Simulation Methods in Financial Mathematics
Computer Lab 15

The aim of this lab is to learn to use two additional methods for computing sensitivities of option prices.

Let us consider European options. For simplicity we also assume that the stock price behaves according to Black-Scholes market model with constant volatility. Then the price of an option with payoff function p is given by

$$H = E[e^{-rT} p(S(T))],$$

where T is the exercise time and the final stock price $S(T)$ is given by

$$S(T) = S_0 e^{(r-D-\sigma^2/2)T+\sigma B(T)}.$$

In addition to the difference quotients method considered in the previous lab, we consider the following two methods.

1. **Pathwise derivative method.** Let θ denote a parameter in the stock price formula. Assume that the function p is continuous with piecewise continuous and bounded derivative, then it can be shown that we can change the order of taking expected value and differentiation when computing the derivative of the price with respect to θ:
$$\frac{\partial H}{\partial \theta} = E[\frac{\partial (e^{-rT} p(S(T)))}{\partial \theta}].$$
For example, if $\theta = S_0$ we have
$$\frac{\partial H}{\partial S_0} = E[e^{-rT} p'(S(T)) \frac{\partial S(T)}{\partial S0}] = E[e^{-rT} p'(S(T)) \frac{S(T)}{S_0}].$$

2. **Likelihood ratio method.** Suppose we know the probability density function f_S of the final stock price. If θ is a market parameter, then the density function depends on θ (is a function of s and θ). Using the density function, we can write
$$H = \int_{-\infty}^{\infty} e^{-rT} p(s) f_S(s,\theta)\, ds.$$
Assuming again, that we can change the order of integration and differentiation, we get now
$$\begin{aligned}\frac{\partial H}{\partial \theta} &= \int_{-\infty}^{\infty} p(s) \left(\frac{\partial}{\partial \theta} (e^{-rT} f_S(s,\theta)) \right) \frac{f_S(s,\theta)}{f_S(s,\theta)}\, ds \\ &= E[p(S(T)) R(S(T),\theta)],\end{aligned}$$
where
$$R(s,\theta) = \frac{\frac{\partial}{\partial \theta}(e^{-rT} f_S(s,\theta))}{f_S(s,\theta)}.$$
If θ is not r or T, we can further write
$$R(s,\theta) = e^{-rT} \frac{\partial}{\partial \theta} \ln f_S(s,\theta).$$

Since according to our assumptions $S(T)$ is log-normally distributed, we have

$$f_S(s) = \frac{1}{s\sigma\sqrt{2\pi T}} \exp\left(-\frac{(\ln\frac{s}{S_0} - (r - D - \frac{\sigma^2}{2})T)^2}{2\sigma^2 T}\right).$$

Thus for $\theta = S_0$ the likelihood ratio method leads to the formula

$$\frac{\partial H}{\partial S_0} = E[e^{-rT} p(S(T)) \frac{\ln(S(T)/S_0) - (r - D - \sigma^2/2)T}{S_0 \sigma^2 T}].$$

Consider two European options: the usual call option and so called binary call option with the payoff function

$$p(s) = \begin{cases} 1, & s \geq E, \\ 0, & s < E. \end{cases}$$

Let $T = 1.0$, $S_0 = 100$, $E = 100$, $\sigma = 0.5$, $r = 0.07$, $D = 0$.

- Task1: Compute the derivative of the usual call option with respect to S_0 with both methods with accuracy (relative error) 1%. Compare the number of generated stock prices.
- Task2: Compute the derivative of the binary call option with respect to S_0 with accuracy 1% by using a suitable method described in the lab.

Homework 8 (Deadline December 22, 2019) Consider an European option with payoff $\max\{0.1(S(T) - 40)(70 - S(T)), 0\}$ at $T = 0.6$. Assume that the stock price follows BS model with $\sigma = 0.6$ and that $r = 0.05,\ D = 0,\ S(0) = 75$. Find the derivative of of the option with respect to r with accuracy 1% of the exact value with all suitable methods (including the method of Lab 14), using $\alpha = 0.1$ for estimating the error. Compare the numbers of generated stock prices. Which method is the best?