**Ülesanne.** Leidke piirväärtus

$$\lim_{n\to\infty}\left(\frac{n^2}{n^3+1}+\frac{n^2}{n^3+2^3}+\ldots+\frac{n^2}{n^3+n^3}\right). \tag{1}$$

*Lahendus.* Saame, et

$$\lim_{n\to\infty}\left(\frac{n^2}{n^3+1}+\frac{n^2}{n^3+2^3}+\ldots+\frac{n^2}{n^3+n^3}\right)= \\ \lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n}\cdot\frac{1}{1+\left(\frac{k}{n}\right)^3}.$$

Paneme tähele, et funktsioon $f(x)=\dfrac{1}{1+x^3}$ on lõigus $[0,1]$ pidev, seega ta on lõigus $[0,1]$ integreeruv. Leiame määramata integraali:

$$\int\frac{\mathrm{d}x}{1+x^3}=\int\left(\frac{1}{(3(x+1)}-\frac{x-2}{3(x^2-x+1)}\right)\mathrm{d}x= \\ =\frac{1}{3}\cdot\ln|x+1|-\frac{1}{6}\cdot\ln(x^2-x+1)+\frac{\sqrt{3}}{3}\cdot\arctan\frac{2x-1}{\sqrt{3}}+C.$$

Seega Newton–Leibnizi valemi kohaselt

$$\int_0^1\frac{\mathrm{d}x}{1+x^3}=\frac{1}{3}\ln 2+\frac{2\sqrt{3}}{3}\cdot\frac{\pi}{6}=:A.$$

Tõestame, et piirväärtus (1) on $A$. Peame näitama, et

$$\forall\varepsilon>0\quad\exists N\in\mathbb{N}\quad:\quad\forall n\in\mathbb{N}\quad n\geqslant N\Rightarrow\left|\sum_{k=1}^{n}\frac{1}{n}\cdot\frac{1}{1+\left(\frac{k}{n}\right)^3}-A\right|<\varepsilon. \tag{2}$$

Olgu $\mathscr{T}$ lõigu $[0,1]$ alajaotuste hulk. Fikseerime $\varepsilon>0$. Et $f$ on integreeruv lõigus $[0,1]$, siis leidub $\delta>0$ omadusega

$$\forall T=T[x_0,\ldots,x_n]\in\mathscr{T}\quad\forall\boldsymbol{\xi}=(\xi_1,\ldots,\xi_n)\in[x_0,x_1]\times\ldots\times[x_{n-1},x_n] \\ \lambda(T)<\delta\quad\Rightarrow\quad\left|\sum_{k=1}^{n}\frac{1}{1+\xi_k^3}\cdot(x_k-x_{k-1})-A\right|<\varepsilon. \tag{3}$$

Valime $N=\left\lceil\dfrac{2}{\delta}\right\rceil$, siis $N\geqslant\dfrac{2}{\delta}$, mistõttu $\dfrac{1}{N}<\dfrac{2}{N}\leqslant\delta$. Olgu $n\geqslant N$. Valime $x_k=\dfrac{k}{n}$, kus $k=0,1,\ldots,n$, nii saame lõigu $[0,1]$ alajaotuse $T$, kusjuures $x_k-x_{k-1}=\dfrac{1}{n}$, seega ka $\lambda(T)=\dfrac{1}{n}\leqslant\dfrac{1}{N}<\delta$. Valime $\xi_k=\dfrac{k}{n}$, siis $\xi_k=x_k\in[x_{k-1},x_k]$, $k=1,\ldots,n$.

Tingimus (3) annab nüüd, et

$$\left|\sum_{k=1}^{n} \frac{1}{1+\left(\frac{k}{n}\right)^3} \cdot \frac{1}{n} - A\right| < \varepsilon.$$

Sellega on lause (2) tõestatud, mistõttu

$$\lim_{n\to\infty}\left(\frac{n^2}{n^3+1} + \frac{n^2}{n^3+2^3} + \ldots + \frac{n^2}{n^3+n^3}\right) = A = \frac{1}{3}\ln 2 + \frac{2\sqrt{3}}{3} \cdot \frac{\pi}{6}.$$