

Simulation Methods in Financial Mathematics

Computer Lab 15

The aim of this lab is to learn to use two additional methods for computing sensitivities of option prices.

Consider a case where the random variable X depends on some parameter θ and therefore the expected value $E(g(X(\theta)))$ depends also on that parameter,

$$V(\theta) = E[g(X(\theta))].$$

For example, stock price (and hence the price of an option) depends on the initial stock price $S(0)$, the volatility σ , risk free interest rate r etc and we may want to know how the price changes if some of the parameters changes. Let us consider European options. For simplicity we also assume that the stock price behaves according to Black-Scholes market model with constant volatility. The price of an option with payoff function p is given by

$$H = \mathbb{E}[e^{-rT}p(S(T))],$$

where T is the exercise time and the final stock price $S(T)$ is given by

$$\begin{aligned} S(T) &= S_0 e^{(r-D-\sigma^2/2)T + \sigma B(T)} \\ &= S_0 e^{(r-D-\sigma^2/2)T + \sigma\sqrt{T}Z}, Z \sim N(0,1), \end{aligned}$$

depends on the initial stock price $S(0)$, the volatility σ , risk free interest rate r etc. In addition to the difference quotients method considered in Lab 14, we consider the following two methods.

1. **Pathwise derivative method.** Let θ denote a parameter in the stock price formula. Assume that the function p is continuous with piecewise continuous and bounded derivative, then it can be shown that we can change the order of taking expected value and differentiation when computing the derivative of the price with respect to θ :

$$\frac{\partial H}{\partial \theta} = \mathbb{E} \left[\frac{\partial(e^{-rT}p(S(T)))}{\partial \theta} \right].$$

For example, if $\theta = S(T)$ (or similarly for $\theta = S_0$) we have

$$\frac{\partial H}{\partial S(T)} = \mathbb{E} \left[e^{-rT} \frac{\partial p(S(T))}{\partial S(T)} \frac{\partial S(T)}{\partial S_0} \right] = \mathbb{E} \left[e^{-rT} p'(S(T)) \frac{S(T)}{S_0} \right].$$

For example, for European call option with strike E

$$\frac{\partial p(s)}{\partial s} = p'(s) = \frac{\partial}{\partial s} \max(0, s - E) = \begin{cases} 1, & s < E \\ 0, & s > E. \end{cases}$$

This derivative fails to exist at $s = E$. But because the event $\{S(T) = E\}$ has probability 0, then H is almost surely differentiable with respect to $S(T)$.

2. **Likelihood ratio method.** Suppose we know the probability density function f_S of the final stock price. If θ is a market parameter, then the density function depends on θ (is a function of s and θ). Using the density function, we can write

$$H = \mathbb{E} \left[\frac{\partial(e^{-rT}p(S(T)))}{\partial \theta} \right] = \int_{-\infty}^{\infty} e^{-rT} p(s) f_S(s, \theta) ds.$$

We can change the order of integration and differentiation,

$$\begin{aligned}\frac{\partial H}{\partial \theta} &= \int_{-\infty}^{\infty} p(s) \left(\frac{\partial}{\partial \theta} (e^{-rT} f_S(s, \theta)) \right) \frac{f_S(s, \theta)}{f_S(s, \theta)} ds \\ &= \mathbb{E}[p(S(T))R(S(T), \theta)],\end{aligned}$$

where

$$R(s, \theta) = \frac{\frac{\partial}{\partial \theta} (e^{-rT} f_S(s, \theta))}{f_S(s, \theta)}. \quad (1)$$

Since according to our assumptions $S(T)$ is log-normally distributed, we have

$$f_S(s) = \frac{1}{s\sigma\sqrt{2\pi T}} \exp\left(-\frac{(\ln \frac{s}{S_0} - (r - D - \frac{\sigma^2}{2})T)^2}{2\sigma^2 T}\right).$$

Note,

- if θ is r or T then in (1) we apply the product rule. Thus, for $\theta = r$,

$$R(s, r) = -Te^{-rT} + e^{-rT} \frac{\partial}{\partial r} \ln f_S(s, r).$$

- if θ is not r or T , $\theta \neq r$, $\theta \neq T$, then (1) simplifies

$$R(s, \theta) = e^{-rT} \frac{\partial}{\partial \theta} \ln f_S(s, \theta).$$

Thus for $\theta = S_0$ the likelihood ratio method leads to the formula

$$\frac{\partial H}{\partial S_0} = \mathbb{E} \left[e^{-rT} p(S(T)) \frac{\ln(S(T)/S_0) - (r - D - \sigma^2/2)T}{S_0 \sigma^2 T} \right].$$

TASKS: Consider two European options: the usual call option and so called binary call option with the payoff function

$$p(s) = \begin{cases} 1, & s \geq E, \\ 0, & s < E. \end{cases}$$

Let $T = 1.0$, $S_0 = 100$, $E = 100$, $\sigma = 0.5$, $r = 0.07$, $D = 0$.

- Task1: Compute the derivative of the usual call option with respect to S_0 with both methods with accuracy 1%. Compare the number of generated stock prices.
- Task2: Compute the derivative of the binary call option with respect to S_0 with accuracy 1% by using a suitable method described in the lab.

Homework 8 (Deadline May 29, 2019) Consider an European option with payoff

$$\max\{0.1(50 - S(T))(S(T) - 80), 0\}$$

at $T = 0.6$. Assume that the stock price follows BS model with $\sigma = 0.6$ and that $r = 0.02$, $D = 0$, $S(0) = 55$. Find the derivative of the option with respect to T with accuracy 1% with all suitable methods (including the method of Lab 14), using $\alpha = 0.1$ for estimating the error. Compare the numbers of generated stock prices. Which method is the best?