MTMM.00.005 Numerical Methods
Practical lesson 3: Newton's method for solving a single equation.

February 26, 2019

To get a positive result, student has to solve all of the exercises given in practical lessons, explain the programming code and answer the questions.

The deadline for the second set of exercises is March 5th.

NB! In order to use the nonstandard packages as *numpy* and *matplotlib* one could use Command Prompt and type

```
H:\nm> "C:\Program Files\Anaconda3\python" <filename>.py
```

Here `H:\nm` is the folder for `<filename>.py`.

Newton's method

Consider the equation $f(x) = 0$. Let $x^* \in [a, b]$ be the solution of this equation, let f be twice continuously differentiable on $[a, b]$, and $x_n \in [a, b]$ just some value. According to Taylor's formula

$$f(x) = f(x_n) + f'(x_n)(x - x_n) + \frac{1}{2} f''(\xi)(x - x_n)^2, \quad \xi \in (x, x_n).$$

If we omit the remainder, then we can replace the initial equation $f(x) = 0$ by its linearization

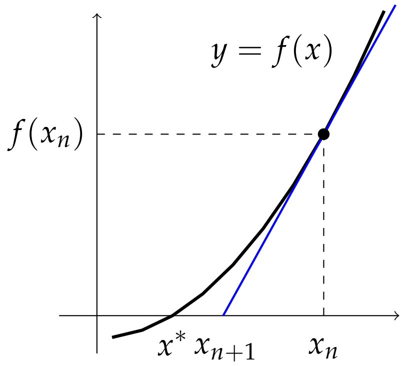
$$f(x_n) + f'(x_n)(x - x_n) = 0.$$

The solution of this equation gives us next approximations

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, \ldots \tag{1}$$

In Newton's method, one initial value, x_0, is given and the next approximations are found according to formula (1). A step can be carried out if $f'(x_n) \neq 0$.

What is the geometrical meaning of Newton's method? The approximation x_{n+1} is the x-intercept of the tangent line $y = f(x_n) + f'(x_n)(x - x_n)$ to the graph of f at $(x_n, f(x_n))$.



Newton's method can be viewed as a special case of the ordinary iteration method $x_{n+1} = g(x_n)$, $n = 0, 1, \ldots$, where

$$g(x) = x - \frac{f(x)}{f'(x)}.$$

For that reason, all the results that we obtained for the ordinary iteration method, can be applied here. Doing so the estimation of

$$|g'(x)| = \left| \frac{f(x) f''(x)}{(f'(x))^2} \right|$$

could be difficult, thus in practice one could use

$$|x_n - x_{n-1}| \le \varepsilon$$

as a stopping criteria for Newton's method. If for some interval $[x^* - \delta, x^* + \delta]$ we have $x_n, x_{n-1} \in [x^* - \delta, x^* + \delta]$ and $|g'(x)| \le q = 0.5$ for all $x \in [x^* - \delta, x^* + \delta]$, then

$$|x_n - x^*| \le \frac{q}{1-q}|x_n - x_{n-1}| = |x_n - x_{n-1}| \le \varepsilon$$

and the precision required is guaranteed.

Theorem. Let f be twice continuously differentiable, $f(x^*) = 0$ and $f'(x^*) \ne 0$, then there exists a $\delta > 0$ such that Newton's method generates a sequence $\{x_n\}_{n=1}^{\infty}$ converging to x^* for any initial approximation $x_0 \in [x^* - \delta, x^* + \delta]$, i.e., under reasonable assumptions, Newton's method converges always provided a sufficiently accurate initial approximation is chosen.

Exercise 1. Function

$$f(x) = \frac{4x-7}{x-2}$$

has a zero at $x^* = 1.75$. Find it by using Newton's method with initial approximations

a) $x_0 = 1.625$; b) $x_0 = 1.875$; c) $x_0 = 1.5$; d) $x_0 = 1.99$; e) $x_0 = 3$; f) $x_0 = 0.5$.

Print out all computed values of x_n. Terminate iteration process if $|x_n - x_{n-1}| \le 10^{-5}$. Explain the results with the help of the graph of f.

For case d) analyze the convergence speed of the method by printing out the ratios

$$\frac{|x_{n+1} - x^*|}{|x_n - x^*|^k}, \quad n = 0, 1, \dots,$$

for different values of k (for example take $k = 1, 2, 3$). Does it hold

$$|x_{n+1} - x^*| \le \lambda |x_n - x^*|^k$$

for some value of $\lambda > 0$?

Exercise 2. Use Newton's method to solve $f(x) = 0$, where

$$f(x) = x^2 - 2xe^{-x} + e^{-2x}, \quad x \in [0, 1].$$

Print out all computed values of x_n. Terminate iteration process if $|x_n - x_{n-1}| \le 10^{-5}$.

Study the multiplicity m of the solution x^* knowing that

$$\frac{x_{n+1} - x_n}{x_n - x_{n-1}} \to 1 - \frac{1}{m} \qquad \text{and/or} \qquad \frac{x_n - x_{n-1}}{-x_{n+1} + 2x_n - x_{n-1}} \to m.$$

Does the Newton-Scröderi method

$$x_{n+1} = x_n - m\frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, \dots$$

improve the convergence? Again, print out all computed values of x_n.