that

$$n, m \geqslant N \quad \Longrightarrow \quad \rho(x_n, x_m) < \varepsilon$$

or, equivalently,

$$n, p \in \mathbb{N},\ n \geqslant N \quad \Longrightarrow \quad \rho(x_n, x_{n+p}) < \varepsilon.$$

Example 6.1. Denote, for all $n \in \mathbb{N}$,

$$e_n = (\underbrace{0, \dots, 0, 1}_{n}, 0, \dots).$$

The sequence $(e_n)_{n=1}^{\infty}$ is not a Cauchy sequence in ℓ_p for any $p \in [0, \infty]$ because, whenever $n < m$,

$$\|e_n - e_m\| = \|(\overbrace{\underbrace{0, \dots, 0, 1}_{n}, 0, \dots, 0, -1}^{m}, 0, \dots)\| = \begin{cases} 2^{\frac{1}{p}}, & \text{if } 1 \leqslant p < \infty; \\ 1, & \text{if } p = \infty. \end{cases}$$

Multiple examples of Cauchy sequences are provided by assertion (a) of the following

Proposition 6.1. (a) *Every convergent sequence is a Cauchy sequence.*

(b) *Every Cauchy sequence is bounded.*

(c) *If a Cauchy sequence has a convergent subsequence, then this sequence converges to the same limit as the subsequence.*

PROOF. Let (x_n) be a sequence in a metric space X.

(a). Suppose that $x_n \xrightarrow[n\to\infty]{} x$ for some $x \in X$. We must show that (x_n) is a Cauchy sequence, which is the case because

$$\rho(x_n, x_m) \leqslant \rho(x_n, x) + \rho(x, x_m) \xrightarrow[n,m\to\infty]{} 0.$$

(b). Let (x_n) be a Cauchy sequence. We must show that the sequence (x_n) is bounded, i.e., the set of its elements is contained in some ball, i.e., there exist $a \in X$ and $r \geqslant 0$ such that $x_n \in \overline{B}(a, r)$ for every $n \in \mathbb{N}$, i.e.,

$$\rho(x_n, a) \leqslant r \quad \text{for every } n \in \mathbb{N}. \tag{6.1}$$

Since (x_n) is a Cauchy sequence, there is an index $N \in \mathbb{N}$ such that

$$n, m \geqslant N \quad \Rightarrow \quad \rho(x_n, x_m) < 1.$$

In particular,

$$\rho(x_n, x_N) < 1 \quad \text{for every } n \geqslant N.$$

But now, (6.1) holds for

$$a := x_N \quad \text{and} \quad r := \max\bigl\{\rho(x_1, x_N), \ldots, \rho(x_{N-1}, x_N), 1\bigr\}.$$

(c). Suppose that (x_n) contains a convergent subsequence, say $(x_{k_n})_{n=1}^{\infty}$, and let $x_{k_n} \xrightarrow[n\to\infty]{} x \in X$. We must show that $x_n \xrightarrow[n\to\infty]{} x$ which is the case because

$$\rho(x_n, x) \leqslant \rho(x_n, x_{k_n}) + \rho(x_{k_n}, x) \xrightarrow[n\to\infty]{} 0$$

(here $\rho(x_n, x_{k_n}) \xrightarrow[n\to\infty]{} 0$ because (x_n) is a Cauchy sequence and $k_n \xrightarrow[n\to\infty]{} \infty$). □

Example 6.2. Non-convergent Cauchy sequences do exist.

Indeed, let X be the interval $(0, 1]$ in $\mathbb{R}$ equipped with the Euclidean distance, and let $x_n = \frac{1}{n} \in X$, $n \in \mathbb{N}$. Then the sequence (x_n) is a Cauchy sequence in X which does not converge in X.

To see this, observe that $x_n \to 0$ in $\mathbb{R}$, thus (x_n) is a convergent sequence in $\mathbb{R}$, hence a Cauchy sequence in $\mathbb{R}$ by Proposition 6.1, (a), and thus also a Cauchy sequence in X. If the sequence (x_n) were convergent in X, say $x_n \to x \in X$ in X, then also $x_n \to x$ in $\mathbb{R}$, and thus (x_n) would have two different limits—0 and x—in $\mathbb{R}$, a contradiction.

The previous example motivates the following

Definition 6.2. A metric space X is said to be *complete* if every Cauchy sequence in X converges (in X).

A complete normed space is called a *Banach space.*

Example 6.3. The space $\mathbb{R}$ is complete (with respect to the Euclidean distance).

Indeed, let (x_n) be a Cauchy sequence in $\mathbb{R}$. By Proposition 6.1, (b), the sequence (x_n) is bounded. Since, by the Bolzano–Weierstrass theorem, every bounded sequence of real numbers has a convergent subsequence, the sequence (x_n) converges by Proposition 6.1, (c).

Example 6.4. The space $\mathbb{C}$ is complete (with respect distance $\rho(x, y) = |x - y|$, $x, y \in \mathbb{C}$).

Indeed, let (x_n) be a Cauchy sequence in $\mathbb{C}$. For every $n \in \mathbb{N}$, let $a_n, b_n \in \mathbb{R}$ be, respectively, the real and imaginary part of x_n, i.e., $x_n = a_n + ib_n$. Since, for all $n, m \in \mathbb{N}$,

$$|x_n - x_m| = |(a_n - a_m) + i(b_n - b_m)| = \sqrt{|a_n - a_m|^2 + |b_n - b_m|^2},$$

one has

$$|a_n - a_m| \leqslant |x_n - x_m| \xrightarrow[n,m\to\infty]{} 0 \quad \text{and} \quad |b_n - b_m| \leqslant |x_n - x_m| \xrightarrow[n,m\to\infty]{} 0,$$

thus the sequences (a_n) and (b_n) are Cauchy sequences in $\mathbb{R}$, and, by the completeness of $\mathbb{R}$, the sequences (a_n) and (b_n) converge in $\mathbb{R}$, i.e., $x_n \xrightarrow[n\to\infty]{} x$ and $x_n \xrightarrow[n\to\infty]{} x$ in $\mathbb{R}$ for some $a, b \in \mathbb{R}$. Putting $x := a + ib$, for the completeness of $\mathbb{C}$, it remains to observe that $x_n \xrightarrow[n\to\infty]{} x$ in $\mathbb{C}$, because

$$|x_n - x| = |(a_n - a) + i(b_n - b)| = \sqrt{|a_n - a|^2 + |b_n - b|^2} \xrightarrow[n\to\infty]{} 0.$$

In fact, every metric space in § 5 is complete. In these notes, we shall prove the completeness for just a few of them.

Example 6.5. For every $n \in \mathbb{N}$ and $1 \leqslant p \leqslant \infty$, the space ℓ_p^n is complete.

Indeed, let $n \in \mathbb{N}$, let $1 \leqslant p \leqslant \infty$, and let $(x_k)_{k=1}^\infty = \big((\xi_j^k)_{j=1}^n\big)_{k=1}^\infty$ be a Cauchy sequence in ℓ_p^n. Then the sequence $(x_k)_{k=1}^\infty$ is bounded, i.e., there is some $M \geqslant 0$ such that $\|x_k\| \leqslant M$ for all $k \in \mathbb{N}$. By Remark 4.2, for every $j \in \{1, \dots, n\}$ and every $k \in \mathbb{N}$,

$$|\xi_j^k| \leqslant \|x_k\|_\infty \leqslant \|x_k\|_p \leqslant M,$$

thus the sequence $(\xi_j^k)_{k=1}^\infty$ is bounded in $\mathbb{K}$. By the Bolzano-Weierstrass theorem, every bounded sequence in $\mathbb{K}$ has a convergent subsequence, thus

(1) the sequence $(\xi_1^k)_{k=1}^\infty$ has a convergent subsequence $(\xi_1^{i_k^1})_{k=1}^\infty$;

(2) the sequence $(\xi_2^{i_k^1})_{k=1}^\infty$ has a convergent subsequence $(\xi_2^{i_k^2})_{k=1}^\infty$;

(3) the sequence $(\xi_3^{i_k^2})_{k=1}^\infty$ has a convergent subsequence $(\xi_3^{i_k^3})_{k=1}^\infty$;

and so on,

$(n-1)$ the sequence $(\xi_{n-1}^{i_k^{n-2}})_{k=1}^\infty$ has a convergent subsequence $(\xi_{n-1}^{i_k^{n-1}})_{k=1}^\infty$;

(n) the sequence $(\xi_n^{i_k^{n-1}})_{k=1}^\infty$ has a convergent subsequence $(\xi_n^{i_k^n})_{k=1}^\infty$.

Now, the subsequence $(x_{i_k^n})_{k=1}^\infty = \big((\xi_j^{i_k^n})_{j=1}^n\big)_{k=1}^\infty$ (of $(x_k)_{k=1}^\infty$) is coordinatewise convergent, and hence convergent in ℓ_p^n.

Example 6.6. The space ℓ_∞ is complete.

Indeed, let $(x_n)_{n=1}^\infty = \big((\xi_j^n)_{j=1}^\infty\big)_{n=1}^\infty$ be a Cauchy sequence in ℓ_∞. For the completeness of ℓ_∞, it suffices to show that the sequence (x_n) converges in ℓ_∞. To this end, observe that, for every $j \in \mathbb{N}$, the sequence of the j-th coordinates $(\xi_j^n)_{n=1}^\infty$ is a Cauchy sequence in $\mathbb{K}$ because

$$|\xi_j^n - \xi_j^m| \leqslant \|x_n - x_m\|_\infty \xrightarrow[n,m\to\infty]{} 0.$$

By the completeness of the space $\mathbb{K}$, for every $j \in \mathbb{N}$, the sequence $(\xi_j^n)_{n=1}^\infty$ converges in $\mathbb{K}$, say $\xi_j^n \xrightarrow[n\to\infty]{} \xi_j \in \mathbb{K}$. It now suffices to show that $x := (\xi_j) \in \ell_\infty$ and $x_n \xrightarrow[n\to\infty]{} x$ in ℓ_∞.

To this end, letting $\varepsilon > 0$ be arbitrary, it suffices to find an $N \in \mathbb{N}$ such that

$$n \geqslant N \quad \Longrightarrow \quad \|x_n - x\|_\infty = \sup_{j\in\mathbb{N}} |\xi_j^n - \xi_j| \leqslant \varepsilon. \tag{6.2}$$

Indeed, if one would have $x \in \ell_\infty$, then this would imply that $x_n \xrightarrow[n\to\infty]{} x$. But (6.2) also implies that, for $n \geqslant N$, one has $x_n - x \in \ell_\infty$ and thus also $x = x_n - (x_n - x) \in \ell_\infty$.

Since (x_n) is a Cauchy sequence, there is an $N \in \mathbb{N}$ such that

$$n, m \geqslant N \quad \Longrightarrow \quad \|x_n - x_m\|_\infty = \sup_{j\in\mathbb{N}} |\xi_j^n - \xi_j^m| < \varepsilon.$$

In particular, for all $j \in \mathbb{N}$,

$$n, m \geqslant N \quad \Longrightarrow \quad |\xi_j^n - \xi_j^m| < \varepsilon.$$

Letting $m \to \infty$, the latter implies (6.2).

Proposition 6.2. (a) *A complete subspace of any metric space is closed.*

(b) *A closed subspace of a complete metric space is complete.*

The following corollary is straightforward from Proposition 6.2.

Corollary 6.3. *A subspace of a complete metric space is complete if and only if it is closed.*

Proof of Proposition 6.2. Let X be a metric space.

(a). Let Y be a complete subspace of X. In order to show that Y is closed, letting (y_n) be a sequence of elements of Y converging to some $x \in X$ in X, (by Proposition 5.8) it suffices to show that $x \in Y$.

Since the sequence (y_n) converges in X, it is a Cauchy sequence in X and thus also a Cauchy sequence in Y. By the completeness of Y, the sequence (y_n) converges in Y to some $y \in Y$. But then the sequence (y_n) converges to y also in X, and, by the uniqueness of the limit, $x = y \in Y$, as desired.

(b). Assume that X is complete, and let Y be a closed subspace of X. In order to show that Y is complete, letting (y_n) be an arbitrary Cauchy sequence of elements of Y, it suffices to show that the sequence (y_n) converges in Y.

Since (y_n) is a Cauchy sequence also in X, by the completeness of X, the sequence (y_n) converges in X to some $x \in X$. Since Y is closed, one has $x \in Y$ (by Proposition 5.8), and thus the sequence (y_n) is convergent in Y, as desired. □

Example 6.7. The space c_0 is complete.

Indeed, since c_0 is a (normed) subspace of the complete space ℓ_∞, for the completeness of c_0, by Corollary 6.3, it suffices to show that c_0 is a closed subspace of ℓ_∞.

Let $x_n = (\xi_j^n)_{j=1}^\infty \in c_0$, $n \in \mathbb{N}$, and $x = (\xi_j)_{j=1}^\infty \in \ell_\infty$ be such that $x_n \xrightarrow[n\to\infty]{} x$ in ℓ_∞. For the closedness of c_0, it suffices to show that $x \in c_0$, i.e., $\xi_j \xrightarrow[j\to\infty]{} 0$, i.e., for every $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that

$$j \geqslant N \quad \Longrightarrow \quad |\xi_j| < \varepsilon.$$

Fix an arbitrary $\varepsilon > 0$. Observe that for all $j, n \in \mathbb{N}$ one has

$$|\xi_j| \leqslant |\xi_j - \xi_j^n| + |\xi_j^n| \leqslant \|x - x_n\| + |\xi_j^n|.$$

Since $x_n \xrightarrow[n\to\infty]{} x$, we can fix an $n \in \mathbb{N}$ so that $\|x - x_n\| < \frac{\varepsilon}{2}$. Since $x_n \in c_0$, one has $\xi_j^n \xrightarrow[j\to\infty]{} 0$, thus there is an $N \in \mathbb{N}$ such that

$$j \geqslant N \quad \Longrightarrow \quad |\xi_j^n| < \frac{\varepsilon}{2}.$$

Now, whenever $j \geqslant N$, one has

$$|\xi_j| \leqslant \|x - x_n\| + |\xi_j^n| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

6.2. Principle of nested balls

The following characterization of complete metric spaces will be used in the next subsection in the proof of the fundamental *Baire's theorem* 6.6. However, it is not without interest in its own right.

Theorem 6.4 (Principle of nested balls)**.** *Let X be a metric space. The following assertions are equivalent:*

(i) *X is complete;*

(ii) *every nested sequence of closed balls in X whose radii tend to zero have a non-empty intersection, i.e., whenever*

$$\overline{B}(x_1, r_1) \supset \overline{B}(x_2, r_2) \supset \cdots \supset \overline{B}(x_n, r_n) \supset \overline{B}(x_{n+1}, r_{n+1}) \supset \cdots \tag{6.3}$$

are closed balls in X with $r_n \xrightarrow[n\to\infty]{} 0$, one has $\bigcap_{n=1}^{\infty} \overline{B}(x_n, r_n) \neq \varnothing$.

Remark 6.1. Cantor's nested interval's theorem says that, whenever $I_1 \supset I_2 \supset I_3 \supset \cdots$ are closed intervals in $\mathbb{R}$ whose lengths tend to 0, the intersection $\bigcap_{n=1}^{\infty} I_n$ consists of exactly one point. By Theorem 6.4, this is the other way to say that $\mathbb{R}$ is complete. (Remark that the completeness of $\mathbb{R}$ is usually proven (see Example 6.4) using the Bolzano-Weierstrass theorem, whose proof, in turn, relies on Cantor's nested interval's theorem.)

An important observation regarding the Principle of nested balls is

Proposition 6.5. *Let X be a metric space, let $X \supset D_1 \supset D_2 \supset D_3 \supset \cdots$ satisfy $d_n := \operatorname{diam} D_n \xrightarrow[n\to\infty]{} 0$, and let $x \in \bigcap_{n=1}^{\infty} D_n$. Then*

(a) *$\bigcap_{n=1}^{\infty} D_n = \{x\}$, i.e., x is the only point in $\bigcap_{n=1}^{\infty} D_n$;*

(b) *whenever $x_n \in D_n$, one has $x_n \xrightarrow[n\to\infty]{} x$.*

Proof.

Exercise 6.1. Prove Proposition 6.5.

□

Proof of Theorem 6.4. (i)⇒(ii). Assume that X is complete, and let (6.3) be closed balls in X whose radii tend to 0. Then the sequence (x_n) of the centers of the balls is a Cauchy sequence, because, for $n, m \in \mathbb{N}$, $n < m$, one has $x_m \in \overline{B}(x_n, r_n)$ and thus

$$\rho(x_n, x_m) \leqslant r_n \xrightarrow[n\to\infty]{} 0.$$

By the completeness of X, the sequence (x_n) converges to some $x \in X$. But now, $x \in \bigcap_{n=1}^{\infty} \overline{B}(x_n, r_n)$ because, for every $n \in \mathbb{N}$, one has $x_m \in \overline{B}(x_n, r_n)$ whenever $m \geqslant n$, and thus, by the closedness of the ball $\overline{B}(x_n, r_n)$, also the limit $x \in \overline{B}(x_n, r_n)$.

(ii)⇒(i). Assume that (ii) holds and let (x_n) be a Cauchy sequence in X. For the completeness of X, we must show that (x_n) converges in X for which, by Proposition 6.1, (c), it suffices to show that (x_n) contains a convergent subsequence.

Our idea is to find indices $k_1 < k_2 < k_3 < \cdots$ and real numbers $r_n > 0$, $n \in \mathbb{N}$, so that that the closed balls $\overline{B}(x_{k_n}, r_n)$, $n \in \mathbb{N}$, are nested and $r_n \xrightarrow[n\to\infty]{} 0$—in this case, by assumption (ii) and Proposition 6.5, the sequence $(x_{k_n})_{n=1}^{\infty}$ converges to the unique element in $\bigcap_{n=1}^{\infty} \overline{B}(x_{k_n}, r_n)$.

To this end, choose indices $k_1 < k_2 < k_3 < \cdots$ so that, for all $n \in \mathbb{N}$,

$$l, m \geqslant k_n \quad \Longrightarrow \quad \rho(x_l, x_m) < \frac{1}{2^n}.$$

It remains to observe that the balls $\overline{B}(x_{k_n}, \frac{1}{2^{n-1}})$, $n \in \mathbb{N}$, are nested: whenever $n \in \mathbb{N}$ and $x \in \overline{B}(x_{k_{n+1}}, \frac{1}{2^n})$, one also has $x \in \overline{B}(x_{k_n}, \frac{1}{2^{n-1}})$ because

$$\rho(x, x_{k_n}) \leqslant \rho(x, x_{k_{n+1}}) + \rho(x_{k_{n+1}}, x_{k_n}) \leqslant \frac{1}{2^n} + \frac{1}{2^n} = \frac{1}{2^{n-1}}.$$

□

6.3. Baire's theorem

Theorem 6.6 below may seem quite useless at first glance. However, it is one of the cornerstones of the theory complete metric spaces—e.g., a number of results on operators between Banach spaces rely on Baire's theorem.

Definition 6.3. A subset A of a metric space is said to be *dense* (in X) if its closure $\overline{A} = X$.

Theorem 6.6 (Baire's theorem). (a) *A countable intersection of open dense sets in a complete metric space is dense.*

(b) *If a complete non-empty metric space is represented as a countable union of closed sets, then at least one of these sets contains a ball.*

Both assertions (a) and (b) of Theorem 6.6 are referred to as Baire's theorem. In fact, it is easy to derive (b) from (a) (this is exactly how we prove (b) below) and, vice versa, it is not much more difficult to derive (a) from (b).

Before proving Baire's theorem, it may be helpful to clarify the relationship of some of its ingredients.

Exercise 6.2. Let X be a metric space and let $A \subset X$. Prove that the following assertions are equivalent:

(i) A is dense;

(ii) $X \backslash A$ contains no balls.

Now we are in a position to prove Baire's theorem.

Proof of Theorem 6.6. Let X be a complete metric space.

(a). Let $G_n \subset X$, $n \in \mathbb{N}$, be open dense sets, and let B be an open ball in X. In order for the intersection $\bigcap_{n=1}^{\infty} G_n$ to be dense, by Exercise 6.2, it suffices to show that $B \cap \bigcap_{n=1}^{\infty} G_n \neq \varnothing$. To this end, choose $x_1 \in X$ and $r_1 \in (0, 1)$ so that

$$\overline{B}(x_1, r_1) \subset G_1 \cap B$$

(this is possible because $G_1 \cap B$ is open and, since G_1 is dense, $G_1 \cap B \neq \varnothing$ by Exercise 6.2) and proceed by induction: given $n \in \mathbb{N}$, and closed balls

$$\overline{B}(x_1, r_1) \supset \overline{B}(x_2, r_2) \supset \cdots \supset \overline{B}(x_n, r_n),$$

choose $x_{n+1} \in X$ and $r_{n+1} \in \left(0, \frac{1}{n+1}\right)$ so that

$$\overline{B}(x_{n+1}, r_{n+1}) \subset G_{n+1} \cap B(x_n, r_n)$$

(this is possible because $G_{n+1} \cap B(x_n, r_n)$ is open and, since G_{n+1} is dense, $G_{n+1} \cap B(x_n, r_n) \neq \varnothing$ by Exercise 6.2). By Theorem 6.4, there exists an

$$x \in \bigcap_{n=1}^{\infty} \overline{B}(x_n, r_n) \subset B \cap \bigcap_{n=1}^{\infty} G_n.$$

(b). Let $X = \bigcup_{n=1}^{\infty} F_n$ where $F_n \subset X$, $n \in \mathbb{N}$, are closed sets. Suppose for contradiction that no F_n contains a ball. Then, by Exercise 6.2, the completion $G_n := X \backslash F_n$ is dense for every $n \in \mathbb{N}$. By (a), also the intersection $\bigcap_{n=1}^{\infty} G_n$ is dense, hence, by Exercise 6.2, its completion $X \backslash \bigcap_{n=1}^{\infty} G_n$ contains no balls. But, by De Morgan's law,

$$X \backslash \bigcap_{n=1}^{\infty} G_n = \bigcup_{n=1}^{\infty} X \backslash G_n = \bigcup_{n=1}^{\infty} F_n = X,$$

hence X does not contain any balls, a contradiction. □

Exercise 6.3. Prove that the intersection of two open dense sets in any metric space (not necessarily complete!) is again dense.

6.4. Completion of a metric space

Example 6.2 suggests that the reason why a Cauchy sequence in a non-complete metric space may fail to converge is that it has nowhere to converge—a non-complete space does not have enough elements to ensure that every Cauchy sequence has a limit. In this subsection, we observe that this flaw can, in a sense, be removed: any metric space can be nicely embedded into a complete metric space as a dense subspace.

Definition 6.4. Let (X, ρ_X) and (Y, ρ_Y) be metric spaces. A bijection $f\colon X \to Y$ is called an *isometry* if

$$\rho_Y\big(f(x), f(u)\big) = \rho_X(x, u) \quad \text{for all } x, u \in X. \tag{6.4}$$

In this case, one says that X and Y are *isometric* (or Y is isometric to X)

In other words, an isometry between metric spaces is a bijection which preserves the distance between elements. Clearly, Y is isometric to X if and only if X is isometric to Y.

Exercise 6.4. Let (X, ρ_X) and (Y, ρ_Y) be metric spaces. Prove that a mapping $f\colon X \to Y$ satisfying (6.4) is an injection. It follows that a surjection $f\colon X \to Y$ satisfying (6.4) is an isometry.

Exercise 6.5. Let X, Y and Z be metric spaces such that X and Y are isometric, and Y and Z are isometric. Prove that X and Z are isometric.

Definition 6.5. Let X be a metric space. A metric space Z is said to be a *completion* of X if there exists a subspace Y of Z such that

(1) Y is dense in Z;

(2) Y and X are isometric.

Proposition 6.7. *Let X be a subspace of a complete metric space Z. Then the closure $\overline{X}$ is a completion of X.*

PROOF. First observe that the closed subspace $\overline{X}$ of the complete space Z is complete. Since X is isometric to itself considered as a subspace of $\overline{X}$, and X is dense in $\overline{X}$, the closure $\overline{X}$ is a completion of X by definition. □

Theorem 6.8. *For every metric space X, there exists a completion Z. Any two completions of X are isometric.*

PROOF. Let (X, ρ) be a metric space. Define an equivalence relation $\sim$ in the set

$$\mathcal{X} := \Big\{(x_n) := (x_n)_{n=1}^{\infty} \colon\ (x_n) \text{ is a Cauchy sequence in } X\Big\}$$

of all Cauchy sequences in X by

$$(x_n) \sim (y_n) \quad :\Longleftrightarrow \quad \rho(x_n, y_n) \xrightarrow[n\to\infty]{} 0, \qquad (x_n), (y_n) \in \mathcal{X}.$$

Exercise 6.6. Prove that ρ is an equivalence relation.

For an element $(x_n) \in \mathcal{X}$, denote its equivalence class in the quotient space $Z := \mathcal{X}/\!\sim$ by $[(x_n)]$. Define a metric $\overline{\rho}$ in Z by

$$\overline{\rho}\Big([(x_n)], [(y_n)]\Big) = \lim_{n\to\infty} \rho(x_n, y_n), \qquad (x_n), (y_n) \in \mathcal{X}. \tag{6.5}$$

Exercise 6.7. Prove that $\overline{\rho}$ is a metric.

HINT. First show that $\overline{\rho}$ is well-defined, i.e.

(1) the limit in (6.5) exists (to this end, using the quadrangle inequality (see Proposition 1.1,(b)), show that the sequence $\big(\rho(x_n, y_n)\big)_{n=1}^{\infty}$ is Cauchy);

(2) the limit in (6.5) does not depend on the choice of the representatives (x_n) and (y_n) in the equivalence classes of $[(x_n)]$ and $[(y_n)]$, i.e., whenever $(x_n), (u_n), (y_n), (v_n) \in \mathcal{X}$ are such that $[(u_n)] = [(x_n)]$ and $[(v_n)] = [(y_n)]$ (i.e., $(u_n) \sim (x_n)$ and $(v_n) \sim (y_n)$), one has

$$\lim_{n\to\infty} \rho(u_n, v_n) = \lim_{n\to\infty} \rho(x_n, y_n).$$

Define

$$Y := \Big\{[(x, x, x, \dots)] \colon\ x \in X\Big\} \subset Z,$$

i.e., Y is the subspace of Z consisting of equivalence classes of constant sequences in X (note that constant sequences in X do converge, thus they are Cauchy, and therefore constant sequences in X belong to $\mathcal{X}$).

Observe that

(I) Y is dense in Z;

(II) X and Y are isometric.

Exercise 6.8. Prove the assertions (I) and (II).

Hint. For (II), show that the mapping

$$f\colon X \ni x \longmapsto [(x, x, x, \dots)] \in Y$$

is an isometry.

In order to see that Z is a completion of X, it remains to show that

(III) the metric space $(Z, \overline{\rho})$ is a complete.

To this end, let let $(a_k)_{k=1}^{\infty}$ be a Cauchy sequence in Z. For the completeness of Z, it suffices to show that the sequence $(a_k)_{k=1}^{\infty}$ converges in Z.

Since Y is dense in Z, for every $k \in \mathbb{N}$, there is a $b_k \in Y$ such that

$$\overline{\rho}(a_k, b_k) < \frac{1}{k}.$$

For every $k \in \mathbb{N}$,

$$\begin{aligned} a_k &= [(x_n^k)_{n=1}^{\infty}] \quad \text{for some Cauchy sequence } (x_n^k)_{n=1}^{\infty} \text{ in } X; \\ b_k &= [(x_k, x_k, x_k, \dots)] \quad \text{for some } x_k \in X. \end{aligned}$$

Observe that the sequence $(x_k)_{k=1}^{\infty}$ in X is Cauchy. Indeed, whenever $k.l \in \mathbb{N}$, one has, for every $n \in \mathbb{N}$,

$$\rho(x_k, x_l) \leqslant \rho(x_k, x_n^k) + \rho(x_n^k, x_n^l) + \rho(x_n^l, x_l).$$

Since

$$\rho(x_k, x_n^k) \xrightarrow[n\to\infty]{} \overline{\rho}(b_k, a_k) < \frac{1}{k}, \tag{6.6}$$

$$\rho(x_n^k, x_n^l) \xrightarrow[n\to\infty]{} \overline{\rho}(a_k, a_l), \qquad \text{and} \qquad \rho(x_n^l, x_l) \xrightarrow[n\to\infty]{} \overline{\rho}(a_l, b_l) < \frac{1}{l},$$

it follows that (since $(a_k)_{k=1}^{\infty}$ is Cauchy)

$$\rho(x_k, x_l) \leqslant \frac{1}{k} + \overline{\rho}(a_k, a_l) + \frac{1}{l} \xrightarrow[k,l\to\infty]{} 0,$$

and thus $(x_k)_{k=1}^{\infty}$ is a Cauchy sequence in X.

Putting $a := [(x_n)_{n=1}^{\infty}] \in Z$, we are going to show that $a_k \xrightarrow[k\to\infty]{} a$ in Z, i.e., $\overline{\rho}(a_k, a) \xrightarrow[k\to\infty]{} 0$, i.e.,

$$\lim_{n\to\infty} \rho(x_n^k, x_n) \xrightarrow[k\to\infty]{} 0. \tag{6.7}$$

To this end, let $\varepsilon > 0$, and observe that, for all $k, n \in \mathbb{N}$,

$$\rho(x_n^k, x_n) \leqslant \rho(x_n^k, x_k) + \rho(x_k, x_n).$$

By (6.6), there is an $N_1 \in \mathbb{N}$ such that

$$n \geqslant N_1 \quad \Longrightarrow \quad \rho(x_n^k, x_k) < \frac{1}{k}.$$

Since the sequence $(x_n)_{n=1}^{\infty}$ is Cauchy, there is an $N_2 \in \mathbb{N}$ such that

$$k, n \geqslant N_2 \quad \Longrightarrow \quad \rho(x_k, x_n) < \frac{\varepsilon}{2}.$$

Thus, whenever $k \geqslant \max\{\frac{2}{\varepsilon}, N_2\}$, one has, for $n \geqslant \max\{N_1, N_2\}$,

$$\rho(x_n^k, x_n) < \frac{1}{k} + \frac{\varepsilon}{2} \leqslant \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

hence also $\lim_{n\to\infty} \rho(x_n^k, x_n) \leqslant \varepsilon$, and (6.7) follows.

It remains to show that

(IV) any two completions of X are isometric.

To this end, it suffices to prove

Claim. *Let Z and W be complete metric spaces. Suppose that there exist dense subspaces Y of Z and V of W such that Y and V are isometric. Then Z and W are isometric.*

To see that Claim implies (IV), let Z and W be two completions of X, and let Y and V be dense subspaces of Z and W, respectively, such that both Y and Z are isometric to X. Then Y and V are isometric (by Exercise 6.5), and Z and W are isometric by Claim.

Exercise 6.9. Prove Claim.

Hint. Letting $g\colon Y \to V$ be an isometry, define a mapping $f\colon Z \to W$ as follows. Whenever $z \in Z$, let (y_n) be a sequence in Y such that $z = \lim_{n\to\infty} y_n$ (such a sequence (y_n) exists because Y is dense in Z), and define $f(z) := \lim_{n\to\infty} g(y_n) \in W$ (observe that, since the convergent sequence (y_n) is Cauchy, also the sequence $\big(g(y_n)\big)$ is Cauchy (because g is an isometry), and thus the sequence $\big(g(y_n)\big)$ converges in W by the completeness of W). In order to see that f is well-defined, one has to show that its value $f(z) \in W$ for $z \in Z$ does not depend on the choice of the sequence (y_n) in Y converging to Y, i.e., one has to show that whenever (y_n) and $(\hat{y}_n)$ are two sequences in Y such that $\lim_{n\to\infty} y_n = \lim_{n\to\infty} \hat{y}_n$, one has $\lim_{n\to\infty} g(y_n) = \lim_{n\to\infty} g(\hat{y}_n)$.

It remains to show that f is an isometry.

□

§ 7. Continuous operators between metric spaces

If X and Y are metric spaces, then mappings $X \to Y$ are called *operators*. Mappings $X \to \mathbb{K}$ are called *functionals.*

7.1. The notion of continuity

Definition 7.1. Let X and Y be metric spaces. An operator $f\colon X \to Y$ is said to be *continuous at a point* $a \in X$ if, for every $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$\rho(x, a) < \delta \quad \Longrightarrow \quad \rho\big(f(x), f(a)\big) < \varepsilon. \tag{7.1}$$

Equivalently, (7.1) means that $f\big[B(a, \delta)\big] \subset B\big(f(a), \varepsilon\big)$.Thus, the definition of continuity can be rephrased as follows: an operator $f\colon X \to Y$ is said to be *continuous at a point* $a \in X$ if, for every neighbourhood V of $f(a)$, there exists a neighbourhood U of a such that $f[U] \subset V$.

Loosely speaking, f is continuous at a if the values of f at points which are close enough to a are as close as we want to $f(a)$.

Theorem 7.1 (Heine's criterion of continuity)**.** *Let X and Y be metric spaces, let $f\colon X \to Y$, and let $a \in X$. The following assertions are equivalent:*

(i) *f is continuous at a;*

(ii) *whenever a sequence of elements of X converges to a in X, the corresponding sequence of values of f converges to $f(a)$ in Y, i.e.,*

$$x_n \in X,\ n \in \mathbb{N},\ x_n \xrightarrow[n\to\infty]{} a \qquad \Longrightarrow \qquad f(x_n) \xrightarrow[n\to\infty]{} f(a).$$

Proof. (i)$\Rightarrow$(ii). Let f be continuous at a, and let $x_n \in X$, $n \in \mathbb{N}$, be such that $x_n \xrightarrow[n\to\infty]{} a$. We must show that $f(x_n) \xrightarrow[n\to\infty]{} f(a)$. To this end, letting $\varepsilon > 0$ be arbitrary, it suffices to find an index $N \in \mathbb{N}$ such that

$$n \geqslant N \quad \Longrightarrow \quad \rho\big(f(x_n), f(a)\big) < \varepsilon. \tag{7.2}$$

By the continuity of f at a, there is a $\delta > 0$ satisfying (7.1). Since $x_n \xrightarrow[n\to\infty]{} a$, there is an $N \in \mathbb{N}$ such that

$$n \geqslant N \qquad \Longrightarrow \qquad \rho(x_n, a) < \delta.$$

For this N, the implication (7.2) holds.

(ii)$\Rightarrow$(i). Let (ii) hold and suppose that f is not continuous at a. Then there is an $\varepsilon > 0$ such that no $\delta > 0$ satisfies (7.1). Thus, for every $n \in \mathbb{N}$, there is an $x_n \in X$ such that

$$\rho(x_n, a) < \frac{1}{n} \quad \text{and} \quad \rho\big(f(x_n), f(a)\big) \geqslant \varepsilon.$$

But now

$$x_n \xrightarrow[n\to\infty]{} a \quad \text{and} \quad f(x_n) \not\xrightarrow[n\to\infty]{} f(a)$$

which contradicts (ii). $\square$

Exercise 7.1. Let X, Y, and Z be metric spaces, let $f\colon X \to Y$ be contimuous at a point $a \in X$, and let $g\colon Y \to Z$ be continuous at the point $f(a) \in Y$. Prove that the composition operator

$$gf\colon X \ni x \longmapsto g\big(f(x)\big) \in Z$$

is continuous at a.

Definition 7.2. Let X and Y be metric spaces. An operator $f\colon X \to Y$ is said to be *continuous,* if it is continuous at every point $x \in X$.

Theorem 7.2. *Let X and Y be metric spaces. An operator $f\colon X \to Y$ is continuous if and only if, for every open set V in Y, its preimage $f^{-1}[V] := \big\{x \in X\colon f(x) \in V\big\}$ is an open set in X.*

Proof. *Necessity.* Let f be continuous, and let V be an open subset of Y. Letting $x \in f^{-1}[V]$ be arbitrary, it suffices to show that $B(x,\delta) \subset f^{-1}[V]$ for some $\delta > 0$. Since V is open, $f(x) \in V$ is an interior point of V, thus there is an $\varepsilon > 0$ such that $B\big(f(x),\varepsilon\big) \subset V$. By the continuity of f at x, there is a $\delta > 0$ satisfying $f\big[B(x,\delta)\big] \subset B\big(f(x),\varepsilon\big)$. But now

$$B(x,\delta) \subset f^{-1}\left[B\big(f(x),\varepsilon\big)\right] \subset f^{-1}[V].$$

Sufficiency. Assume that, for every open set in Y, its preimage with respect to f is open in X, and let $x \in X$. In order for X to be continuous, it suffices to show that f is continuous an x. To this end, letting $\varepsilon > 0$ be arbitrary, it suffices to show that there is a $\delta > 0$ such that $B(x,\delta) \subset f^{-1}\left[B\big(f(x),\varepsilon\big)\right]$. By our assumption, the preimage $f^{-1}\left[B\big(f(x),\varepsilon\big)\right]$ of the open ball $B\big(f(x),\varepsilon\big)$ is an open set, thus $x \in f^{-1}\left[B\big(f(x),\varepsilon\big)\right]$ is its interior point, hence $B(x,\delta) \subset f^{-1}\left[B\big(f(x),\varepsilon\big)\right]$ for some $\delta > 0$, as desired. □

Corollary 7.3. *Let X and Y be metric spaces. An operator $f\colon X \to Y$ is continuous if and only if, for every closed set H in Y, its preimage $f^{-1}[H]$ is a closed set in X.*

Proof. First observe that, for any subset D of Y, one has

$$f^{-1}[Y \backslash D] = X \backslash f^{-1}[D]. \tag{7.3}$$

Exercise 7.2. Prove the equality (7.3).

Now, since open sets are complements of closed sets and complements of closed sets are open,

$$\begin{aligned} f \text{ is continuous} \quad &\iff \quad f^{-1}[V] \text{ is an open set for every open set } V \subset Y \\ &\iff \quad f^{-1}[Y\backslash H] \text{ is an open set for every closed set } H \subset Y \\ &\iff \quad X\backslash f^{-1}[H] \text{ is an open set for every closed set } H \subset Y \\ &\iff \quad f^{-1}[H] \text{ is a closed set for every closed set } H \subset Y. \end{aligned}$$

□

Remark 7.1. In general,

- the image of an open set under a continuous function need not be open;
- the image of a closed set under a continuous function need not be closed.

Example 7.1. The natural inclusion map

$$j\colon\ [0,1) \ni x \longmapsto x \in \mathbb{R}$$

where both $[0,1)$ and $\mathbb{R}$ are equipped with the Euclidean metric is obviously continuous. The set $[0,1)$ is both open and closed in the domain space $[0,1)$ while its image $j\big[[0,1)\big] = [0,1)$ is neither open nor closed in the range space $\mathbb{R}$.

Example 7.2. Let X be any metric space and let $z \in \mathbb{R}$. The constant mapping

$$f\colon\ X \ni x \longmapsto z \in \mathbb{R}$$

is obviously continuous. The domain space X is an open set in itself while its image $f[X] = \{z\}$ is not open in $\mathbb{R}$.

7.2. Lipschitz condition

An important class of continuous operators is comprised by

Definition 7.3. Let X and Y be metric spaces. An operator $f\colon\ X \to Y$ is said to satisfy the *Lipschitz condition* if there is an $L \geqslant 0$ such that

$$\rho\big(f(x), f(z)\big) \leqslant L\,\rho(x,z) \quad \text{for all } x, z \in X.$$

In this case one also says that f is a *Lipschitz function.* The constant L is called a *Lipschitz constant* for f.

Proposition 7.4. *Every Lipschitz operator is continuous.*

Proof. Let X and Y be metric spaces, let $f\colon\ X \to Y$ be a Lipschitz function with Lipschitz constant L, and let $x \in X$. It suffices to show that f is continuous at x. To this end, letting $x_n \in X$, $n \in \mathbb{N}$, be such that $x_n \xrightarrow[n\to\infty]{} x$, it suffices to show that $f(x_n) \xrightarrow[n\to\infty]{} f(x)$. The latter holds because

$$\rho\big(f(x_n), f(x)\big) \leqslant L\rho(x_n, x) \xrightarrow[n\to\infty]{} 0.$$

□

§ 8. Banach fixed point theorem

Throughout this section, X will be a metric space.

Definition 8.1. A point $x_0 \in X$ is called a *fixed point* of an operator $f\colon X \to X$ if $f(x_0) = x_0$.

In other words, $x_0 \in X$ is a fixed point of f if it is a solution of the equation $f(x) = x$.

Definition 8.2. An operator $f\colon X \to X$ is called a *contraction* if there exists a non-negative $q < 1$ such that

$$\rho\big(f(x), f(z)\big) \leqslant q\,\rho(x, z) \quad \text{for all } x, z \in X. \tag{8.1}$$

In other words, a contraction is a mapping in a metric space satisfying the Lipschitz condition with constant < 1. In particular, it follows that any contraction is continuous.

The following *Banach fixed point theorem* is also often referred to as the *Banach fixed point principle* or the *contraction principle.*

Theorem 8.1 (The Banach fixed point principle)**.** *A contraction in a complete metric space has exactly one fixed point.*

Proof. Let X be a complete metric space, and let $f\colon X \to X$ be a contraction, i.e., there is a $q \in (0, 1)$ satisfying (8.1).

Letting $x_0 \in X$ be arbitrary, inductively define a sequence $(x_n) = (x_n)_{n=1}^{\infty}$ in X by setting

$$x_n := f(x_{n-1}) \quad \text{for every } n \in \mathbb{N}.$$

It suffices to show that

(1) (x_n) is a Cauchy sequence (and thus, by the completeness of X, a convergent in X sequence);

(2) putting $z_0 := \lim_{n\to\infty} x_n$, the point z_0 is a fixed point of f, i.e., $f(z_0) = z_0$;

(3) z_0 is the only fixed point of f.

(1). Whenever $n, p \in \mathbb{N}$,

$$\rho(x_n, x_{n+p}) \leqslant \rho(x_n, x_{n+1}) + \rho(x_{n+1}, x_{n+2}) + \cdots + \rho(x_{n+p-1}, x_p) = \sum_{j=n}^{n+p-1} \rho(x_j, x_{j+1}).$$

For every $j \in \mathbb{N}$,

$$\begin{aligned}
\rho(x_j, x_{j+1}) &= \rho\big(f(x_{j-1}), f(x_j)\big) \\
&\leqslant q\rho(x_{j-1}, x_j) = q\rho\big(f(x_{j-2}), f(x_{j-1})\big) \\
&\leqslant q^2\rho(x_{j-2}, x_{j-1}) = q^2\rho\big(f(x_{j-3}), f(x_{j-2})\big) \\
&\cdots\cdots\cdots\cdots \\
&\leqslant q^j\rho(x_0, x_1),
\end{aligned}$$

thus, putting $a := \rho(x_0, x_1)$,

$$\rho(x_n, x_{n+p}) \leqslant \sum_{j=n}^{n+p-1} q^j\, a = a \sum_{j=n}^{n+p-1} q^j \leqslant a\, q^n \sum_{i=0}^{\infty} q^i = \frac{a\, q^n}{1-q} \xrightarrow[n\to\infty]{} 0,$$

and it follows that (x_n) is a Cauchy sequence.

Indeed, letting $\varepsilon > 0$ be arbitrary, one can pick $N \in \mathbb{N}$ so that

$$n \geqslant N \quad \Longrightarrow \quad \frac{a\, q^n}{1-q} < \varepsilon,$$

but now also $\rho(x_n, x_{n+p}) < \varepsilon$ whenever $n, p \in \mathbb{N}$ with $n \geqslant N$.

(2). By the continuity of f,

$$f(z_0) = f\bigl(\lim_{n\to\infty} x_n\bigr) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} x_{n+1} = z_0.$$

(3). In order to prove that z_0 is the only fixed point of f, letting $z \in X$ be any fixed point of X, i.e., $f(z) = z$, it suffices to show that $z = z_0$. One has

$$0 \leqslant \rho(z, z_0) = \rho\bigl(f(z), f(z_0)\bigr) \leqslant q\rho(z, z_0)$$

which, since $0 \leqslant q < 1$, implies that $\rho(z, z_0) = 0$, i.e., $z = z_0$, as desired. □

§ 9. Compact sets in metric spaces

9.1. The notion of compactness.

Let X be a metric space and let K be a subset of X.

Definition 9.1. The subset K is said to be *relatively compact* if every sequence of its elements contains a convergent subsequence.

Observe that the convergent subsequence in the preceding definition need not converge to an element of K.

Definition 9.2. The subset K is said to be *compact* if it is relatively compact and closed.

Proposition 9.1. *Let X be a metric space and let K be a subset of X. The following assertions are equivalent.*

(i) *K is compact;*

(ii) *every sequence of elements of K has a subsequence which converges to an element of K.*

PROOF.

Exercise 9.1. Prove Proposition 9.1.

□

Definition 9.3. A metric space X is said to be compact if it is a compact subset of itself, i.e., every sequence in X contains a convergent subsequence.

Since any metric space X is a closed subset of itself, for X the notions of relative compactness and compactness coincide.

Proposition 9.2. *A compact metric space is complete*

PROOF. Let X be a compact metric space. For the completeness of X, we must show that every Cauchy sequence in X is convergent. So, let (x_n) be a Cauchy sequence in X. By the compactness of X the sequence (x_n) has a convergent subsequence, and it follows that the sequence (x_n) is convergent itself (because, by Proposition 6.1, (c), whenever a Cauchy sequence has a convergent subsequence, the sequence is convergent itself). □

Example 9.1. Every finite set in a metric space is compact.

Indeed, given a sequence of elements of a finite set in a metric space, at least one of the elements of this finite set must occur infinitely many times in this sequence, thus the sequence has a constant subsequence whose members are equal to this element of this set, and this constant subsequence converges to this element of this set.

Example 9.2. Whenever $n \in \mathbb{N}$ and $1 \leqslant p \leqslant \infty$, every bounded subset of ℓ_p^n is relatively compact.

Indeed, in Example 6.5, we essentially proved that every bounded sequence in ℓ_p^n ($n \in \mathbb{N}$, $1 \leqslant p \leqslant \infty$) has a bounded subsequence, and it easily follows that bounded sets in ℓ_p^n are relatively compact.

Proposition 9.3. *A relatively compact set is bounded.*

Proof. Let X be a metric space and let A be a relatively compact subset in X. Suppose, for contradiction, that A is not bounded. Then, letting $a \in A$ being arbitrary, for every $n \in \mathbb{N}$, there is an $x_n \in A \backslash B(a, n)$ (because, otherwise, we would have $A \subset B(a, n)$ and thus A would be bounded, a contradiction). Since A is relatively compact, the sequence $(x_n)_{n=1}^{\infty}$ in A has a convergent subsequence, say $(x_{k_n})_{n=1}^{\infty}$. Letting $x := \lim_{n\to\infty} x_n$, one has, by the continuity of the metric (Proposition 1.2, (d)),

$$\rho(x_n, a) \xrightarrow[n\to\infty]{} \rho(x, a).$$

One the other hand, since $x_n \notin B(a, n)$ for every $n \in \mathbb{N}$,

$$\rho(x_n, a) \geqslant n \xrightarrow[n\to\infty]{} \infty,$$

and thus $\rho(x_n, a) \xrightarrow[n\to\infty]{} \infty$, a contradiction. □

Corollary 9.4. *Let $n \in \mathbb{N}$ and $1 \leqslant p \leqslant \infty$.*

(a) *A subset of ℓ_p^n is relatively compact if and only if it is bounded.*

(b) *A subset of ℓ_p^n is compact if and only if it is bounded and closed.*

9.2. Hausdorff's theorem

Definition 9.4. Let X be a metric space, let A and B be subsets of X, and let $\varepsilon > 0$.

The set B is said to be an *ε-net* for A if

$$A \subset \bigcup_{b\in B} B(b, \varepsilon),$$

i.e., for every $a \in A$, there is a $b \in B$ such that $\rho(b, a) < \varepsilon$.

Theorem 9.5 (Hausdorff's theorem). *Let A be a subset of a metric space X. In order for A to be relatively compact, it is necessary and, if X is complete, also sufficient that, for every $\varepsilon > 0$, the set A admits a finite ε-net.*

Proof. *Necessity.* Assume that A is relatively compact, and let $\varepsilon > 0$. Suppose, for contradiction, that, for some $\varepsilon > 0$, the set A does not have any finite ε-nets. Letting $x_1 \in A$ be arbitrary, inductively choose a sequence (x_n) in A as follows: given $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in A$, choose an $x_{n+1} \in A \backslash \bigcup_{i=1}^{n} B(x_i, \varepsilon)$ (such an element x_{n+1}

exists because otherwise one would have $\bigcup_{i=1}^{n} B(x_i, \varepsilon) \supset A$, i.e., $\{x_1, \ldots, x_n\}$ would be a finite ε-net for A).

The sequence (x_n) obtained in this process does not contain any Cauchy subsequences, because, whenever $n, m \in \mathbb{N}$, $m > n$, one has $x_m \notin \bigcup_{i=1}^{m-1} B(x_i, \varepsilon)$, thus $x_m \notin B(x_n, \varepsilon)$, i.e., $\rho(x_m, x_n) \geqslant \varepsilon$. Thus the sequence (x_n) (of elements of A) does not contain any convergent subsequences which is a contradiction, because A is relatively compact by our assumption.

Sufficiency. Assume that X is complete and that, for every $\varepsilon > 0$, the set A admits a finite ε-net. Let $(x_n) = (x_n)_{n=1}^{\infty}$ be a sequence in A. We must show that (x_n) has a convergent subsequence. Choose $\varepsilon_n > 0$, $n \in \mathbb{N}$, so that $\varepsilon_n \xrightarrow[n\to\infty]{} 0$.

By our assumption, there exists a finite ε_1-net for A, i.e., there exists a finite set $A_1 \subset X$ such that

$$A \subset \bigcup_{a \in A_1} B(a, \varepsilon_1).$$

Choose an $a_1 \in A_1$ so that the ball $B(a_1, \varepsilon_1)$ contains infinitely many members of the sequence (x_n) (such an a_1 exists!), and an index $k_1 \in \mathbb{N}$ so that $x_{k_1} \in B(a_1, \varepsilon_1)$.

Proceed inductively as follows: provided $n \in \mathbb{N}$, elements $a_1, \ldots, a_n \in X$ and indices $k_1 < k_2 < \cdots < k_n$ such that the intersection $\bigcap_{i=1}^{n} B(a_i, \varepsilon_i)$ contains infinitely many elements of the sequence (x_n) and $x_{k_j} \in \bigcap_{i=1}^{j} B(a_i, \varepsilon_i)$ for every $j \in \{1, \ldots, n\}$; by our assumption, there exists a finite ε_{n+1}-net for $A \cap \bigcap_{i=1}^{n} B(a_i, \varepsilon_i)$, i.e., there exists a finite set $A_{n+1} \subset X$ such that

$$A \cap \bigcap_{i=1}^{n} B(a_i, \varepsilon_i) \subset \bigcup_{x \in A_{n+1}} B(a, \varepsilon_{n+1});$$

now choose an $a_{n+1} \in A_{n+1}$ so that the intersection

$$\left(\bigcap_{j=1}^{n} B(a_i, \varepsilon_i) \right) \cap B(a_{n+1}, \varepsilon_{n+1}) = \bigcap_{j=1}^{n+1} B(a_i, \varepsilon_i)$$

contains infinitely many members of the sequence (x_n) and an index $k_{n+1} \in \mathbb{N}$ so that $k_{n+1} > k_n$ and $x_{k_{n+1}} \in \bigcap_{j=1}^{n+1} B(a_i, \varepsilon_i)$.

In order to see that the subsequence $(x_{k_n})_{n=1}^{\infty}$ obtained by this process is convergent, by the completeness of X it suffices to show that this subsequence is Cauchy which is the case, because, whenever $n, p \in \mathbb{N}$, one has $x_{k_n}, x_{k_{n+p}} \in B(a_n, \varepsilon_n)$, and thus

$$\rho(x_{k_n}, x_{k_{n+p}}) \leqslant \rho(x_{k_n}, a_n) + \rho(a_n, x_{k_{n+p}}) < 2\varepsilon_n \xrightarrow[n\to\infty]{} 0.$$

□

9.3. Continuous functionals on compact sets.

The following exercise says that continuous operators take relatively compact sets into relatively compact ones and compact sets into compact ones.

Exercise 9.2. Let X and Y be metric spaces, let $f\colon X \to Y$ be a continuous operator, and let S be a subset of X. Prove that

(a) if S is relatively compact, then $f[S]$ is relatively compact;

(b) if S is compact, then $f[S]$ is compact.

Exercise 9.3. Let X and Y be metric spaces, let $f\colon X \to Y$ be a continuous operator, and let S be a subset of X. Prove that

(a) $f[\overline{S}] \subset \overline{f[S]}$;

(b) if S is relatively compact, then $f[\overline{S}] = \overline{f[S]}$.

Theorem 9.6. *Let K be a compact metric space, and let $f\colon K \to \mathbb{K}$ be a continuous functional. Then*

(a) *f is bounded, i.e., there exists an $L \geqslant 0$ such that*

$$|f(x)| \leqslant L \quad \text{for every } x \in K; \tag{9.1}$$

(b) *if $\mathbb{K} = \mathbb{R}$, then f attains its minimum and maximum, i.e., there exist $z_1, z_2 \in K$ such that*

$$f(z_1) = \min_{x \in K} f(x) \quad \text{and} \quad f(z_2) = \max_{x \in K} f(x).$$

PROOF. (a). Suppose for contradiction that f is not bounded, i.e., there is no $L \geqslant 0$ satisfying (9.1). Then, for every $n \in \mathbb{N}$, there exists an $x_n \in K$ such that

$$|f(x_n)| > n.$$

By the compactness of K, the sequence $(x_n)_{n=1}^{\infty}$ has a convergent to some $z \in K$ subsequence $(x_{k_n})_{n=1}^{\infty}$, i.e., $x_{k_n} \xrightarrow[n\to\infty]{} z$. The functional $K \ni x \mapsto |f(x)| \in \mathbb{R}$ is continuous (because it is the composition of the continuous functionals $f\colon K \to \mathbb{K}$ and $|\cdot|\colon \mathbb{K} \ni \alpha \mapsto |\alpha| \in \mathbb{R}$), thus $|f(x_{k_n})| \xrightarrow[n\to\infty]{} |f(z)|$. On the other hand,

$$|f(x_{k_n})| \geqslant k_n \xrightarrow[n\to\infty]{} \infty,$$

thus $|f(x_{k_n})| \xrightarrow[n\to\infty]{} \infty$, a contradiction.

(b). Let $\mathbb{K} = \mathbb{R}$. We only show that f attains its maximum (the proof that f attains its minimum is analogous). By (a),

$$M := \sup_{x \in K} f(x) < \infty,$$

We must find a $z_1 \in K$ such that $f(z_1) = M$.

For every $n \in \mathbb{N}$, there is an $x_n \in K$ such that

$$f(x_n) > M - \frac{1}{n}.$$

By the compactness of K, the sequence $(x_n)_{n=1}^{\infty}$ has a convergent subsequence $(x_{k_n})_{n=1}^{\infty}$. Put $z_1 := \lim_{n\to\infty} x_n$. Since

$$M - \frac{1}{k_n} < f(x_{k_n}) \leqslant M \quad \text{for every } n \in \mathbb{N},$$

letting $n \to \infty$, it follows that $f(x_{k_n}) \xrightarrow[n\to\infty]{} M$. On the other hand, since f is continuous, $f(x_{k_n}) \xrightarrow[n\to\infty]{} f(z_1)$, and thus $f(z_1) = M$ by the uniqueness of the limit. □

9.4. The Arzelà–Ascoli theorem.

Definition 9.5. Let $a, b \in \mathbb{R}$, $a < b$, and let C be a set of continuous functions $[a, b] \to \mathbb{R}$. The set C is said to be *uniformly bounded* if there exists an $M \geqslant 0$ such that

$$|x(t)| \leqslant M \quad \text{for all } t \in [a, b] \text{ and all } x \in C.$$

The set C is said to be *equicontinuous* if, for every $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$t_1, t_2 \in [a, b],\ |t_1 - t_2| < \delta \qquad \Longrightarrow \qquad |x(t_1) - x(t_2)| < \varepsilon \quad \text{for all } x \in C. \tag{9.2}$$

Clearly,

- a set of continuous functions $[a, b] \to \mathbb{R}$ is uniformly bounded if and only if it is bounded when considered as a subset of the metric space $C[a, b]$;
- every function of an equicontinuous set of functions $[a, b] \to \mathbb{R}$ is uniformly continuous on $[a, b]$; for a set consisting of a single function, its equicontinuity is equivalent to the uniform continuity of this function (and thus to the continuity of this function by Cantor's theorem).

Theorem 9.7 (The Arzelà–Ascoli theorem)**.** *Let C be a subset of the metric space $C[a, b]$. The following assertions are equivalent:*

(i) *C is relatively compact;*

(ii) *C is uniformly bounded and equicontinuous.*

Corollary 9.8 (The Arzelà–Ascoli theorem)**.** *Let C be a subset of the metric space $C[a, b]$. The following assertions are equivalent:*

(i) *C is compact;*

(ii) *C is closed, uniformly bounded and equicontinuous.*

Proof of the Arzelà–Ascoli theorem 9.7. (i)⇒(ii). Assume that $C \neq \varnothing$ is relatively compact. Then C is a bounded subset of $C[a,b]$, i.e., it is a uniformly bounded set, and it remains to show that C is equicontinuous, i.e., for every $\varepsilon > 0$, there is a $\delta > 0$ satisfying (9.2). Let $\varepsilon > 0$ be arbitray. Since C is relatively compact, by Hausdorff's theorem, there exists a finite $\frac{\varepsilon}{3}$-net $B \subset C[a,b]$ for C. Every function $z \in B$ is uniformly continuous on $[a,b]$ by Cantor's theorem, thus there is a $\delta_z > 0$ such that

$$t_1, t_2 \in [a,b],\ |t_1 - t_2| < \delta_z \qquad \Longrightarrow \qquad |z(t_1) - z(t_2)| < \frac{\varepsilon}{3}.$$

Put $\delta := \min_{z \in B} \delta_z$, and let $x \in C$ be arbitrary. Since B is an $\frac{\varepsilon}{3}$-net for C, there is a $z \in B$ such that $\|x - z\| < \frac{\varepsilon}{3}$. Whenever $t_1, t_2 \in [a,b]$ satisfy $|t_1 - t_2| < \delta$, one also has $|t_1 - t_2| < \delta_z$, and thus

$$\begin{aligned} |x(t_1) - x(t_2)| &= |x(t_1) - z(t_1) + z(t_1) - z(t_2) + z(t_2) - x(t_2)| \\ &\leqslant |x(t_1) - z(t_1)| + |z(t_1) - z(t_2)| + |z(t_2) - x(t_2)| \\ &\leqslant \|x - z\| + \frac{\varepsilon}{3} + \|x - z\| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

(ii)⇒(i) will be omitted in this course. □

§ 10. Separable metric spaces

Recall that a subset A of a metric space X is said to be *dense* (in X), if $\overline{A} = X$.

The following proposition is a direct consequence of Proposition 5.11.

Proposition 10.1. *Let X be a metric space and let $A \subset X$. The following assertions are equivalent:*

(i) *A is dense (in X), i.e., $\overline{A} = X$;*

(ii) *for every $x \in X$ and every $\varepsilon > 0$, there is an $a \in A$ such that $\rho(x, a) < \varepsilon$, i.e., $x \in B(a, \varepsilon)$;*

(iii) *for every $x \in X$, there is a sequence (x_n) of elements of A such that $x_n \xrightarrow[n\to\infty]{} x$.*

Definition 10.1. A metric space X is said to be *separable* if there exists an at most countable dense subset of X.

In fact, all the metric spaces introduced in § 4, except ℓ_∞ and $L_\infty(a, b)$, are separable. In these notes, we shall only prove the separability of quite a few of them.

Example 10.1. The space $\mathbb{K}$ (endowed with the natural metric $d(x, y) = |x - y|$, $x, y \in \mathbb{K}$) is separable: the subset $\mathbb{K}_\mathbb{Q}$ of $\mathbb{K}$ where

$$\mathbb{R}_\mathbb{Q} := \mathbb{Q} \qquad \text{and} \qquad \mathbb{C}_\mathbb{Q} := \{\alpha + i\beta \colon \alpha, \beta \in \mathbb{Q}\}$$

is countable and $\overline{\mathbb{K}_\mathbb{Q}} = \mathbb{K}$.

Indeed, suppose that $\mathbb{K} = \mathbb{R}$. The set $\mathbb{R}_\mathbb{Q} = \mathbb{Q}$ is countable. Whenever $x \in \mathbb{R}$ and $\varepsilon > 0$, there is an $\alpha \in \mathbb{Q}$ such that $d(x, \alpha) = |x - \alpha| < \varepsilon$, hence $\mathbb{R}_\mathbb{Q} = \mathbb{Q}$ is dense in $\mathbb{R}$.

Now let $\mathbb{K} = \mathbb{C}$. The set $\mathbb{C}_\mathbb{Q}$ is countable, because the mapping $\mathbb{Q} \times \mathbb{Q} \ni (\alpha, \beta) \mapsto \alpha + i\beta \in \mathbb{C}_\mathbb{Q}$ is a bijection and the set $\mathbb{Q} \times \mathbb{Q}$ is countable. To see that $\overline{\mathbb{C}_\mathbb{Q}} = \mathbb{C}$ is dense in $\mathbb{C}$, let $z = x + iy \in \mathbb{C}$ (here $x, y \in \mathbb{R}$) and $\varepsilon > 0$ be arbitrary. By the density of $\mathbb{Q}$ in $\mathbb{R}$, there are $\alpha, \beta \in \mathbb{Q}$ such that $|x - \alpha|, |y - \beta| < \frac{\varepsilon}{\sqrt{2}}$. Putting $a := \alpha + i\beta \in \mathbb{C}_\mathbb{Q}$, one has

$$d(z, a) = |z - a| = |(x - \alpha) + i(y - \beta)| = \sqrt{|x - \alpha|^2 + |y - \beta|^2} < \sqrt{\frac{\varepsilon^2}{2} + \frac{\varepsilon^2}{2}} = \varepsilon.$$

and $\overline{\mathbb{C}_\mathbb{Q}} = \mathbb{C}$ is dense in $\mathbb{C}$ by Proposition 10.1.

Example 10.2. Whenever $n \in \mathbb{N}$ and $p \in [1, \infty]$, the space ℓ_p^n is separable: the set

$$\underbrace{\mathbb{K}_\mathbb{Q} \times \cdots \times \mathbb{K}_\mathbb{Q}}_{n} = \big\{(\kappa_1, \ldots, \kappa_n) \colon \kappa_1, \ldots, \kappa_n \in \mathbb{K}_\mathbb{Q}\big\}$$

is countable and dense in ℓ_p^n.

Exercise 10.1. Prove that the set $\underbrace{\mathbb{K}_\mathbb{Q} \times \cdots \times \mathbb{K}_\mathbb{Q}}_{n}$ is dense in ℓ_p^n.

HINT. It suffices to show that $\underbrace{\mathbb{K}_{\mathbb{Q}} \times \cdots \times \mathbb{K}_{\mathbb{Q}}}_{n}$ is dense in ℓ_1^n: the density of $\underbrace{\mathbb{K}_{\mathbb{Q}} \times \cdots \times \mathbb{K}_{\mathbb{Q}}}_{n}$ in ℓ_p^n for arbitrary $p \in [1, \infty]$ follows from its density in ℓ_1^n, because $\|x\|_p \leqslant \|x\|_1$ for every $x \in \underbrace{\mathbb{K} \times \cdots \times \mathbb{K}}_{n}$ by Proposition 3.4.

Proposition 10.2. *A compact metric space is separable.*

PROOF. Let X be a compact metric space. By Hausdorff's theorem 9.5, for every $n \in \mathbb{N}$, there exists a finite $\dfrac{1}{n}$-net $B_n \subset X$ of X. The union $B := \bigcup_{n=1}^{\infty} B_n$ is at most countable and dense in X, i.e., $\overline{B} = X$. To prove the latter, letting $x \in X$ and $\varepsilon > 0$ be arbitrary, it suffices to find a $b \in B$ such that $\rho(x, b) < \varepsilon$. To this end, pick an $n \in \mathbb{N}$ such that $\dfrac{1}{n} < \varepsilon$; since B_n is a $\dfrac{1}{n}$-net for X, there exists a $z \in B_n \subset B$ such that $\rho(x, b) < \dfrac{1}{n} < \varepsilon$, as desired. □

Definition 10.2. Let X be a linear space and let A be a subset of X. The smallest linear subspace of X containing A is called the *linear span* of A and is denoted by $\operatorname{span} A$.

Thus, $\operatorname{span} \varnothing = \{0\}$ and, for $A \neq \varnothing$,

$$\operatorname{span} A = \left\{ \sum_{k=1}^{n} a_k x_k \colon \ n \in \mathbb{N}, \, x_1, \ldots, x_n \in X, \, a_1, \ldots, a_n \in \mathbb{K} \right\},$$

i.e., for $A \neq \varnothing$, the linear span of A is the linear subspace of all linear combinations of elements of A.

Definition 10.3. Let X be a normed space. A subset A of X is said to be *total* (in X), if its linear span is dense in X, i.e. $\overline{\operatorname{span} A} = X$.

Proposition 10.3. *A normed space is separable if and only if it has an at most countable total subset.*

Remark 10.1. In § 12, we shall prove, using Propostion 10.3, that the normed spaces c_0, c, and ℓ_p where $1 \leqslant p < \infty$ are separable.

Also, Propostion 10.3 yields the separability of ℓ_p^n for all $n \in \mathbb{N}$ and $p \in [1, \infty]$ (i.e., the result of Example 10.2), because, defining

$$e_k := (\overbrace{\underbrace{0, \ldots, 0, 1}_{k}, 0, \ldots, 0}^{n}) \quad \text{for every } k \in \{1, \ldots, n\},$$

the subset $\{e_1, \ldots, e_n\} \subset \ell_p^n$ is total in ℓ_p^n: in fact, $\operatorname{span}\{e_1, \ldots, e_n\} = \ell_p^n$.

For the proof of the sufficiency in Proposition 10.3, it is appropriate to point out the following easy

Lemma 10.4. *Let Y be a dense subset of a metric space X and let Z be a dense (in Y) subset of Y. Then Z is dense in X.*

PROOF.

Exercise 10.2. Prove lemma 10.4.

□

PROOF OF PROPOSITION 10.3. *Necessity* is obvious, because any dense subset of X is total (therefore, if X is a separable normed space and A is an at most countable dense subset of S, the set A is a total; thus X has an at most countable total subset).

Sufficiency. Suppose that a normed space X has an at most countable total subset A, i.e. $\overline{\operatorname{span} A} = X$. For the separability of X, by Lemma 10.4, it suffices to show that $\operatorname{span} A$ has an at most countable dense (in $\operatorname{span} A$) subset. We may assume that $X \neq \{0\}$. Then $A \neq \varnothing$ and (since A is at most countable) we can write $A = \{x_k \colon k \in \mathbb{N}\}$ where $x_k \in A$ for every $k \in \mathbb{N}$ (with the possibility that $x_k = x_l$ for some $k \neq l$ (this happens when A is finite)). Now it suffices to show that the subset

$$\operatorname{span}_{\mathbb{Q}} A := \left\{ \sum_{k=1}^{n} \alpha_k x_k \colon\ n \in \mathbb{N},\ \alpha_1, \ldots, \alpha_n \in \mathbb{K}_{\mathbb{C}} \right\} \subset \operatorname{span} A$$

is an at most countable set which is dense in $\operatorname{span} A$.

Exercise 10.3. Prove that $\operatorname{span}_{\mathbb{Q}} A$ is at most countable and $\overline{\operatorname{span}_{\mathbb{Q}} A} \supset \operatorname{span} A$.

HINT. For every $n \in \mathbb{N}$, put $B_n := \left\{ \sum_{k=1}^{n} \alpha_k x_k \colon\ \alpha_1, \ldots, \alpha_n \in \mathbb{K}_{\mathbb{C}} \right\}$; then $\operatorname{span}_{\mathbb{Q}} A = \bigcup_{n=1}^{\infty} B_n$. Thus, in order that $\operatorname{span}_{\mathbb{Q}} A$ were at most countable, letting $n \in \mathbb{N}$ be arbitrary, it suffices to show that B_n is at most countable. To this end, it suffices to observe that the mapping $\underbrace{\mathbb{K}_{\mathbb{Q}} \times \cdots \times \mathbb{K}_{\mathbb{Q}}}_{n} \ni (\alpha_1, \ldots, \alpha_n) \mapsto \sum_{k=1}^{n} \alpha_k x_k$ is a surjection while the set $\underbrace{\mathbb{K}_{\mathbb{Q}} \times \cdots \times \mathbb{K}_{\mathbb{Q}}}_{n}$ is countable.

□

Example 10.3. The space ℓ_∞ is not separable.

Indeed, let A be a dense subset of ℓ_∞. For the non-separability of ℓ_∞, it suffices to show that A is uncountable (i.e., A is neither finite nor countable). To this end, denote by D the subset of ℓ_∞ consisting of sequences having only 1 and 0 as their terms. The set D is known to be uncountable. Since A is dense in ℓ_∞, for every $x \in D$, there is an $a_x \in A$ such that $x \in B\Big(a_x, \dfrac{1}{2}\Big)$. Whenever $x, z \in D$ and such that $x \neq z$, one has $a_x \neq a_z$, because otherwise $x, z \in B\Big(a_x, \dfrac{1}{2}\Big)$ and thus

$$\|x - z\| = \|x - a_x + a_x - z\| \leqslant \|x - a_x\| + \|a_x - z\| < \frac{1}{2} + \frac{1}{2} = 1 = \|x - z\|,$$

a contradiction (here $\|x - z\| = 1$ because the sequence $x - z$ can have only 1, 0 and -1 as its terms, and, since, $x \neq z$, at least one of the terms of $x - z$ is either 1 or -1). It follows that the subset $\{a_x \colon\ x \in D\}$ of A is uncountable, and thus also A itself is uncountable.

***Exercise 10.4.** Prove that any subspace of a separable metric space is separable.

§ 11. Topological spaces

11.1. The notion of a topological space

Definition 11.1. Let X be a set. A collection τ of subsets of X is called a *topology* if

1° $\varnothing, X \in \tau$;

2° any finite intersection of sets in τ belongs to τ, i.e., whenever $n \in \mathbb{N}$ and $A_1, \ldots, A_n \in \tau$, also the intersection $\bigcap_{j=1}^{n} A_j \in \tau$;

3° any union of sets in τ belongs to τ, i.e., whenever I is a set of indices and $A_j \in \tau$ for every $j \in I$, also the union $\bigcup_{j \in I} A_j \in \tau$.

The pair (X, τ) (or just X when the role of the topology τ is well understood from the context) is called a *topological space.* The sets in the collection τ are called *open* sets in X.

Example 11.1. The collection of open subsets in a metric space X is a topology (by Proposition 5.3). This topology is referred to as the *topology induced by the metric* of X.

Example 11.2. Let X be a set. The collection $\mathcal{P}(X)$ of all subsets of X is clearly a topology. This topology, called the *discrete topology* on X, is induced by the discrete metric on X (recall that (see Exercise 5.6) every subset of X is an open set with respect to the discrete metric).

Example 11.3. Let X be a set. The collection $\{\varnothing, X\}$ is a topology on X. This topology is referred to as the weakest topology on X.

Remark 11.1. Not every topology is induced by a metric. For example, letting $X = \{a, b\}$ (i.e., X is a set consisting of two elements) and $\tau := \Big\{\varnothing, X, \{A\}\Big\}$, the collection τ is clearly a topology. However, this topology is not induced by any metric.

Exercise 11.1. Prove that the topology τ is not induce by any metric.

Definition 11.2. A set A in a topological space (X, τ) is said to be closed if its complement $X \backslash A$ is open, i.e., $X \backslash A \in \tau$.

Recall that, in a metric space, a set is closed if and only if its complement is open (see Proposition 5.6). Thus closed sets in a metric space are exactly the sets that are closed with respect to the topology induced by the metric.

The following properties of closed sets in a topological space follow immediately from Definitions 11.2 and 11.1 by De Morgan's laws.

Proposition 11.1. *Let X be a topological space. Then*

(a) *$\varnothing$ and X are closed sets;*

(b) *any finite union of closed sets is a closed set, i.e., whenever* $n \in \mathbb{N}$ *and* $A_1, \ldots, A_n \subset X$ *are closed sets, also their union* $\bigcup_{j=1}^{n} A_j$ *is a closed set;*

(c) *any intersection of closed sets is a closed set, i.e., whenever* I *is a set of indices and* A_j, $j \in I$, *are closed sets, also their intersection* $\bigcap_{j \in I} A_j$ *is a closed set.*

11.2. Convergence of sequences in topological spaces

Definition 11.3. Let (X, τ) be a topological space, and let $x \in X$. A set $G \subset X$ is called a *neighbourhood* of the point x (in the topology τ) if there is a set $u \in \tau$ such that $x \in U \subset G$.

Thus, the neighbourhoods of a point in a metric spaces are the same as its neighbourhoods in the topology induced by the metric.

Definition 11.4. Let (X, τ) be a topological space. A sequence $(x_n)_{n=1}^{\infty}$ in X is said to *converge* to an element $x \in X$ (with respect to the topology τ) if, for every neighbourhood U of x, there is an index $N \in \mathbb{N}$ such that

$$n \geqslant \mathbb{N} \quad \Longrightarrow \quad x_n \in U.$$

Thus, the convergence of a sequence in a metric space is the same as the convergence of this sequence with respect to the topology induced by the metric.

Notice that, in general, the limit of a sequence in a topological space need not be unique: for instance, when X is any set and τ is the weakest topology on X, i.e., $\tau := \{\varnothing, X\}$, every sequence in X is converges to any element in X.

Remark that, in topological spaces, the convergence of sequences does not play as an important role as in metric spaces: the role played by sequences in the theory of metric spaces is, in topological spaces, performed by *nets* (a concept more general than that of a *sequence*).

§ 12. Series in normed spaces

12.1. Series in normed spaces

Definition 12.1. Let X be a normed space and let $x_k \in X$, $k \in \mathbb{N}$.

The formal infinite sum

$$x_1 + x_2 + x_3 + \cdots =: \sum_{k=1}^{\infty} x_k \tag{12.1}$$

is called a *series*. The elements $x_k \in X$, $k \in \mathbb{N}$, are called the *terms* of the series (12.1). The sums

$$\sum_{k=1}^{n} x_k, \quad n \in \mathbb{N},$$

are called the *partial sums* of the series (12.1).

If the sequence $\left(\sum_{k=1}^{n} x_k\right)_{n=1}^{\infty}$ of the partial sums of the series (12.1) converges, then its limit is called the *sum* of this series, and the series (12.1) is said to be *convergent* (to its sum). Otherwise the series (12.1) is said to be *divergent.*

If the series (12.1) converges, then its sum is denoted by $\sum_{k=1}^{\infty} x_k$ (as the series (12.1) itself). Thus the sum of the series (12.1) is

$$\sum_{k=1}^{\infty} x_k = \lim_{n\to\infty} \sum_{k=1}^{n} x_k.$$

First properties of convergent series are collected in

Proposition 12.1. *Let $\sum_{k=1}^{\infty} x_k$ and $\sum_{k=1}^{\infty} y_k$ be convergent series in a normed space X, and let $\alpha, \beta \in \mathbb{K}$. Then*

(a) *the series $\sum_{k=1}^{\infty}(\alpha x_k + \beta y_k)$ is convergent and its sum*

$$\sum_{k=1}^{\infty}(\alpha x_k + \beta y_k) = \alpha \sum_{k=1}^{\infty} x_k + \beta \sum_{k=1}^{\infty} y_k;$$

(b) $x_k \xrightarrow[k\to\infty]{} 0$*;*

(c) $\sum_{k=n+1}^{\infty} x_k \xrightarrow[n\to\infty]{} 0$*;*

(d) $\left\|\sum_{k=1}^{\infty} x_k\right\| \leqslant \sum_{k=1}^{\infty} \|x_k\|$.

The assertions (b) and (c) of Proposition 12.1 say, respectively, that

- the sequence of the terms of a convergent series converges to 0;
- the *remainder term* of a convergent series converges to 0.

PROOF OF PROPOSITION 12.1. Put $x := \sum_{k=1}^{\infty} x_k$ and $y := \sum_{k=1}^{\infty} y_k$.

(a). We must show that

$$\sum_{k=1}^{n}(\alpha x_k + \beta y_k) \xrightarrow[n\to\infty]{} \alpha x + \beta y. \tag{12.2}$$

Exercise 12.1. Verify (12.2).

(b). One has

$$x_n = \sum_{k=1}^{n} x_k - \sum_{k=1}^{n-1} x_k \xrightarrow[n\to\infty]{} x - x = 0.$$

(c). One has

$$\sum_{k=n+1}^{\infty} x_k \overset{(\bullet)}{=} x - \sum_{k=1}^{n} x_k \xrightarrow[n\to\infty]{} 0.$$

Exercise 12.2. Prove the equality $(\bullet)$.

(d). One has, by the continuity of the norm and by the triangle inequality,

$$\left\|\sum_{k=1}^{\infty} x_k\right\| = \left\|\lim_{n\to\infty}\sum_{k=1}^{n} x_k\right\| = \lim_{n\to\infty}\left\|\sum_{k=1}^{n} x_k\right\| \leqslant \lim_{n\to\infty}\sum_{k=1}^{n}\|x_k\| = \sum_{k=1}^{\infty}\|x_k\|.$$

□

Exercise 12.3. Put

$$e_k = (\underbrace{0,\ldots,0,1}_{k},0,\ldots),\ \ k \in \mathbb{N}, \qquad \text{and} \qquad e = (1,1,1,\ldots).$$

(a) Does the series $\displaystyle\sum_{k=1}^{\infty}\frac{1}{k}\,e_k$ converge in the space

(a1) ℓ_1;

(a2) ℓ_p where $1 < p < \infty$;

(a3) ℓ_∞?

(b) Is it true that $x = \displaystyle\sum_{k=1}^{\infty}\xi_k\,e_k$ in X

(b1) for every $x = (\xi_k) \in X = \ell_p$ where $1 \leqslant p < \infty$;

(b2) for every $x = (\xi_k) \in X = c_0$;

(b3) for every $x = (\xi_k) \in X = c$?

(c) Let $x = (\xi_k)_{k=1}^{\infty} \in c$ with $\xi_k \to \xi$. Prove that

$$x = \xi\,e + \sum_{k=1}^{\infty}(\xi_k - \xi)\,e_k.$$

Remark 12.1. Suppose that $1 \leqslant p < \infty$. From Exercise 12.3, (b1), it follows that the countable set $\{e_k\colon\ k \in \mathbb{N}\}$ is total in the space ℓ_p; thus ℓ_p is separable by Proposition 10.3. Indeed, for the totality of $\{e_k\colon\ k \in \mathbb{N}\}$ in ℓ_p, observe that, whenever $x = (\xi_k)_{k=1}^{\infty} \in \ell_p$, one has

$$\sum_{k=1}^{n} \xi_k e_k \xrightarrow[n\to\infty]{} \sum_{k=1}^{\infty} \xi_k e_k = x$$

while $\sum_{k=1}^{n} \xi_k e_k \in \operatorname{span}\{e_k\colon\ k \in \mathbb{N}\}$ for all $n \in \mathbb{N}$; thus $\overline{\operatorname{span}\{e_k\colon\ k \in \mathbb{N}\}} = \ell_p$.

Similarly, from Exercise 12.3, (b2) and (c), one deduces that the countable sets $\{e_k\colon\ k \in \mathbb{N}\}$ and $\{e\} \cup \{e_k\colon\ k \in \mathbb{N}\}$ are total in the spaces c_0 and c, respectively; thus the spaces c_0 and c are separable by Proposition 10.3.

Definition 12.2. A series $\sum_{k=1}^{\infty} x_k$ in a normed space X is said to be *absolutely convergent* if the series $\sum_{k=1}^{\infty} \|x_k\|$ converges in $\mathbb{R}$, i.e.,

$$\sum_{k=1}^{\infty} \|x_k\| < \infty.$$

In order to prove the completeness of a normed space, it is often convenient to use

Theorem 12.2. *A normed space X is complete (i.e., a Banach space) if and only if every absolutely convergent series in X converges in X.*

Proof. *Necessity.* Let X be a Banach space, and let $\sum_{k=1}^{\infty} x_k$ be an absolutely convergent series in X. In order that the series $\sum_{k=1}^{\infty} x_k$ were convergent, courtesy of the completeness of X, it suffices to show that the sequence $(S_n)_{n=1}^{\infty}$ of its partial sums is a Cauchy sequence, i.e., $\|S_n - S_m\| \xrightarrow[n,m\to\infty]{} 0$. To this end, observe that, for $n, m \in \mathbb{N}$, $n > m$, one has

$$\|S_n - S_m\| = \left\| \sum_{k=1}^{n} x_k - \sum_{k=1}^{m} x_k \right\| = \left\| \sum_{k=m+1}^{n} x_k \right\| \leqslant \sum_{k=m+1}^{n} \|x_k\| \leqslant \sum_{k=m+1}^{\infty} \|x_k\| \xrightarrow[m\to\infty]{} 0,$$

because the remainder term of the convergent (in $\mathbb{R}$) series $\sum_{k=1}^{\infty} \|x_k\|$ converges to 0.

Sufficiency. Let X be a normed space such that every absolutely convergent series in X is convergent, and let $(x_n)_{n=1}^{\infty}$ be a Cauchy sequence in X. In order for X to be complete, it suffices to show that the sequence $(x_n)_{n=1}^{\infty}$ converges in X. To this end, by Proposition 6.1, (b), it suffices to show that $(x_n)_{n=1}^{\infty}$ has a convergent subsequence. To this end, pick indices $0 = m_0 < m_1 < m_2 < m_3 < \cdots$ so that, for all $k \in \mathbb{N}$,

$$n, m \geqslant m_k \quad \Longrightarrow \quad \|x_n - x_m\| \leqslant \frac{1}{2^k} \tag{12.3}$$

(this is possible because (x_n) is a Cauchy sequence). Now the series

$$\sum_{k=1}^{\infty} (x_{m_k} - x_{m_{k-1}})$$

is absolutely convergent because, for every $k \geqslant 2$, one has $\|x_{m_k} - x_{m_{k-1}}\| \leqslant \dfrac{1}{2^{k-1}}$ by (12.3) and thus

$$\sum_{k=1}^{\infty} \|x_{m_k} - x_{m_{k-1}}\| \leqslant \|x_{m_1}\| + \sum_{k=2}^{\infty} \frac{1}{2^{k-1}} = \|x_{m_1}\| + 1 < \infty.$$

By our assumption on X, the series $\sum_{k=1}^{\infty}(x_{m_k} - x_{m_{k-1}})$ is convergent, i.e., the sequence of its partial sums is convergent. But, for every $n \in \mathbb{N}$,

$$\sum_{k=1}^{n}(x_{m_k} - x_{m_{k-1}}) = x_{m_n},$$

thus the sequence $(x_{m_n})_{n=1}^{\infty}$ is convergent. □

Example 12.1. The space ℓ_1 is complete.

Indeed, let $\sum_{k=1}^{\infty} x_k$ be an absolutely convergent series in ℓ_1. For the completeness of ℓ_1, by Theorem 12.2, it suffices to show that $\sum_{k=1}^{\infty} x_k$ converges in ℓ_1.

To this end, letting $x_k = (\xi_j^k)_{j=1}^{\infty}$ for every $k \in \mathbb{N}$, observe that, by the absolute convergence of $\sum_{k=1}^{\infty} x_k$,

$$\sum_{k=1}^{\infty} \|x_k\| = \sum_{k=1}^{\infty}\sum_{j=1}^{\infty} |\xi_j^k| = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} |\xi_j^k| < \infty.$$

In particular, for every $j \in \mathbb{N}$, one has $\sum_{k=1}^{\infty} |\xi_j^k| < \infty$, i.e., the series $\sum_{k=1}^{\infty} \xi_j^k$ is absolutely convergent (in $\mathbb{K}$) and hence convergent (in $\mathbb{K}$). It remains to observe that $x := \left(\sum_{k=1}^{\infty} \xi_j^k\right)_{j=1}^{\infty} \in \ell_1$ (because $\sum_{j=1}^{\infty}\left|\sum_{k=1}^{\infty} \xi_j^k\right| \leqslant \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} |\xi_j^k| < \infty$) and $\sum_{k=1}^{\infty} x_k = x$ in ℓ_1:

$$\begin{aligned}\left\|x - \sum_{k=1}^{n} x_k\right\| &= \left\|\left(\sum_{k=1}^{\infty} \xi_j^k\right)_{j=1}^{\infty} - \left(\sum_{k=1}^{n} \xi_j^k\right)_{j=1}^{\infty}\right\| = \left\|\left(\sum_{k=n+1}^{\infty} \xi_j^k\right)_{j=1}^{\infty}\right\| \\ &= \sum_{j=1}^{\infty}\left|\sum_{k=n+1}^{\infty} \xi_j^k\right| \leqslant \sum_{j=1}^{\infty}\sum_{k=n+1}^{\infty} |\xi_j^k| = \sum_{k=n+1}^{\infty}\sum_{j=1}^{\infty} |\xi_j^k| \xrightarrow[n\to\infty]{} 0\end{aligned}$$

(because the remainder term $\sum_{k=n+1}^{\infty}\sum_{j=1}^{\infty} |\xi_j^k|$ of the convergent series $\sum_{k=1}^{\infty}\sum_{j=1}^{\infty} |\xi_j^k|$ tends to 0).

§ 13. Continuous linear operators between normed spaces

13.1. Continuous linear operators between normed spaces

Definition 13.1. Let X and Y be linear spaces over the same scalar field $\mathbb{K}$ (where $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$).

A mapping $A\colon X \to Y$ is said to be *linear* if

1° A is *additive,* i.e.,

$$A(x+z) = Ax + Az \quad \text{for all } x, y \in X;$$

2° A is *homogenous,* i.e.,

$$A(\alpha x) = \alpha Ax \quad \text{for all } x \in X \text{ and all } \alpha \in \mathbb{K}.$$

Loosely speaking, the linearity of a mapping means that it preserves the linear structure of its domain space.

Exercise 13.1. Let X and Y be linear spaces and let $A\colon X \to Y$. Prove that the following assertions are equivalent:

1° A is linear;

2° $A(\alpha x + \beta z) = \alpha Ax + \beta Az$ for all $x, y \in X$ and all $\alpha, \beta \in \mathbb{K}$;

3° $A(\alpha x + z) = \alpha Ax + Az$ for all $x, y \in X$ and all $\alpha \in \mathbb{K}$.

The following proposition says that a linear operator between normed spaces is continuous already when it is continuous at a single point.

Proposition 13.1. *Let X and Y be normed spaces and let $A\colon X \to Y$ be a linear operator. The following assertions are equivalent:*

(i) *A is continuous;*

(ii) *A is continuous at 0;*

(iii) *there exists a point in X at which A is continuous.*

Proof. (i)⇒(ii)⇒(iii) is obvious.

(iii)⇒(i). Suppose that A is continuous at a point $z \in X$, and let $x, x_n \in X$, $n \in \mathbb{N}$, be such that $x_n \xrightarrow[n\to\infty]{} x$. For the continuity of A, it suffices to show that $Ax_n \xrightarrow[n\to\infty]{} Ax$. By the linearity of A,

$$Ax_n = A(x_n - x + z) - Az + Ax \xrightarrow[n\to\infty]{} Az - Az + Ax = Ax$$

(because $x_n - x + z \xrightarrow[n\to\infty]{} z$, and thus $A(x_n - x + z) \xrightarrow[n\to\infty]{} Az$ by the continuity of A at z). □

Definition 13.2. Let X and Y be normed spaces. A linear operator $A\colon X \to Y$ is said to be *bounded,* if there exists an $M \geqslant 0$ such that

$$\|Ax\| \leqslant M\,\|x\| \quad \text{for all } x \in X. \tag{13.1}$$

The term "bounded" linear operator is justified by

Proposition 13.2. *Let X and Y be normed spaces and let $A\colon X \to Y$ be a linear operator. The following assertions are equivalent:*

(i) *A is bounded;*

(ii) *A maps bounded sets in X into bounded sets in Y, i.e., whenever B is a bounded subset of X, its image $A[B] := \{Ax\colon x \in B\}$ is a bounded subset of Y.*

PROOF.

Exercise 13.2. Prove Proposition 13.2.

SOLUTION. (i)⇒(ii). Suppose that A is bounded, and let B be a bounded subset of X. We must show that $A[B] = \{Ax\colon x \in B\}$ is a bounded subset of Y, i.e. (see Proposition 2.2), there is a $K \geqslant 0$ such that

$$\|y\| \leqslant K \quad \text{for all } y \in A[B], \qquad \text{i.e.,} \qquad \|Ax\| \leqslant K \quad \text{for all } x \in B.$$

Since A is a bounded operator, there is an $M \geqslant 0$ satisfying (13.1). Since B is a bounded set, there is an $L \geqslant 0$ such that $\|x\| \leqslant L$ for all $x \in B$. Now, for all $x \in B$,

$$\|Ax\| \leqslant M\|x\| \leqslant M\,L.$$

(ii)⇒(i). Let the images under A of bounded sets in X be bounded sets in Y. Then, in particular, the image $A[S_X] = \{Ax\colon x \in S_X\}$ of the unit sphere S_X of X is a bounded set in Y; thus there exists an $M \geqslant 0$ such that

$$\|y\| \leqslant M \quad \text{for all } y \in A[S_X], \qquad \text{i.e.,} \qquad \|Ax\| \leqslant M \quad \text{for all } x \in S_X.$$

For all $x \in X\backslash\{0\}$, one has $\dfrac{x}{\|x\|} \in S_X$, thus

$$\frac{1}{\|x\|}\|Ax\| = \left\|\frac{1}{\|x\|}\,Ax\right\| = \left\|A\Big(\frac{x}{\|x\|}\Big)\right\| \leqslant M,$$

i.e., $\|Ax\| \leqslant M\|x\|$. The latter clearly holds also for $x = 0$, thus A is bounded.

□

Theorem 13.3. *A linear operator between normed spaces is continuous if and only if it is bounded.*

PROOF. *Sufficiency.* A bounded linear operator between normed spaces satisfies the Lipschitz condition, hence it is continuous.

Necessity. Let X and Y be normed spaces and let $A\colon X \to Y$ be a continuous linear operator. Suppose for contradiction that A is not bounded. Then, for every $n \in \mathbb{N}$, there is an $x_n \in X$ such that

$$\|Ax_n\| > n\|x_n\|.$$

Now $z_n := \dfrac{x_n}{n\|x_n\|} \xrightarrow[n\to\infty]{} 0$, because

$$\|z_n\| = \left\|\frac{x_n}{n\|x_n\|}\right\| = \frac{1}{n\|x_n\|}\,\|x_n\| = \frac{1}{n} \xrightarrow[n\to\infty]{} 0$$

On the other hand, for every $n \in \mathbb{N}$,

$$\|Az_n\| = \left\|A\Big(\frac{x_n}{n\|x_n\|}\Big)\right\| = \left\|\frac{1}{n\|x_n\|}\,Ax_n\right\| = \frac{1}{n\|x_n\|}\,\|Ax_n\| > \frac{1}{n\|x_n\|}\,n\|x_n\| = 1,$$

thus $Az_n \underset{n\to\infty}{\not\longrightarrow} 0$. This contradicts the continuity of A at 0. □

13.2. The norm of an operator

Let X and Y be normed spaces (over the same scalar field $\mathbb{K}$). In this subsection, we shall give the set of continuous linear operators from X to Y the structure of a normed space.

First, letting $A, B\colon X \to Y$ be continuous linear operators and $\alpha \in \mathbb{K}$, one defines the sum $A + B\colon X \to Y$ and scalar multiple $\alpha A\colon X \to Y$ by

$$(A + B)(x) := Ax + Bx \quad \text{and} \quad (\alpha A)x := \alpha(Ax), \qquad x \in X. \tag{13.2}$$

It is straightforward to verify that $A + B$ and αA are linear and continuous.

Exercise 13.3. Prove that the operators $A + B$ and αA are linear and bounded.

It is also straightforward to verify that the set of continuous linear operators $X \to Y$ is a linear space with respect to the operations (13.2). This linear space is denoted by $\mathcal{L}(X, Y)$.

Define, for $A \in \mathcal{L}(X, Y)$,

$$\|A\| := \sup_{x\in B_X} \|Ax\|. \tag{13.3}$$

Observe that $\|A\| < \infty$. Indeed, since A is bounded, there is an $M \geqslant 0$ such that $\|Ax\| \leqslant M\|x\|$ for all $x \in X$. For such M,

$$\|A\| = \sup_{x\in B_X} \|Ax\| \leqslant \sup_{x\in B_X} M\,\|x\| = M.$$

Proposition 13.4. *$\mathcal{L}(X, Y)$ is a normed space with respect to the norm* (13.3).

Proof.

Exercise 13.4. Prove that (13.3) is a norm in $\mathcal{L}(X, Y)$.

□

Exercise 13.5. Let $A \in \mathcal{L}(X, Y)\backslash\{0\}$ and suppose that $\|A\| = \|Ax\|$ for some $x \in B_X$. Prove that $x \in S_X$, i.e., $\|x\| = 1$.

Exercise 13.6. Let $X \neq 0$. Prove that, for every $A \in \mathcal{L}(X, Y)$,

$$\|A\| := \sup_{x\in B_X} \|Ax\| \overset{(\bullet)}{=} \sup_{x\in B_X^\circ} \|Ax\| = \sup_{x\in S_X} \|Ax\|.$$

Notice that the equality $(\bullet)$ holds also if $X = 0$ while $S_X = \varnothing$ in this case.

The following proposition is a handy tool for estimating the norm of a continuous linear operator from above.

Proposition 13.5. *Let $A \in \mathcal{L}(X, Y)$. Then*

$$\|A\| = \min\big\{M \geqslant 0\colon\ \|Ax\| \leqslant M\|x\| \ \textit{for all } x \in X\big\}$$

Proof. In the paragraph following (13.3), we proved that

- $\|A\| \leqslant M$ whenever $M \geqslant 0$ satisfies $\|Ax\| \leqslant M\|x\|$ for all $x \in X$.

Thus it remains to prove that

$$\|Ax\| \leqslant \|A\|\,\|x\| \quad \text{for all } x \in X.$$

If $x = 0$, then clearly $\|Ax\| = \|A0\| = \|0\| = 0 \leqslant \|A\|\,\|x\|$. If $x \neq 0$, then $\dfrac{x}{\|x\|} \in B_X$ and thus

$$\|Ax\| = \frac{1}{\|x\|}\|Ax\|\,\|x\| = \left\|\frac{1}{\|x\|}Ax\right\|\|x\| = \left\|A\Big(\frac{x}{\|x\|}\Big)\right\|\|x\| \leqslant \|A\|\,\|x\|.$$

□

Definition 13.3. The normed space $\mathcal{L}(X, \mathbb{K})$ of continuous linear functionals $X \to \mathbb{K}$ is called the *dual space of* X, and denoted by X^*:

$$X^* := \mathcal{L}(X, \mathbb{K}).$$

Exercise 13.7. Prove that convergence in $\mathcal{L}(X, Y)$ implies *pointwise convergence,* i.e., whenever $T_n, T \in \mathcal{L}(X, Y)$, $n \in \mathbb{N}$, are such that $T_n \xrightarrow[n\to\infty]{} T$ in $\mathcal{L}(X, Y)$, i.e., $\|T_n - T\| \xrightarrow[n\to\infty]{} 0$, then $T_n \xrightarrow[n\to\infty]{} T$ pointwise, i.e.,

$$T_n x \xrightarrow[n\to\infty]{} Tx \quad \text{for all } x \in X.$$

13.3. Completeness of the space of linear operators

It is natural to ask when the space $\mathcal{L}(X, Y)$ is complete. This question is answered by

Theorem 13.6. *Let $X \neq \{0\}$. Then $\mathcal{L}(X, Y)$ is a Banach space if and only if Y is a Banach space.*

In the trivial case $X = \{0\}$, the space $\mathcal{L}(X, Y) = \{0\}$ is, of course, always complete.

In particular, Theorem 13.6 implies that, since $X^* = \mathcal{L}(X, \mathbb{K})$ and the space $\mathbb{K}$ is complete, the *dual space X^* of a normed space X is always complete.*

The proof of the necessity in Theorem 13.6 relies on

Theorem 13.7. *Whenever $X \neq \{0\}$, the dual space X^* contains non-zero functionals. Moreover, for every $x \in X$, there exists an $x^* \in S_{X^*}$ such that $x^*(x) = \|x\|$.*

Theorem 13.7 is a consequence of the *Hahn-Banach theorem* which is not included in this introductory course; we omit its proof here. (Note, that the proof of the Hahn–Banach theorem, of course, does not make use of Theorem 13.6).

For the proof of Theorem 13.6 it is also convenient to point out the following exercise.

Exercise 13.8. Let $x^* \in X^*$ and $y \in Y$. Define an operator

$$x^* \otimes y \colon\ X \ni x \longmapsto x^*(x)\, y \in Y.$$

Prove that $x^* \otimes y \in \mathcal{L}(X,Y)$ with $\|x^* \otimes y\| = \|x^*\|\,\|y\|$.

Proof of Theorem 13.6. *Necessity.* Suppose that $\mathcal{L}(X,Y)$ is complete, and let $(y_n)_{n=1}^{\infty}$ be a Cauchy sequence in Y. We must show that the sequence (y_n) converges in Y.

Pick an $x \in S_X$. By Theorem 13.7, there is an $x^* \in S_{X^*}$ such that $x^*(x) = \|x\| = 1$. Observe that the operators $x^* \otimes y_n$, $n \in \mathbb{N}$ (see Exercise 13.8), form a Cauchy sequence:

$$\|x^* \otimes y_n - x^* \otimes y_m\| = \|x^* \otimes (y_n - y_m)\| = \|x^*\|\,\|y_n - y_m\| = \|y_n - y_m\| \xrightarrow[n,m\to\infty]{} 0$$

(because (y_n) is a Cauchy sequence). Since $\mathcal{L}(X,Y)$ is complete, the Cauchy sequence $(x^* \otimes y_n)$ converges in $\mathcal{L}(X,Y)$, say $x^* \otimes y_n \xrightarrow[n\to\infty]{} T$ for some $T \in \mathcal{L}(X,Y)$. In particular, $(x^* \otimes y_n)x \xrightarrow[n\to\infty]{} Tx$, i.e.,

$$\|(x^* \otimes y_n)x - Tx\| = \|x^*(x\,)y_n - Tx\| = \|y_n - Tx\| \xrightarrow[n\to\infty]{} 0,$$

i.e., $y_n \to Tx$ in Y.

Sufficiency. Let Y be a Banach space, and let $\sum_{k=1}^{\infty} T_k$ be an absolutely convergent series in $\mathcal{L}(X,Y)$. In order that $\mathcal{L}(X,Y)$ were a Banach space, by Theorem 12.2, it suffices to show that the series $\sum_{k=1}^{\infty} T_k$ converges in $\mathcal{L}(X,Y)$. To this end observe that, for every $x \in X$,

$$\sum_{k=1}^{\infty} \|T_k x\| \leqslant \sum_{k=1}^{\infty} \|T_k\|\,\|x\| = \|x\| \sum_{k=1}^{\infty} \|T_k\| < \infty,$$

i.e., the series $\sum_{k=1}^{\infty} T_k x$ is absolutely convergent in Y, and, hence, since Y is a Banach space, the series $\sum_{k=1}^{\infty} T_k x$ converges in Y by Theorem 12.2. Define an operator $T\colon X \to Y$ by

$$Tx := \sum_{k=1}^{\infty} T_k x, \quad x \in X.$$

It is straightforward to verify that $T \in \mathcal{L}(X,Y)$.

Exercise 13.9. Prove that T is linear and bounded.

It remains to observe that $\sum_{k=1}^{\infty} T_k = T$ in $\mathcal{L}(X,Y)$, i.e., $\left\|T - \sum_{k=1}^{n} T_k\right\| \xrightarrow[n\to\infty]{} 0$:

$$\left\|T - \sum_{k=1}^{n} T_k\right\| = \sup_{x\in B_X}\left\|Tx - \sum_{k=1}^{n} T_k x\right\| = \sup_{x\in B_X}\left\|\sum_{k=n+1}^{\infty} T_k x\right\| \leqslant \sup_{x\in B_X}\sum_{k=n+1}^{\infty} \|T_k x\|$$
$$\leqslant \sup_{x\in B_X}\sum_{k=n+1}^{\infty} \|T_k\|\,\|x\| = \sum_{k=n+1}^{\infty} \|T_k\| \xrightarrow[n\to\infty]{} 0$$

because the remainder term $\sum_{k=n+1}^{\infty} \|T_k\|$ of the covergent (in $\mathbb{R}$) series $\sum_{k=1}^{\infty} \|T_k\|$ tends to 0 as $n \to \infty$. □

Exercise 13.10. Let $\sum_{k=1}^{\infty} x_k$ be a convergent series in X, and let $T \in \mathcal{L}(X,Y)$. Prove that

$$T\Big(\sum_{k=1}^{\infty} x_k\Big) = \sum_{k=1}^{\infty} T x_k.$$

§ 14. The "geometric series formula" for operators

14.1. The composition of continuous linear operators

Proposition 14.1. *Let X, Y, and Z be normed spaces, and let $A\colon X \to Y$ and $B\colon Y \to Z$ be continuous linear operators, i.e., $A \in \mathcal{L}(X,Y)$ and $B \in \mathcal{L}(Y,Z)$. Then the composition $BA\colon X \to Z$ is linear and continuous, i.e., $BA \in \mathcal{L}(X,Z)$, and $\|BA\| \leqslant \|B\|\,\|A\|$.*

Proof. The linearity of the composition BA is (or, at least, ought to be) known from the introductory course in linear algebra.

Exercise 14.1. Prove that the composition $BA\colon X \to Z$ is linear.

For every $x \in X$,

$$\|(BA)x\| = \|B(Ax)\| \leqslant \|B\|\|Ax\| \leqslant \|B\|\|A\|\|x\|,$$

thus BA is bounded and $\|BA\| \leqslant \|B\|\|A\|$. □

Proposition 14.2. *Let X, Y, and Z be normed spaces, and let $A, A_n \in \mathcal{L}(X,Y)$ and $B, B_n \in \mathcal{L}(Y,Z)$, $n \in \mathbb{N}$, be such that*

$$A_n \xrightarrow[n\to\infty]{} A \quad \text{in } \mathcal{L}(X,Y) \qquad \text{and} \qquad B_n \xrightarrow[n\to\infty]{} B \quad \text{in } \mathcal{L}(Y,Z). \tag{14.1}$$

Then $B_nA_n \xrightarrow[n\to\infty]{} BA$ in $\mathcal{L}(X,Z)$.

Proof. For every $n \in \mathbb{N}$,

$$\begin{aligned}\|B_nA_n - BA\| &= \|B_nA_n - B_nA + B_nA - BA\| = \|B_n(A_n - A) + (B_n - B)A\| \\ &\leqslant \|B_n(A_n - A)\| + \|(B_n - B)A\| \leqslant \|B_n\|\|A_n - A\| + \|B_n - B\|\|A\|.\end{aligned}$$

The convergent sequence $(B_n)_{n=1}^{\infty}$ is bounded in $\mathcal{L}(Y,Z)$, i.e., there is an $M \geqslant 0$ such that

$$\|B_n\| \leqslant M \quad \text{for every } n \in \mathbb{N}.$$

Now,

$$\begin{aligned}\|B_nA_n - BA\| &\leqslant \|B_n\|\|A_n - A\| + \|B_n - B\|\|A\| \\ &\leqslant M\|A_n - A\| + \|B_n - B\|\|A\| \xrightarrow[n\to\infty]{} 0,\end{aligned}$$

because $\|A_n - A\| \xrightarrow[n\to\infty]{} 0$ and $\|B_n - B\| \xrightarrow[n\to\infty]{} 0$ by the assumption (14.1). □

14.2. The "geometric series formula" for operators

For a normed space X, a linear operator $A\colon X \to X$, and a number $n \in \{0\} \cup \mathbb{N}$, the *$n$-th power* A^n of the operator A is defined by

$$A^0 := I, \qquad A^1 := A, \qquad \text{and} \qquad A^n = AA^{n-1} \quad \text{for } n \geqslant 2.$$

(here I denotes the identity operator on X). From Proposition 14.1, it follows by induction that

$$\|A^n\| \leqslant \|A\|^n \quad \text{for all } n \in \{0\} \cup \mathbb{N}.$$

Theorem 14.3 (“the geometric series formula” for operators)**.** *Let X be a Banach space (i.e. a complete normed space), and let $A \in \mathcal{L}(X,X)$.*

(a) *Suppose that $\|A\| < 1$. Then the operator $I - A$ is invertible and $(I-A)^{-1}$ is continuous (i.e., $(I-A)^{-1} \in \mathcal{L}(X,X)$). More precisely,*

$$(I-A)^{-1} = \sum_{n=0}^{\infty} A^n$$

where the latter series converges in $\mathcal{L}(X,X)$.

(b) *Suppose that $\|I - A\| < 1$. Then the operator A is invertible and A^{-1} is continuous (i.e., $A^{-1} \in \mathcal{L}(X,X)$). More precisely,*

$$A^{-1} = \sum_{n=0}^{\infty} (I-A)^n$$

where the latter series converges in $\mathcal{L}(X,X)$.

Proof. (b) follows immediately from (a), because $A = I - (I - A)$.

(a). First observe that the series $\sum_{n=0}^{\infty} A^n$ is absolutely convergent, because

$$\sum_{n=0}^{\infty} \|A^n\| \leqslant \sum_{n=0}^{\infty} \|A\|^n < \infty$$

(the latter series is a geometric series where the multiplier $\|A\| < 1$). Since the space X is complete, also $\mathcal{L}(X,X)$ is complete (by Theorem 13.6), and thus the absolutely convergent series $\sum_{n=0}^{\infty} A^n$ converges in $\mathcal{L}(X,X)$ (by Theorem 12.2). Put $B := \sum_{n=0}^{\infty} A^n \in \mathcal{L}(X,X)$. In order to prove that $B = (I-A)^{-1}$, it suffices to show that

$$B(I-A) = I \quad \text{and} \quad (I-A)B = I. \tag{14.2}$$

To this end, putting, for every $m \in \mathbb{N}$,

$$B_m := \sum_{n=0}^{m} A^n,$$

it suffices to show that

$$B_m(I-A) \xrightarrow[m\to\infty]{} I \quad \text{and} \quad (I-A)B_m \xrightarrow[m\to\infty]{} I \qquad \text{in } \mathcal{L}(X,X), \tag{14.3}$$

because, since $B_m \xrightarrow[m\to\infty]{} B$, by Proposition 14.2,

$$B_m(I-A) \xrightarrow[m\to\infty]{} B(I-A) \quad \text{and} \quad (I-A)B_m \xrightarrow[m\to\infty]{} (I-A)B \qquad \text{in } \mathcal{L}(X,X);$$

therefore, if (14.3) holds, then one has (14.2) by the uniqueness of the limit.

It remains to verify the conditions (14.3):

$$(I-A)B_m = (I-A)\sum_{n=0}^{m} A^n = \sum_{n=0}^{m} A^n - A\sum_{n=0}^{m} A^n = \sum_{n=0}^{m} A^n - \sum_{n=0}^{m} AA^n$$
$$= \sum_{n=0}^{m} A^n - \sum_{n=0}^{m} A^{n+1} = \sum_{n=0}^{m} A^n - \sum_{n=1}^{m+1} A^n = A^0 - A^m = I - A^m;$$

since $\|A\| < 1$, one has $\|A^m\| \leqslant \|A\|^m \xrightarrow[m\to\infty]{} 0$, thus $A^m \xrightarrow[m\to\infty]{} 0$ and

$$(I-A)B_m = I - A^m \xrightarrow[m\to\infty]{} I;$$

similarly one obtains that $B_m(I-A) \xrightarrow[m\to\infty]{} I$, and the proof is complete. □

Corollary 14.4. *Let X and Y be normed spaces with one of them being complete, and let $A, B \in \mathcal{L}(X,Y)$ be such that A is invertible and the inverse A^{-1} is continuous (i.e., $A^{-1} \in \mathcal{L}(Y,X)$), and*

$$\|B - A\| < \frac{1}{\|A^{-1}\|}.$$

Then also B is invertible and the inverse B^{-1} is continuous, i.e., $B^{-1} \in \mathcal{L}(Y,X)$.

PROOF. [I] First consider the case when Y is complete. In this case, one can write

$$B = \big(I - (A-B)A^{-1}\big)\, A.$$

Since

$$\|(A-B)A^{-1}\| \leqslant \|A-B\|\, \|A^{-1}\| < \frac{1}{\|A^{-1}\|}\, \|A^{-1}\| = 1$$

and the space Y is complete, by "the geometric series formula" (Theorem 14.3), the operator $C := I - (A-B)A^{-1} \in \mathcal{L}(Y,Y)$ is invertible and $C^{-1} \in \mathcal{L}(Y,Y)$. Thus also $B = CA$ in invertible and $B^{-1} = A^{-1}C^{-1} \in \mathcal{L}(Y,X)$.

[II] In the case when X is complete, write

$$B = A\,\big(I - A^{-1}(A-B)\big).$$

The rest of the argument is symmetric to that in (I). □

§ 15. Finite dimensional normed spaces

15.1. Pairwise isomorphness of n-dimensional normed spaces

Definition 15.1. Let X and Y be normed spaces. An operator $T \in \mathcal{L}(X, Y)$ is called an *isomorphism,* if it is a bijection whose inverse is continuous, i.e., $T^{-1} \in \mathcal{L}(Y, X)$. The spaces X and Y are said to be *isomorphic,* if there exists an isomorphism $T \in \mathcal{L}(X, Y)$. In this case, one also says that X *is isomorphic to* Y or that Y *is isomorphic to* X.

Proposition 15.1. *Let X and Y be normed spaces. A linear surjection $T\colon X \to Y$ is an isomorphism if and only if there are constants $\alpha, \beta > 0$ such that, for every $x \in X$,*

$$\alpha\|x\| \leqslant \|Tx\| \leqslant \beta\|x\|. \tag{15.1}$$

Proof.

Exercise 15.1. Prove Proposition 15.1.

□

Definition 15.2. Let $n \in \mathbb{N}$. One says that a normed space X is *n-dimensional* and writes $\dim X = n$ if X is n-dimensional as a linear space, i.e., there are $e_1, \ldots, e_n \in X$ such that every $x \in X$ admits a unique representation

$$x = \sum_{j=1}^{n} \xi_j e_j \quad \text{where } \xi_1, \ldots, \xi_n \in \mathbb{K}.$$

The system $\{e_1, \ldots, e_n\}$ is called a *basis* for X.

A normed space X is said to be *finite dimensional* if it is finite dimensional as a linear space, i.e., either X is n-dimensional for some $n \in \mathbb{N}$ or $X = \{0\}$. In this case, one writes $\dim X < \infty$.

A normed space X is said to be *infinite dimensional* if it is not finite dimensional. In this case, one writes $\dim X = \infty$.

Remark 15.1. From the introductory course in linear algebra, one remembers that *a linear space X is n-dimensional if and only if there exists a linearly independent system in X consisting of n vectors whereas every system consisting of $n+1$ vectors in X is linearly dependent.*

Example 15.1. The spaces ℓ_p^n, where $n \in \mathbb{N}$ and $1 \leqslant p \leqslant \infty$, are n-dimensional: the system

$$e_1 = (\underbrace{1, 0, \ldots, 0}_{n}), \quad e_2 = (\underbrace{0, 1, 0, \ldots, 0}_{n}), \quad \ldots\ldots, \quad e_n = (\underbrace{0, \ldots, 0, 1}_{n})$$

is a basis for ℓ_p^n.

All the other normed spaces introduced in Section 4 are infinite dimensional.

In this subsection, we shall show that *any two n-dimensional ($n \in \mathbb{N}$) normed spaces are isomorphic.* Our first step towards this result is the following lemma.

Lemma 15.2. *Let $n \in \mathbb{N}$ and let X be an n-dimensional normed space with a basis $\{e_1, \ldots, e_n\}$. The mapping*

$$T\colon\ \ell_\infty^n \ni (\xi_j)_{j=1}^n \longmapsto \sum_{j=1}^n \xi_j e_j \in X \tag{15.2}$$

is an isomorphism. Thus every n-dimensional normed space is isomorphic to ℓ_∞^n.

PROOF. Clearly T is a linear bijecton. In order to see that T is an isomorphism, it remains to find $\alpha, \beta > 0$ satisfying (15.1) for every $x \in \ell_\infty^n$.

On one hand, for every $x = (\xi_j)_{j=1}^n \in \ell_\infty^n$, one has

$$\|Tx\| = \left\|\sum_{j=1}^n \xi_j e_j\right\| \leqslant \sum_{j=1}^n |\xi_j|\|e_j\| \leqslant \left(\sum_{j=1}^n \|e_j\|\right) \max_{1\leqslant j\leqslant n} |\xi_j| = \left(\sum_{j=1}^n \|e_j\|\right)\|x\|,$$

i.e., the second inequality in (15.1) holds for $\beta := \sum_{j=1}^n \|e_j\|$.

On the other hand, if $x = 0$, the first inequality in (15.1) holds for every $\alpha > 0$. If $x \neq 0$, this inequality is equivalent to $\alpha \leqslant \dfrac{1}{\|x\|}\|Tx\|$, i.e., $\alpha \leqslant \left\|\dfrac{1}{\|x\|}Tx\right\|$, i.e.,

$$\left\|T\left(\frac{x}{\|x\|}\right)\right\| \geqslant \alpha.$$

Thus, denoting by S the *unit sphere of ℓ_∞^n*, i.e. $S := \{x \in \ell_\infty^n\colon\ \|x\| = 1\}$, in order to find the desired $\alpha > 0$, it suffices to show that

$$\inf_{x\in S} \|Tx\| > 0, \tag{15.3}$$

because, in this case, one can take $\alpha := \inf_{x\in S} \|Tx\|$ (observe that $\dfrac{x}{\|x\|} \in S$ whenever $x \in \ell_\infty^n \backslash \{0\}$).

It remains to prove (15.3). To this end, first observe that the function

$$f\colon\ S \ni x \longmapsto \|Tx\| \in \mathbb{R}$$

is continuous (because it is the composition of the continuous mappings $T|_S\colon\ S \to X$ and $\|\cdot\|\colon\ X \to \mathbb{R}$). The unit sphere S is relatively compact (because, by Example 9.2, bounded sets in ℓ_∞^n are relatively compact) and closed (because every sphere in a metric space is closed); thus S is compact. By Theorem 9.6, the continuous function f on the compact set S attains its infimum, i.e., there is an $x_0 = (\xi_j^0)_{j=1}^n \in S$ such that

$$f(x_0) = \inf_{x\in S} f(x) = \inf_{x\in S} \|Tx\|.$$

It remains to show that $f(x_0) > 0$. Suppose for contradiction that $f(x_0) = 0$, i.e., $\|Tx_0\| = \left\|\sum_{j=1}^n \xi_j^0 e_j\right\| = 0$, i.e. $\sum_{j=1}^n \xi_j^0 e_j = 0$. Since $\{e_1, \ldots, e_n\}$ is a basis for X, it follows that $\xi_1^0 = \cdots = \xi_n^0 = 0$. On the other hand, $\|x_0\| = \max_{1\leqslant j\leqslant n} |\xi_j^0| = 1$, a contradiction. □

Corollary 15.3. *Let $n \in \mathbb{N}$. Any two n-dimensional normed spaces are isomorphic.*

PROOF. By Lemma 15.2, every n-dimensional normed space is isomorphic to ℓ_∞^n. Thus it suffices to solve the following exercise.

Exercise 15.2. Let X, Y, and Z be normed spaces. Prove that

(a) if $T\colon X \to Y$ and $S\colon Y \to Z$ are isomorphisms, then also the composition $ST\colon X \to Z$ is an isomorphism;

(b) if X is isomorphic to Y, and Y is isomorphic to Z, then X is isomorphic to Z

□

Besides Corollary 15.3, Lemma 15.2 has some more nice corollaries.

Corollary 15.4. *Any finite dimensional normed space is complete.*

PROOF. By Lemma 15.2, every n-dimensional normed space is isomorphic to ℓ_∞^n. The space ℓ_∞^n is complete (by Example 6.5). Thus it remains to solve the following exercise.

Exercise 15.3. Prove that if one of two isomorphic normed spaces is complete, then so is the other.

□

Corollary 15.5. *A finite dimensional subspace of a normed space is closed.*

PROOF. By Corollary 15.4, any finite dimensional subspace of a normed space is complete. By Proposition 6.2, (a), any complete subspace of a metric space is closed, thus any finite dimensional subspace of a normed space is closed. □

Corollary 15.6. *Any bounded set in a finite dimensional normed space is relatively compact.*

PROOF. Let B be a bounded subset of an n-dimensional normed space, and let $T\colon \ell_\infty^n \to X$ be the isomorphism from Lemma 15.2. Then $T^{-1}(B) = \{T^{-1}z\colon\ z \in B\}$ is a bounded subset of ℓ_∞^n (because the bounded operator T^{-1} maps the bounded set B into a bounded set by Proposition 13.2), thus $T^{-1}(B)$ is relatively compact (because every bounded subset of ℓ_∞^n is relatively compact by Example 9.2). It follows that also $B = T\big(T^{-1}(B)\big)$ is relatively compact (because the continuous operator T maps the relatively compact set $T^{-1}(B)$ to a relatively compact set by Exercise 9.2). □

Definition 15.3. Two norms $\|\cdot\|$ and $|||\cdot|||$ on a linear space X are said to be *equivalent* if there are constants $\alpha, \beta > 0$ such that

$$\alpha\|x\| \leqslant |||x||| \leqslant \beta\|x\| \quad \text{for every } x \in X.$$

By Proposition 15.1, it is clear that the equivalence of the norms $\|\cdot\|$ and $|||\cdot|||$ means that the formal identity operator

$$\big(X, \|\cdot\|\big) \ni x \longmapsto x \in \big(X, |||\cdot|||\big)$$

is an isomorphism.

Corollary 15.7. *Any two norms on a finite-dimensional linear space are equivalent.*

Proof. Let X be an n-dimensional linear space with basis $\{e_1, \dots, e_n\}$. It suffices to show that the formal identity operator

$$J\colon\ \big(X, \|\cdot\|\big) \ni x \longmapsto x \in \big(X, |||\cdot|||\big)$$

is an isomorphism. To this end, recall that, by Lemma 15.2, the mappings

$$T\colon\ \ell_\infty^n \ni (\xi_j)_{j=1}^n \longmapsto \sum_{j=1}^n \xi_j e_j \in \big(X, \|\cdot\|\big) \quad \text{and} \quad S\colon\ \ell_\infty^n \ni (\xi_j)_{j=1}^n \longmapsto \sum_{j=1}^n \xi_j e_j \in \big(X, |||\cdot|||\big)$$

are isomorphisms. Observing that $J = ST^{-1}$, it remains to apply Exercise 15.2, (a). □

Corollary 15.8. *Let X and Y be normed spaces. Suppose that* $\dim X < \infty$. *Then every linear operator $T\colon X \to Y$ is continuous.*

Proof.

***Exercise 15.4.** Prove Corollary 15.8.

□

Exercise 15.5. Prove that a continuous linear operator between normed spaces X an Y remains continuous if the original norms in X and Y are replaced by equivalent norms.

15.2. Riesz's lemma. Non-compactness of the unit sphere in infinite dimensional normed spaces

By Corollary 15.6, every bounded set in a finite dimensional normed space is relatively compact. In particular, the unit sphere of a finite dimensional normed space is relatively compact. In this subsection, we shall see that this is a characteristic feature of finite dimensional normed spaces.

The crucial step in showing that the unit sphere of an infinite dimensional normed space is always non-compact is the following theorem which is of interest in its own right.

Theorem 15.9 (Riesz's lemma). *Let X be a normed space, let Y be a proper closed subspace of X (i.e., Y is a closed subspace of X such that $Y \neq X$), and let* $0 < \varepsilon < 1$. *Then there is an $x_\varepsilon \in X$ with $\|x_\varepsilon\| = 1$ such that*

$$\|x_\varepsilon - y\| \geqslant 1 - \varepsilon \quad \textit{for every } y \in Y, \tag{15.4}$$

i.e., $\rho(x_\varepsilon, Y) \geqslant 1 - \varepsilon$.

Recall that in a metric space X, the distance $\rho(x, A)$ between a point $x \in X$ and a subset $A \subset X$ is defined by $\rho(x, A) := \inf_{y \in A} \rho(x, y)$. It is clear that $\rho(x, A) = 0$ if and only if $x \in \overline{A}$.

Proof of Theorem 15.9. Since Y is a proper subspace, there is an $x \in X \backslash Y$. Since Y is closed, $\overline{Y} = Y$, therefore $x \notin \overline{Y}$, hence $\rho(x, Y) > 0$. Thus one can choose a $z \in Y$ so that $\|x - z\| < \dfrac{\rho(x, Y)}{1 - \varepsilon}$. Putting $x_\varepsilon := \dfrac{x - z}{\|x - z\|}$, one has $\|x_\varepsilon\| = 1$, and it remains to verify (15.4): whenever $y \in Y$, one has

$$\begin{aligned}\|x_\varepsilon - y\| &= \left\| \frac{x - z}{\|x - z\|} - y \right\| = \left\| \frac{x - z - \|x - z\|\, y}{\|x - z\|} \right\| \\ &= \frac{1}{\|x - z\|} \left\| x - z - \|x - z\|\, y \right\| = \frac{1}{\|x - z\|} \left\| x - \big(z + \|x - z\|\, y\big) \right\| \\ &\geqslant \frac{\rho(x, Y)}{\|x - z\|} > 1 - \varepsilon.\end{aligned}$$

□

Corollary 15.10. *The unit sphere of an infinite dimensional normed space is not compact.*

Proof. Let X be an infinite dimensional normed space. It suffices to construct a sequence $(x_k)_{k=1}^{\infty}$ in the unit sphere $S_X := \{x \in X\colon \|x\| = 1\}$ of X such that, for all $k, l \in \mathbb{N}$ with $k \neq l$,

$$\|x_k - x_l\| \geqslant \frac{1}{2}, \tag{15.5}$$

because such a sequence (x_k) does not contain any Cauchy subsequences, thus it does not contain any convergent subsequences, and the non-compactness of the unit sphere S_X follows.

In order to construct the desired sequence (x_k), first choose an arbitrary $x_1 \in S_X$ and proceed inductively as follows:

- given $n \in \mathbb{N}$ and $\{x_1, \ldots, x_n\}$ satisfying (15.5) for all $k, l \in \{1, \ldots, n\}$ with $k \neq l$, put $Y := \operatorname{span}\{x_1, \ldots, x_n\}$ and choose an $x_{n+1} \in S_X$ so that $\rho(x_{n+1}, Y) \geqslant \dfrac{1}{2}$ (since, by Corollary 15.5, the subspace Y is closed, such an x_{n+1} exists by Riesz's lemma).

The sequence (x_k) obtained in this process clearly satisfies (15.5) for all $k, l \in \mathbb{N}$ with $k \neq l$. □

Corollary 15.11. *Let X be a normed space. The following assertions are equivalent:*

(i) *X is finite dimensional;*

(ii) *every bounded set in X is relatively compact;*

(iii) *the unit sphere $S_X := \{x \in X\colon \|x\| = 1\}$ is compact.*

Proof. (i)⇒(ii) is Corollary 15.6.

(ii)⇒(iii). Assume that (ii) holds. Then S_X is relatively compact (because it is bounded). Since S_X is closed (because every sphere in a metric space is closed), it is compact.

(iii)⇒(i) follows from Corollary 15.10.

□