MTMM.00.005 Numerical Methods
Practical lesson 2: Fixed-point iteration for solving a single equation.

February 19, 2019

To get a positive result, student has to solve all of the exercises given in practical lessons, explain the programming code and answer the questions.

The deadline for the second set of exercises is February 26th.

NB! In order to use the nonstandard packages as *numpy* and *matplotlib* one could use Command Prompt and type

```
H:\nm> "C:\Program Files\Anaconda3\python" <filename>.py
```

Here `H:\nm` is the folder for `<filename>.py`.

Fixed-point iteration

Consider equations of the form $f(x) = 0$, where f is any continuous function. We replace $f(x) = 0$ with an equivalent equation

$$x = g(x),$$

at least in the neighbourhood of the solution x^*. This could be done in several ways, for example by taking $g(x) = x - \psi(x)f(x)$, where ψ is a continuous function which does not have zeros in the neighbourhood of the solution x^*. The easiest way is to use a constant function $\psi(x) = \beta = \text{const} \neq 0$.

Description of the method. Choose initial value x_0. Find

$$x_{n+1} = g(x_n), \quad n = 0, 1, \dots .$$

If g is continuous and iteration method converges then the sequence x_n converges to the solution of initial equation.

Theorem (Convergence Theorem). Assume that $g : [a, b] \to [a, b]$ and g is a contraction on interval $[a, b]$, i.e., exists $q < 1$ such that $|g(x_1) - g(x_2)| \le q|x_1 - x_2|$ for all $x_1, x_2 \in [a, b]$. Then equation $x = g(x)$ has exactly one solution x^* in interval $[a, b]$. For every $x_0 \in [a, b]$ it holds $x_n \to x^*$ with the estimate

$$|x_n - x^*| \le \frac{q^n}{1-q}|x_1 - x_0|.$$

Remark. In case of differentiable g the requirement of contraction in theorem may be replaced by the stronger assumption

$$|g'(x)| \le q < 1 \quad \forall x \in [a, b].$$

Stopping criteria for fixed-point iteration. Under the assumptions of Convergence Theorem we have

$$|x_n - x^*| = |g(x_{n-1}) - g(x^*)| \le q|x_{n-1} - x^*| \le q(|x_n - x_{n-1}| + |x_n - x^*|).$$

Thus

$$|x_n - x^*| \le \frac{q}{1-q}|x_n - x_{n-1}|, \quad n = 1, 2, \dots .$$

One could use a stopping criteria

$$\frac{q}{1-q}|x_n - x_{n-1}| \le \varepsilon.$$

Exercise. Use the fixed-point iteration method to solve the following equation

$$(10+\alpha)x - (1+0.001\alpha)^x = 0.$$

The value of α is unique for each student and given by the lecturer. Find smaller solution with accuracy 10^{-6} and greater solution with accuracy 10^{-3}. How many iterations does it take to find the solutions? Print out all computed approximate values.

Tip: This equation has two solutions, smaller of them is in $[0,1]$. Replace the equation with an equivalent equation $x = g(x)$ for which there exists $[a,b]$ such that $g([a,b]) \subset [a,b]$ and $|g'(x)| \le q < 1$ for all $x \in [a,b]$.