MTMM.00.005 Numerical Methods
Practical lesson 1: Errors in calculations.

February 12, 2019

To get a positive result, student has to solve all of the exercises given in practical lessons, explain the programming code and answer the questions.

The deadline for the first set of exercises is February 19th.

Python

In practical lessons we use Python or any other programming language. Here are some simple examples. For further assistance see `https://docs.python.org/3/tutorial/`.

```
v0=1
t=10
g=9.8
s=v0*t+0.5*g*t**2
print(s)
```

```
C = -20
dC = 5
while C <= 40:
   F = (9.0/5)*C + 32
   print(C,F)
   C = C + dC
```

```
Ckraadid = [-20,-15,-10,-5,0,5,10,15,20,25,30,35,40]
for C in Ckraadid:
   F = (9.0/5)*C + 32
   print(C,F)
```

```
import math
r = 2.3
pindala = math.pi*r*r
print(pindala)
```

Package numpy

Quite often we use the *numpy* package. Short introduction could be found here `https://docs.scipy.org/doc/numpy/user/quickstart.html`. And again some simple examples and exercises with possible solutions.

```
import numpy as np

def f(x):   #defineerib ühe muutuja funktsiooni f
```

```
   return np.exp(-x)   #tagastab e^{-x}

a=np.linspace(-1,1,5)   #loob 5-komponendilise vektori
                        #[-1,-0.5,0,0.5,1]
b=np.zeros(len(a))   #loob nullidest koosneva vektori,
                     #mille pikkus on võrdne vektori a pikkusega
a[0]   #tagastab vektori a esimese elemendi
a[1:-1]   #tagastab vektori a elemendid teisest kuni eelviimaseni
f(a)   #rakendab funktsiooni f vektori a kõikidele elementidele
       #(tulemus on vektor)
np.zeros((3,5))   #loob nullidest koosneva 3*5 maatriksi
np.ones((3,5))   #loob ühtedest koosneva 3*5 maatriksi
A=np.ones((3,5))
A*a   #maatriksite A ja a korrutis elementide kaupa
      #(ei ole maatriksite korrutamine)
B=np.eye(2)   #loob teist järku ühikmaatriksi
C=np.array([[0.0,-1.0],[1.0,0.0]])
D=np.dot(C,C)   #maatriksite C ja C korrutis
```

Exercise 1. Find the sum of the first n natural numbers.

```
def s(n):
   s = 0
   for i in range(n+1):
      s = s+i
   return s
```

Graphs in Pythonis

There are several options to draw graphs, one of them is the package *Matplotlyb* (see `https://matplotlib.org/users/index.html`).

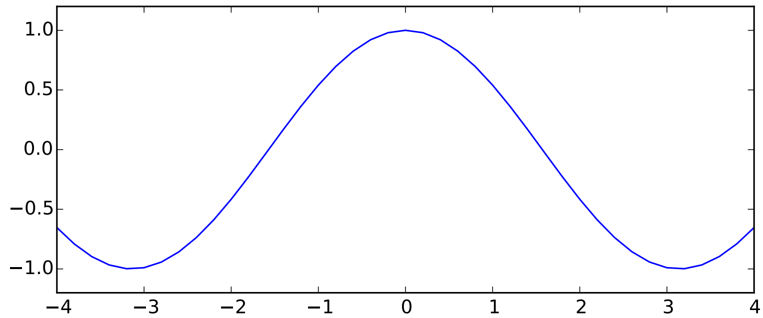
Exercise 2. Plot the function $cos(x)$ for $x \in [-4; 4]$.

```
import numpy as np
import matplotlib.pyplot as plt

def h(t):
   return np.cos(t)

t=np.linspace(-4,4,41)   #loome argumentide vektori
y=h(t)   #rakendame funktsiooni h vektorile t

plt.plot(t,y)   #joonistab murdjoone (t_i,y_i), i=0..len(t)-1
plt.show()   #näitab joonist
```



Exercise 3. Plot the functions $f_1(t) = t^2 e^{-t^2}$ and $f_2(t) = t^4 e^{-t^2}$, $t \in [0, 3]$, in the same graph. Save the result to the file Fig1.png.

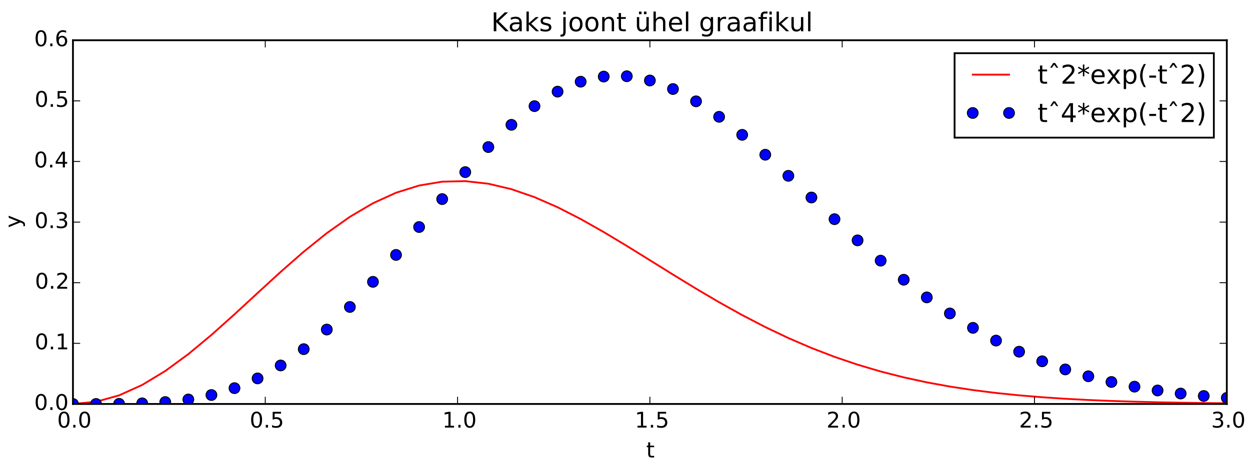
```
import numpy as np
import matplotlib.pyplot as plt

def f1(t):   #defineerib funktsiooni f1
   return t**2*np.exp(-t**2)

def f2(t):   #defineerib funktsiooni f2
   return t**2*f1(t)

t = np.linspace(0,3,51)   #loob sõlmed
y1 = f1(t)   #funktsiooni f1 väärtused sõlmedes
y2 = f2(t)   #funktsiooni f2 väärtused sõlmedes

plt.plot(t,y1,'r-')   #f1 graafik: punane (r) joon (-)
plt.plot(t,y2,'bo')   #f2 graafik: sinised (b) ringid (o)
plt.xlabel('t')   #x-telg
plt.ylabel('y')   #y-telg
plt.legend(['t^2*exp(-t^2)','t^4*exp(-t^2)'])   #joonte tähendused
plt.title('Kaks joont ühel graafikul')   #graafiku nimi
plt.savefig ('Fig1.png')  #salvestab graafiku faili Fig1.png
plt.show()
```



Exercise 4. Compute the first 100 terms of the array $a_k = \sin(k)$, $k \in \mathbb{N}$, and plot them as a histogram. Find the first index k for which $a_k > d$, where $0.9 < d < 1$ is some fixed number.

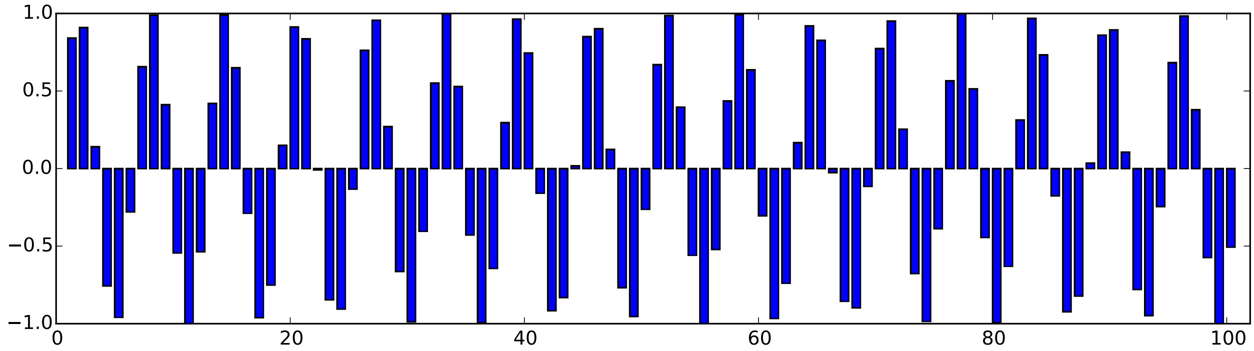
```
import numpy as np
import matplotlib.pyplot as plt

k = np.linspace(1,100,100)
a = np.sin(k)
def ind(d):
   i=0
   while a[i]<=d:
      i=i+1
   return i+1

print('Enter the value of d: ')
d = float(input())
print('a_',ind(d),'>',d)

fig = plt.figure()   # Kõigepealt loome joonist tähistava objekti
ax = fig.add_subplot(1,1,1)   # ja lisame joonisele joonestusala
ax.set_ylim(-1,1)   # Määrame y-telje nähtavuspiirkonna
ax.set_xlim(0,102)   # Määrame x-telje nähtavuspiirkonna
ax.bar(k,a,0.7)   # Lisame tulbad (0.7 on tulba laius)
plt.show()
```

Above program code has an error – it fails for some values of d. How to fix this problem?



Errors in calculations

Consider a number a, which is approximately equal to number A. Call number A exact. The **actual error** is $\Delta a = A - a$. The **absolute error** of an approximate number a is any number $\Delta > 0$ that satisfies the inequality

$$|\Delta a| = |A - a| \leq \Delta.$$

Smaller Δ gives better estimate. Since $a - \Delta \leq A \leq a + \Delta$, this situation is also denoted by $A = a \pm \Delta$.

The **relative error** of an approximate number a is any number $\delta > 0$ that satisfies the inequality

$$\left|\frac{\Delta a}{a}\right| = \left|\frac{A - a}{a}\right| \leq \delta.$$

One can take $\delta = \dfrac{\Delta}{|a|}$.

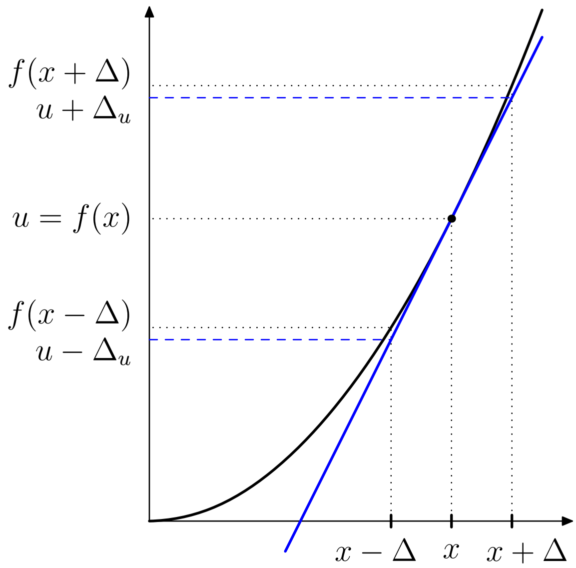
The approximate values $x_1, \ldots, x_n$ and corresponding absolute errors $\Delta_1, \ldots, \Delta_n$ are given. Assume that the function $u = f(x_1, \ldots, x_n)$ is differentiable and the numbers Δ_i are small. Then the absolute error of u is

$$\Delta_u = \sum_{i=1}^{n} \left|\frac{\partial u}{\partial x_i}\right| \Delta_i$$

and the relative error of u is

$$\delta_u = \frac{\Delta_u}{|u|} = \sum_{i=1}^{n} \left|\frac{\partial \ln |u|}{\partial x_i}\right| \Delta_i.$$

In case of addition and subtraction the absolute errors are added, in case of multiplication and division the relative errors are added.



Example. Consider the first example from `http://www.ima.umn.edu/~arnold/455.f96/disasters.html` The Patriot Missile battery's internal clock measured the time in tenths of second. The value

$$\frac{1}{10} = \frac{1}{2^4} + \frac{1}{2^5} + \frac{1}{2^8} + \frac{1}{2^9} + \frac{1}{2^{12}} + \frac{1}{2^{13}} + \ldots = \frac{\frac{1}{2^4}}{1 - \frac{1}{2^4}} + \frac{\frac{1}{2^5}}{1 - \frac{1}{2^4}} = \frac{1}{15} + \frac{1}{30}$$

has a non-terminating binary expansion 0.00011001100110011001100110011 .... Due to the 24 bit fixed point register the number 0.00011001100110011001100 was used in calculations introducing an error of $0.0000000000000000000000011001 1\ldots \approx 9.5 \cdot 10^{-8}$. The Patriot battery had been up around 100 hours, the resulting time error due to the magnified chopping error was $9.5 \cdot 10^{-8} \cdot 100 \cdot 60 \cdot 60 \cdot 10 = 0.34$ seconds. An Iraqi Scud travels at about 1676 meters per second. In 0.34 seconds it travels about 570 meters. This was far enough that the incoming Scud was outside the "range gate" that the Patriot tracked.

Compulsory exercises

Exercise 5. Find the volume of a layer on the Earth. The thickness of the layer is $d = 35$ cm and the Earth's equatorial radius is $R = 6378.135$ km.
Estimate the absolute error and the relative error of the result. Would the result be different if instead of the formula

$$V = \frac{4}{3}\pi(3R^2d + 3Rd^2 + d^3)$$

you would use the formula

$$V = \frac{4}{3}\pi(R+d)^3 - \frac{4}{3}\pi R^3?$$

Exercise 6. Determine the number of terms necessary to approximate $\cos(x)$ at $x = \pi$ with precision 10^{-8} using the Taylor series expansion

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots.$$

Remark 1. The **Taylor series** of a function f that is infinitely differentiable at a point a is

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \dots.$$

In our exercise $a = 0$ and

$$\begin{aligned}
\cos(x) &= \cos(0) + \cos'(x)|_{x=0}\frac{x}{1!} + \cos''(x)|_{x=0}\frac{x^2}{2!} + \cos'''(x)|_{x=0}\frac{x^3}{3!} + \cos''''(x)|_{x=0}\frac{x^4}{4!} + \dots \\
&= 1 - \sin(x)|_{x=0}\frac{x}{1!} - \cos(x)|_{x=0}\frac{x^2}{2!} + \sin(x)|_{x=0}\frac{x^3}{3!} + \cos(x)|_{x=0}\frac{x^4}{4!} + \dots \\
&= 1 - 0 - \frac{x^2}{2!} + 0 + \frac{x^4}{4!} + \dots
\end{aligned}$$

When $a = 0$, the series is also called a Maclaurin series.

Märkus 2. The degree n **Taylor polynomial** for f at a is

$$P_n(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

In exercise 6 you have to find a number n such that $|\cos(\pi) - P_n(\pi)| \leq 10^{-8}$.