Simulation Methods in Financial Mathematics
Computer Lab 10
April 16, 2019

Goal of the lab: To learn to use Monte-Carlo method for pricing **Asian options**.

An Asian option is an option which payoff depends on the average stock price. Let $A(T)$ be the average stock price for the period $[0, T]$ i.e.

$$A(T) = \frac{1}{T} \int_0^T S(t)\, dt.$$

The most typical payoff functions for Asian options are

- **Average strike** call options $p(s, a) = \max(s - a, 0)$. At time T the value of the option
$$p(S(T), A(T)) = \max(S(T) - A(T), 0).$$
- **Average strike** put option $p(s, a) = \max(a - s, 0)$. At time T the value of the option
$$p(S(T), A(T)) = \max(A(T) - S(T), 0).$$
- **Average price** call options $p(s, a) = \max(a - E, 0)$ with strike price E. At time T the value of the option
$$p(S(T), A(T)) = \max(A(T) - E, 0).$$
- **Average price** put options, $p(s, a) = \max(E - a, 0)$ with strike price E. At time T the value of the option
$$p(S(T), A(T)) = \max(E - A(T), 0).$$

When the stock price is governed by the Black-Scholes market model, then the price of all the named options at time $t = 0$ can be calculated as the expected value

$$Price = E[e^{-rT} p(S(T), A(T))],$$

where $S(T)$ corresponds to the Black-Scholes market model with trend $\mu = r - D$. Thus to use the MC method we need to generate (in addition to the stock prices at time T) the average values of stock prices which depend also on the intermediate values of the stock price paths. The simplest way of calculating the average is using the average of $S(i\frac{T}{m}),\ i = 0, 1, \ldots, m - 1$,

$$A(T) = \frac{1}{m} \sum_{i=1}^{m} S_i. \tag{1}$$

A better approximation can be calculated as

$$A(T) \approx \frac{1}{m} \sum_{i=1}^{m} S_{i-1} \left(1 + (r - D)\frac{T}{2m} + \frac{\sigma}{2}(B(t_i) - B(t_{i-1}))\right). \tag{2}$$

The idea leading to the improved formula is to write

$$\frac{1}{T}\int_0^T S(t)\,dt = \frac{1}{T}\sum_{i=1}^{m}\int_{t_{i-1}}^{t_i} S(t)\,dt = \frac{1}{T}\sum_{i=1}^{m}\int_{t_{i-1}}^{t_i}\left(S(t_{i-1}) + \int_{t_{i-1}}^{t} dS(\tau)\right)dt,$$

using the equation for $dS(\tau)$ and to approximate the resulting double integrals by replacing the integrands with their values at the beginning of the integration intervals.

Tasks:
We will assume that the Black-Scholes market model with constant parameters holds and fix $r = 0.1$, $D = 0$, $\sigma = 0.4$, $T = 0.5$, $S(0) = 100$. We will consider average strike calls and average price calls with strike price $E = 100$.

1. Write a generator which for a given value of n would generate n pairs of (terminal) stock price and average stock price. Calculate the stock prices using the exact formula (also use it when calculating the average stock prices); calculate the average as the mean of $S(i\frac{T}{m}),\ i = 0, 1, \ldots, m-1$, given by Equation (1). Find the weak convergence rate depending on m (by using the values $5, 10, 20, 40$ for m and a small enough MC error). For both options find m for which the error caused by the choice of m is less than 0.1 (using the result obtained for the weak rate of convergence).

2. Repeat the task when the average price is calculated according to the improved formula, given by Equation (2). To study the rate of weak convergence use the values $2, 4, 6, 8$ for m and take 0.01 for the MC error.