Simulation Methods in Financial Mathematics
Computer Lab 4

Goal of the lab:

- To learn to use higher order discretization methods for stock price generation and study their error rates

There are many numerical schemes of different strong and weak convergence rates for solving stochastic differential equations. For example, a stochastic differential equation of the form

$$dS(t) = S(t)\big[\mu(t, S(t))\, dt + \sigma(t, S(t))\, dB(t)\big] \tag{1}$$

can be solved by using the following methods:

Euler's method

$$S_{i+1} = S_i\big[1 + \mu(t_i, S_i)\Delta t + \sigma(t_i, S_i)X_i\big], \tag{2}$$

Milstein's method

$$S_{i+1} = S_i\big[1 + \mu(t_i, S_i)\Delta t + \sigma(t_i, S_i)X_i\big] + \frac{1}{2}(X_i^2 - \Delta t)L_2\sigma(t_i, S_i) \tag{3}$$

and
A weakly second order method

$$\begin{aligned} S_{i+1} &= S_i\big[1 + \mu(t_i, S_i)\Delta t + \sigma(t_i, S_i)X_i)\big] \\ &+ \frac{1}{2}\left\{\Delta t^2 L_1\mu(t_i, S_i) + \Delta t\, X_i\left[L_2\mu(t_i, S_i) + L_1\sigma(t_i, S_i)\right] + (X_i^2 - \Delta t)L_2\sigma(t_i, S_i)\right\}, \end{aligned} \tag{4}$$

where random variables X_i are independent and have distribution $N(0, \sqrt{\Delta t})$ and the operators L_1 and L_2 are defined by

$$L_1 f(t, s) = s\frac{\partial f}{\partial t}(t, s) + s\mu(t, s)\left[f(t, s) + s\frac{\partial f}{\partial s}(t, s)) + \frac{s^2\sigma(t, s)^2}{2}(2\frac{\partial f}{\partial s}(t, s) + s\frac{\partial^2 f}{\partial s^2}(t, s)\right],$$

$$L_2 f(t, s) = s\sigma(t, s)\left[f(t, s) + s\frac{\partial f}{\partial s}(t, s)\right].$$

L_1 and L_2 are called operators because they act on functions of two variables and the result is also a function of two variables. For example, for each function f the result of applying L_1 to f the outcome is a new function (denoted by $L_1 f$) of two variables. A more concrete example: if $f(t, s) = s^2$ then $\frac{\partial f}{\partial s}(t, s) = 2s$ and therefore the function $L_2 f$ is given by

$$L_2 f(t, s) = s \cdot \sigma(t, s) \cdot (s^2 + s \cdot 2s) = 3s^3\sigma(t, s).$$

Similarly, if $\mu(t, s) = \mu$ (a constant), then $L_1\mu(t, s) = s \cdot \mu^2$, $L_2\mu(t, s) = s \cdot \sigma(t, s) \cdot \mu$.

Tasks:

1. Modify Euler's method so that it works also in the case of variable volatility (and define a function $\sigma(t, s) = 0.6$).

2. Program methods (3) and (4) for generating solutions of the SDE (1) at time T when the Black-Scholes market model is assumed. Assume that in additions to functions $\mu(t,s)$ and $\sigma(t,s)$ also the functions $L_1\mu(t,s), L_1\sigma(t,s)$, $L_2\mu(t,s)$ and $L_2\sigma(t,s)$ are defined outside of the generators. Hint: Before using the method, the functions $\sigma(t,s)$, $\frac{\partial\sigma}{\partial s}(t,s)$, $\frac{\partial\sigma}{\partial t}(t,s)$ and $\frac{\partial^2\sigma}{\partial s^2}(t,s)$ should be defined. Then derive and define $L_1\mu(t,s), L_1\sigma(t,s)$, $L_2\mu(t,s)$ and $L_2\sigma(t,s)$. Using those functions the generator of the Milstein and the weakly second order methods for generating the stock prices can be easily defined.

3. Experimentally find the strong convergence rates of (3) and (4) using procedure `nls` in the case when $S(0) = 100$, $T = 0.5, \mu = 0.1$ and $\sigma(t,s) = 0.6$.