Simulation Methods in Financial Mathematics
Computer Lab 6

Goal of the lab:

- To learn to use antithetic variates and control variates for variance reduction in the Monte Carlo method.

Antithetic variates. Suppose we know, how to generate Y and $\tilde{Y}$ that have the same distribution but are negatively correlated. Then it is reasonable to generate a random variable $Z = \frac{Y+\tilde{Y}}{2}$ as the variance of it is smaller than the variances of Y and $\tilde{Y}$ (show it!). This means that the Monte Carlo method will work faster. In option pricing the easiest way to introduce antithetic variates is generating pairs of stock prices $(S(T), \tilde{S}(T))$, where the normally distributed random variables used to generate the second one are the ones used for the first stock price but with changed sign. Assuming the payoff function is monotone, it is reasonable to hope (but is not always true) that it's values are negatively correlated for those price pairs and we can calculate

$$Price = E[e^{-rT}\frac{p(S(T)) + p(\tilde{S}(T))}{2}].$$

Control variates. Control variates method can be used when in addition to the random variable of interest, Y, we also know how to generate another random variable Z for which the expected value is known and which is highly correlated with Y. This means that we are also able to generate $Y_1 = Y - a(Z - EZ)$ which clearly has the same expected value as Y and if Y and Z are strongly correlated and a is chosen wisely, Y_1 can have a variance that is significantly smaller than the variance of Y. The constant a should approximate $\frac{\text{cov}(Y,Z)}{Var(Z)}$ (Prove it! It is a simple exercise!).
In this lab the example of finding the price of an European call option with strike price $E = 43$ at time $t = 0$ is considered in the case when Black-Scholes market model with $r = 0.02,\ D = 0.01,\ T = 0.5,\ S(0) = 49$ and $\sigma(t, s) = 0.3 + \frac{0.4}{1+0.01\cdot(s-50)^2}$.
Tasks:

1. Make sure that the function for calculating expected values with Monte-Carlo method outputs the number of generated random variables used to achieve a specific precision together with the estimate of the expected value. Use the function for pricing the call option by generating stock prices with Euler's method with $m = 20$ and precision 0.05.

2. Enhance the generator of stock prices using Euler's method so that it would output a matrix with two columns consisting of pairs of antithetic stock prices. Function g also needs to be updated so that it is able to use correctly the output of the generator of antithetic stock prices. Compare the number of generations needed (when using pairs the number of generations is actually double the amount of the counter value)

3. Modify the generator that uses the Euler's method so that it would use the same Brownian motion to output the stock prices for a market model with constant volatility and a market model with a non-constant volatility. For the former use the exact stock price formula. Let S be the stock price corresponding to the non-constant volatility and $\bar{S}$ the stock price corresponding to the constant volatility. Use the Monte Carlo method (with the settings of this lab and constant volatility $\sigma_1 = 0.4$) to find the expected value of

$$e^{-rT}p(S) - a(e^{-rT}p(\bar{S}) - h).$$

Here h is the exact price of the option for the market model with constant volatility and a is the estimate of

$$\frac{\text{cov}(e^{-rT}p(S), e^{-rT}p(\bar{S}))}{\text{var}(e^{-rT}p(\bar{S}))}$$

based on 1000 generations. Function p is the payoff function for an European call option. Compare the number of generations needed with the corresponding number of the previous task.