Simulation Methods in Financial Mathematics
Computer Lab 11
April 23, 2019

Goal of the lab: To learn to use **stratified sampling** for speeding up the pricing of **Asian options**.

We will consider the case where the average stock price is calculated using the formula

$$A(T) \approx \frac{1}{m}\sum_{i=1}^{m} S_{i-1}\left(1 + (r-D)\frac{T}{2m} + \frac{\sigma(t_{i-1}, S_{i-1})}{2}(B(t_i) - B(t_{i-1}))\right), \tag{1}$$

where $t_i = i\frac{T}{m}$. We already know, how to generate the increments of the Brownian motion $\Delta B = (B(t_1) - B(t_0), B(t_2) - B(t_1), \ldots, B(t_m) - B(t_{m-1}))'$ (here $\mathbf{a}'$ denotes the transpose of $\mathbf{a}$) so that given a vector $\mathbf{v} = (v_1, \ldots, v_m)'$ the stratifying would be based on the values of $\mathbf{v}'\Delta B$:

- Generate the value of W from the desired stratum of $N(0, \|\mathbf{v}\|\sqrt{\frac{\mathbf{T}}{\mathbf{m}}})$.
- Generate a random vector $\mathbf{Z} = (Z_1, \ldots, Z_m)'$, where $Z_i \sim N(0, \sqrt{\frac{T}{m}})$
- Calculate

$$\Delta B = \frac{1}{\|\mathbf{v}\|^{\mathbf{2}}} W\mathbf{v} + \mathbf{Z} - \frac{1}{\|\mathbf{v}\|^{\mathbf{2}}}\mathbf{v}(\mathbf{v}'\mathbf{Z}).$$

When we want to generate a matrix with dimensions $m \times n$ so that each column would represent the increments of the Brownian motion of a corresponding generated value of W (thus in total there are n values of W), then we need to modify the formula as follows: W must be generated as a row vector with n components, $\mathbf{Z}$ must be a $m \times n$ matrix and the increment matrix can then be calculated as

$$\Delta B = \frac{1}{\|\mathbf{v}\|^{\mathbf{2}}}\mathbf{v}\mathbf{W} + \mathbf{Z} - \frac{1}{\|\mathbf{v}\|^{\mathbf{2}}}(\mathbf{v}\mathbf{v}'\mathbf{Z}).$$

Using this formula there are several possibilities for stratification:

- When we want the the strata based on the values of $B(T)$, we take

$$\mathbf{v} = (1, 1, \ldots, 1)'. \tag{2}$$

  This stratification should be appropriate for European options.

- When we want the the strata based on the average values of the Brownian motion $\frac{1}{T}\int_0^T B(t)\,dt$, we may approximate the integral by

$$\frac{1}{T}\int_0^T B(t)\,dt \approx \sum_{i=0}^{m-1}\frac{B(t_i)}{m} = \sum_{i=1}^{m-1}\sum_{j=1}^{i}\Delta B_j = \sum_{j=1}^{m-1}\frac{m-j}{m}\Delta B_j$$

  and hence should use stratification with

$$\mathbf{v} = \left(\frac{m-1}{m}, \frac{m-2}{m}, \ldots, \frac{m-m}{m}\right)'. \tag{3}$$

  This stratification should be appropriate for Asian options when the payoff does not depend on the stock price at time T.

- When we want the strata based on differences of $B(T)$ and the average values of the Brownian motion, we ta ke

$$\mathbf{v} = \left(\frac{1}{m}, \frac{2}{m}, \dots, \frac{m}{m}\right)'. \tag{4}$$

  This stratification should be appropriate for **average strike** options.

- In general, if the payoff is a function of a linear combination of $S(T)$ and $A(T)$, a good choice for stratification vector v is the same linear combination of vectors

$$\mathbf{v_1} = (1, 1, \dots, 1)' \text{ and } \mathbf{v_2} = \left(\frac{m-1}{m}, \frac{m-2}{m}, \dots, \frac{m-m}{m}\right)'. \tag{5}$$

Tasks:

1. Let $m = 20$. Consider the pricing of Asian options with payoff functions $p(s, a) = \max(50 - a, 0)$ and $p(s, a) = \max(s - a, 0)$ using the MC method with MC error 0.01 in the case when $S(0) = 49$, $r = 0.05$, $D = 0.02$, $T = 0.5$, $\sigma = 0.5$. Compare the number of generations required when not using any variance reduction methods and when using the optimal stratified sampling with the number of strata $k = 40$ with stratifications given by (2), (3) and (4).

2. **Homework 6** (deadline 01.05.2019). When volatility is not constant, then we have to use a numerical method for generating the stock prices and an approximation for the average stock price. Implement MC for pricing Asian options by using Euler's method for generating the option prices and formula (1) for computing the values of the average stock price. Note that combination of those methods gives us a method with weak convergence rate $q = 1$. Use this method for computing the price of the option with pay-off function

$$p(s, a) = \max(14 + 2.3a - 0.9s, 0)$$

   in the case $S0 = 20$, $r = 0.02$, $D = 0.01$, $T = 0.5$,

$$\sigma(t, s) = \frac{\ln s^3 - 0.3}{\ln s + 25} - 0.08,$$

   with the total error that is less than 0.01 with the probability 0.95 . Use the optimal stratified sampling variance reduction method for computing the final answer.