# MTMM.00.005 Numerical Methods
# Practical lesson 8–9: Interpolation.

April 2–9, 2019

To get a positive result, student has to solve all of the exercises given in practical lessons, explain the programming code and answer the questions.

The deadline for the eighth set of exercises is April 16th.

NB! In order to use the nonstandard packages as *numpy* and *matplotlib* one could use Command Prompt and type

```
H:\nm> "C:\Program Files\Anaconda3\python" <filename>.py
```

Here `H:\nm` is the folder for `<filename>.py`.

### Interpolation problem

For given interpolation knots $x_0, \dots, x_n$, where $x_i \neq x_j$, if $i \neq j$, and corresponding numbers $f_0, \dots, f_n$ find a function $f$ such that $f(x_i) = f_i$.

### Lagrange's interpolation formula

**The Lagrange fundamental polynomials** corresponding to the knots $x_0, \dots, x_n$ are

$$l_{ni}(x) = \prod_{\substack{j=0 \\ j\neq i}}^{n} \frac{x - x_j}{x_i - x_j} = \frac{w_n(x)}{(x - x_i)\, w_n'(x_i)}, \quad i = 0, \dots, n,$$

where $w_n(x) = (x - x_0) \dots (x - x_n)$. These polynomials satisfy the conditions

$$l_{ni}(x_j) = \begin{cases} 1, & \text{if } j = i, \\ 0, & \text{if } j \neq i, \end{cases} \quad j = 0, \dots, n.$$

**The Lagrange interpolation polynomial** is of the form

$$P_n(x) = \sum_{i=0}^{n} f_i\, l_{ni}(x).$$

It satisfies the interpolation conditions $P_n(x_j) = f_j$, $j = 0, \dots, n$.

### Newton's interpolation formula

Let it be given the pairwise different knots $x_0, \dots, x_n$ and corresponding function values $f(x_0), \dots, f(x_n)$. We define the **divided differences of the first order** as

$$f(x_i, x_j) = \frac{f(x_i) - f(x_j)}{x_i - x_j}, \quad i \neq j,$$

the **divided differences of the second order** as

$$f(x_i, x_j, x_k) = \frac{f(x_i, x_j) - f(x_j, x_k)}{x_i - x_k}, \quad i \neq j, j \neq k, i \neq k,$$

and the **divided differences of the $k$th order** as

$$f(x_{i_0}, \dots, x_{i_k}) = \frac{f(x_{i_0}, \dots, x_{i_{k-1}}) - f(x_{i_1}, \dots, x_{i_k})}{x_{i_0} - x_{i_k}}.$$

It holds

$$f(x_0,\dots,x_n) = \sum_{i=0}^{n} \frac{f(x_i)}{\prod\limits_{\substack{j=0\\ j\neq i}}^{n} (x_i - x_j)} = \sum_{i=0}^{n} \frac{f(x_i)}{w_n'(x_i)}.$$

**Newton's interpolation polynomial** is a function of the form

$$\begin{aligned} P_n(x) &= f(x_0) + f(x_0,x_1)(x-x_0) + f(x_0,x_1,x_2)(x-x_0)(x-x_1) + \ldots + \\ &\quad + f(x_0,\dots,x_n)(x-x_0)\ldots(x-x_{n-1}). \end{aligned}$$

It can be shown that $P_n(x_i) = f(x_i)$, $i = 0,\dots,n$.

Polynomial interpolation is numerically unstable. It means that small changes in initial data can cause large changes for polynomial interpolant.

**Exercise 1.** Fill the next table, round the function values to four decimal places.

| $x_i$ | 105 | 115 | 120 | 125 | 130 | 140 |
|---|---|---|---|---|---|---|
| $f(x) = \sqrt[3]{x+\alpha}$ | | | | | | |

Find $P_n(x)$, $n = 1,\dots,5$, where the knots are the arguments in the table which are closest to $x_\alpha = 121 + 0.1\alpha$.
Draw the graphs of functions $P_n(x)$ and $f(x)$, $x \in [40, 200]$, on the same figure for $n = 1,\dots,5$.
Find $P_n(x_\alpha) - f(x_\alpha)$, $n = 1,\dots,5$.

a) Use Lagrange's interpolation formula;
b) Use Newton's interpolation formula.

**Exercise 2.** Approximate the function $f(x) = |x|$, $x \in [-1,1]$, with interpolation polynomials. Use knot values $x_i = -1 + \frac{2}{n}i$, $i = 0,\dots,n$.
Estimate the error $\varepsilon_n = \max\limits_{-1\le x\le 1} |P_n(x) - f(x)|$ on a ten times denser grid by finding

$$\bar{\varepsilon}_n = \max_{0\le i\le 10n} |P_n(z_i) - f(z_i)|, \quad z_i = -1 + \frac{2}{10n}i.$$

Find $\bar{\varepsilon}_n$ for at least ten different values of $n$ ($n \le 30$) including $n = 30$. Draw the graphs of functions $P_n(x)$, $n = 4, 8, 12$, and $f(x)$ on the same figure.