Simulation Methods in Financial Mathematics
Computer Lab 5

Reminder from Lab 4:

1. Find the prices of an European put option with **Milstein's method**

$$S_{i+1} = S_i \cdot (1 + \mu(t_i, S_i)\Delta t + \sigma(t_i, S_i)X_i) + \frac{1}{2}L_2\sigma(t_i, S_i)(X_i^2 - \Delta t) \tag{1}$$

and **weakly second order method**

$$\begin{aligned} S_{i+1} &= S_i \cdot (1 + \mu(t_i, S_i)\Delta t + \sigma(t_i, S_i)X_i) \\ &+ \frac{1}{2}\left(L_1\mu(t_i, S_i)\Delta t^2 + (L_2\mu(t_i, S_i) + L_1\sigma(t_i, S_i))\Delta t\, X_i + L_2\sigma(t_i, S_i)(X_i^2 - \Delta t)\right), \end{aligned}$$

where random variables X_i are independent and have distribution $N(0, \sqrt{\Delta t})$ and the operators L_1 and L_2 are defined by

$$\begin{aligned} L_1 f(t,s) &= s\frac{\partial f}{\partial t}(t,s) + s \cdot \mu(t,s) \cdot (f(t,s) + s\frac{\partial f}{\partial s}(t,s)) + \frac{s^2\sigma(t,s)^2}{2}(2\frac{\partial f}{\partial s}(t,s) + s\frac{\partial^2 f}{\partial s^2}(t,s)), \\ L_2 f(t,s) &= s\sigma(t,s)(f(t,s) + s\frac{\partial f}{\partial s}(t,s)). \end{aligned}$$

in the cases $m = 4, 8, 16$ with MC error less than 0.01 (computed for $\alpha = 0.05$) when $E = 55$, $T = 0.5$, $r = 0.05$, $S(0) = 50$, $D = 0$, $\sigma(s) = 0.5 + 0.2 \cdot \sin(0.1s - 5)$. For each method, look at the differences of prices. If MC error is small enough (more than 3 times smaller than the differences), then the differences describe the weak convergence rate - if m is multiplied by 2 for each new computation, the ratios of consecutive differences should be approximately 2^q (if we divide previous difference with the next one), where q is the weak convergence rate (can you figure out, why this is true?). Is the faster convergence rate of the weakly second order method observable?

Goal of the Lab 5:

- To compute option prices with a given accuracy when using a numerical method for generating stock prices and to learn to use Euler's method for systems of stochastic differential equations.

Compute option prices with a given accuracy: We know several methods for generating stock prices and we know that for pricing options the weak convergence rate of the method used is important. It is known that if p is continuous and has bounded first derivative, then Euler's method and Milstein's method are weakly convergent with rate $q = 1$ and we also know one method that has weak convergence rate $q = 2$ for sufficiently nice pay-off functions. Next we consider, how this information can be used for computing the price of and option with a given accuracy when we have to use a numerical method for generating the stock prices. Let V be the price of an European option with the expiration date T and pay-off function p, then

$$V = E(\exp(-rT)p(S(T))),$$

where $S(t)$, $0 \le t \le T$ follows certain stochastic differential equation (SDE). If the SDE can not be solved exactly, then instead of $S(T)$ we use S_m, thus we use Monte-Carlo method to compute an approximate value V_m of V, where

$$V_m = E[e^{-rT}p(S_m)].$$

Often we know the weak convergence rate of a numerical method. This means that

$$|V - V_m| = \frac{C}{m^q} + o(\frac{1}{m^q})$$

for some $q > 0$. Here C is a constant that does not depend on m and $m^q \cdot o(\frac{1}{m^q}) \to 0$ as $m \to \infty$. Actually, usually a more precise relation

$$V - V_m = \frac{C_1}{m^q} + o(\frac{1}{m^q}),$$

holds (and thus the previous estimate for the absolute value of the error holds with here $C = |C_1|$) and we use that later for estimating the coefficient C. Thus, if we use S_m instead of $S(T)$ and use Monte-Carlo method with allowed error ε at a specified allowed error probability α, then the total error of the computed number $\hat{V}_{m,\varepsilon}$ is

$$|V - \hat{V}_{m,\epsilon}| \le |V - V_m| + |V_m - \hat{V}_{m,\epsilon}| \le \frac{C}{m^q} + o(\frac{1}{m^q}) + \epsilon.$$

The last term is the error of the Monte-Carlo method and can be chosen by us. So, in order to compute the option price V with a given error ε, we should choose large enough m (so that the term $\frac{C}{m^q}$ is small enough, for example less than $\frac{\varepsilon}{2}$) and then use MC method with allowed error $\epsilon = \frac{\varepsilon}{2}$). There is one trouble: we do not know C. One possibility to estimate C is as follow:

1. Choose some values for m_0, ϵ_0 for m and MC error ϵ. The value of m_0 should not be too small, but very large values take too much computation time; the value of the allowed error ϵ_0 should be sufficiently small (we discuss it in more detail in the next step). In practice we usually use $m_0 = 5$ or $m_0 = 10$.
2. Use MC method twice to compute $\hat{V}_{m_0,\epsilon_0}$ and $\hat{V}_{2m_0,\epsilon_0}$. The value ϵ_0 is small enough if the results differ significantly more than by ϵ_0. If we use too large value of ϵ_0, then we overestimate the value of C and hence the final value of m in the following steps and our final computations may take too much time.
3. Estimate the value of C. We use the inequality

$$|V_{m_0} - V_{2m_0}| \le |V_{m_0} - \hat{V}_{m_0,\epsilon_0}| + |V_{2m_0} - \hat{V}_{2m_0,\epsilon_0}| + |\hat{V}_{m_0,\epsilon_0} - \hat{V}_{2m_0,\epsilon_0}|.$$

   If we use the more precise information about V_m and V_{2m} and assume that the terms $o(\frac{1}{m_0^q})$ and $o(\frac{1}{(2m_0)^q})$ are practically zero, then it follows that

$$\begin{aligned} C &\le \frac{(2m_0)^q}{2^q - 1} \cdot (|\hat{V}_{m_0,\epsilon_0} - \hat{V}_{2m_0,\epsilon_0}| + |V_{m_0} - \hat{V}_{m_0,\epsilon_0}| + |V_{2m_0} - \hat{V}_{2m_0,\epsilon_0}|) \\ &\le \frac{(2m_0)^q}{2^q - 1} \cdot (|\hat{V}_{m_0,\epsilon_0} - \hat{V}_{2m_0,\epsilon_0}| + 2\epsilon_0) =: \bar{C}. \end{aligned}$$

4. Choose (express) m_1 such that

$$\frac{\bar{C}}{m_1^q} \le \frac{\varepsilon}{2}$$

   and compute $\hat{V}_{m_1,\frac{\varepsilon}{2}}$. The last result is an approximation of the true option price which satisfies the desired error estimate. If the starting value of m_0 was large enough so that the additional error terms of order $o(\frac{1}{m^q})$ are practically equal to zero. In this course we do not consider methods of determining if the starting value of m_0 was sufficiently large and take the result of the last computation to be the desired answer.

Task:

1. Find the value of an European call option with strike price $E = 98$ at time $t = 0$ with precision 0.1, when $\alpha = 0.05$, $r = 0.05$, $D = 0$, $T = 0.5$, $S(0) = 100$ and $\sigma(t,s) = 0.7 - 0.7\,e^{-0.01s}$. To solve the problem we have to choose an m so that the error due to m would be sufficiently small.

Euler's method for systems of stochastic differential equations: For solving a stochastic differential equation (SDE) of the form

$$dY(t) = \alpha(t, Y(t))\,dt + \beta(t, Y(t))\,dB(t),\ \ Y(0) = Y_0$$

the **Euler's method** gives

$$Y_{k+1} = Y_k + \alpha(t_k, Y_k)(t_{k+1} - t_k) + \beta(t_k, Y_k)X_k,\ \ k = 0, 1, \ldots, m-1$$

where

$$X_k \sim N(0, \sqrt{t_{k+1} - t_k})$$

are independent random variables and Y_k are the approximate values of $Y(t_k)$. Typically we take

$$t_k = k \cdot \frac{T}{m},$$

which in turn means that $t_{k+1} - t_k = \Delta t = \frac{T}{m}$. More generally, the Euler's method for solving a system of SDE's of the form

$$dY_i(t) = \alpha(t, Y_1(t), \ldots, Y_N(t))\,dt + \beta(t, Y_1(t), \ldots, Y_N(t))\,dB_i(t),\ \ Y_i(0) = Y_{i0}, i = 1, \ldots, N$$

is

$$Y_{i,k+1} = Y_{ik} + \alpha(t_k, Y_{1k}, \ldots, Y_{Nk})(t_{k+1} - t_k) + \beta(t_k, Y_{1k}, \ldots, Y_{Nk})X_{ik},\ i = 1, \ldots, N,\ k = 0, 1, \ldots, m-1,$$

where the vectors $(X_{1k}, \ldots, X_{Nk})$ are independent random vectors with the same n-dimensional normal distribution as $(B_1(t_{k+1}) - B_1(t_k), \ldots, B_N(t_{k+1}) - B_N(t_k))$. In the particular case when all Brownian motions are independent and $t_k = k \cdot \frac{T}{m}$, all values X_{ik} are iid random variables with the distribution $N(0, \sqrt{\frac{T}{m}})$.

Tasks:

1. In reality, the future risk free interest rate is not a constant and can be considered to be a random variable. In the case Black-Scholes model with a random interest rate R the prices of European options can be computed as
$$Price = E[e^{-\int_0^T R(t)\,dt} p(S(T))],$$
where
$$dS(t) = S(t)((R(t) - D)\,dt + \sigma(t, s)\,dB_1(t))$$
and $R(t)$ follows a suitable stochastic differential equation. We consider so called Cox-Ingersoll-Ross model
$$dR(t) = a(b - R(t))\,dt + \sigma_2\sqrt{R(t)}\,dB_2(t),$$
where a, b are constants, $B_1(t)$ and $B_2(t)$ are independent Brownian motions. So we have a system of stochastic differential equations for S and R. When computing the option prices we can replace the integral $\int_0^T R(t)$ with the product of the mean value or the interest rate and T. So, for computing the option price we should write a function that for a given values of parameters $D, \sigma, \sigma_2, T, a, b, m$ and n generates n pairs of the future stock prices $S(T)$ and mean values of the interest rates corresponding to the same trajectory. Compute the price of the call option described in the previous problem in the case of stochastic interest rate by using Euler's method with $m = 60$ time steps for solving the system of SDEs. Assume that $R(0) = 0.02,\ a = 0.5,\ b = 0.05,\ \sigma_2 = 0.2$.

2. **Homework** (Deadline 20.03.2019). Assume that the price at $t = 0$ of an European option with pay-off function p is given by
$$Price = E[e^{-0.06 \cdot T} p(S_1(T), S_2(T))],$$
where $r = 0.06$ is risk-neutral interest rate, T is the exercise time (or duration of the option) and $S_i(T),\ i = 1, 2$ correspond to the solution of the system of stochastic differential equations
$$dS_1(t) = S_1(t)[0.06\,dt + 0.8(dB_1(t) + 0.3\,dB_2(t))],$$
$$dS_2(t) = S_2(t)\left[0.06\,dt + \left(0.4 + \frac{0.3}{1 + 0.02(S_1(t) - 35)^2}\right)\,dB_2(t)\right],$$
satisfying the initial conditions $S_1(0) = 35,\ S_2(0) = 47$. Here B_1 and B_2 are independent Brownian motions. Find the price of the European option with the exercise time $T = 1$ and the pay-off function
$$p(s_1, s_2) = \max(s_1 - 35, s_2 - 35, 0)$$
with total Monte-Carlo error less than 0.01 at the confidence level $\alpha = 0.06$.