Simulation Methods in Financial Mathematics
Computer Lab 7
26 March 2019

Goal of the lab: To learn to use importance sampling for speeding the numerical option pricing process.

The idea of **importance sampling** comes from the fact that we can compute instead of the expected value of $g(X)$ in the case of random variable X with probability density function f_X the expected value of $\frac{g(Y)f_X(Y)}{f_Y(Y)}$ using a random variable Y with the probability density function f_Y:

$$E(g(X)) = \int_{-\infty}^{\infty} g(x)f_X(x)\,dx = \int_{-\infty}^{\infty} \frac{g(y)f_X(y)}{f_Y(y)} f_Y(y)\,dy = E[\frac{g(Y)f_X(Y)}{f_Y(Y)}].$$

This idea is very useful when X is such that $g(X)$ has large values with low probability and Y increases the chances of seeing the large (important) values of g.

Since in the case of option pricing problems we work mainly with normally distributed random variables (increments of Brownian motion), we consider in detail what happens if we change the parameters of the normal distribution.

If $X \sim N(0,b)$ is a normally distributed random variable and we replace it with $Y \sim N(a,b)$ distributed random variable, then direct computation gives us

$$\frac{f_X(y)}{f_Y(y)} = \frac{e^{-\frac{y^2}{2b^2}}}{e^{-\frac{(y-a)^2}{2b^2}}} = e^{\frac{-2ay+a^2}{2b^2}},$$

thus

$$E[g(X)] = E[g(Y)e^{\frac{-2aY+a^2}{2b^2}}].$$

But we can represent $Y \sim N(a,b)$ as $Y = X + a$, where $X \sim N(0,b)$, thus we can write this equality as

$$E[g(X)] = E[g(X+a)e^{\frac{-2aX-a^2}{2b^2}}].$$

Similarly we can change the standard deviation: for $X \sim N(0,b)$ we have

$$E[g(X)] = E\left[c\,g(cX)e^{-\frac{X^2}{2b^2}(c^2-1)}\right].$$

It is also possible to combine those changes to obtain a formula (for $X \sim N(0,b)$)

$$E[g(X)] = E\left[c\,g(cX+a)e^{-\frac{X^2(c^2-1)+2acX+a^2}{2b^2}}\right].$$

Changing the mean value is used more often than changing the standard deviation.

For example, in the case of BS model with constant volatility, we know that the price of an European options with payoff function p is given by

$$price = E[e^{-rT}p\left(S(0)e^{(r-D-\sigma^2/2)T+\sigma B(T)}\right)].$$

By using $B(T)+\eta T$ instead of $B(T)$, we get from the previous formula (where $a=\eta T,\ b=\sqrt{T}$) that for any η the price can be computed as

$$price = E\left[e^{-rT}p\left(S(0)e^{(r-D-\sigma^2/2+\eta\sigma)T+\sigma B(T)}\right)e^{-\eta B(T)-\frac{\eta^2 T}{2}}\right].$$

More generally, the stock price depends on the full trajectory of the Brownian motion $B(t),\ 0 \le t \le T$ and if we change it into drifted Brownian motion $B(t)+\eta t$, it corresponds to changing each increment $dB(t)$ in our market model to $dB(t)+\eta\,dt$, which produces a factor $e^{-\eta\,dB(t)-\frac{\eta^2}{2}\,dt}$ under the expected value sign for each small interval dt. By multiplying them together, we get that even in the case of complicated market models with non-constant or stochastic volatility, we can change the trend in the equation for $S(t)$ by replacing $dB(t)$ with $dB(t) + \eta\, dt$, but this introduces an additional factor $e^{-\eta B(T)-\frac{\eta^2 T}{2}}$ under the expected value sign. For example, for BS market model with variable volatility it means that for any value of η we have

$$Price = E[e^{-rT-\eta B(T)-\frac{\eta^2 T}{2}} p(S(T))],$$

where the stock price $S(T)$ corresponds to the market model

$$dS(t) = S(t)((r - D + \eta\sigma(t, S(t))\, dt + \sigma(t, S(t))dB(t)).$$

This is especially useful for out-of-the-money options when there is relatively low probability of the payoff function of becoming positive.

To reduce the variance of the random variable expectation of which is to be found, we must choose η so that $S(0) \cdot e^{(r-D-\frac{\sigma^2}{2}+\eta\sigma)T}$ would be in the region where the payoff function p is not equal to zero. This kind of methodology is known as importance sampling because we increase the probability of generating important values of the random variable.

Tasks:

1. Consider BS market model with constant volatility $\sigma = 0.5$ and European call option with exercise time $T = 0.5$ and exercise price $E = 90$. Assume that $r = 0.1$, $D = 0$ and $S(0) = 50$. Use usual (no variance reduction)) approach and then importance sampling to find the price of the option with MC method with precision 0.01 at the error probability $\alpha = 0.05$. Use the value of η for which we have

$$S(0) \cdot e^{(r-D-\frac{\sigma^2}{2}+\eta\sigma)T} = 1.1 \cdot E.$$

   Compare the speed (the number of prices generated) of the methods.

2. Consider the same market model as in the first exercise and the option that pays the owner 30, if $20 <= S(T) <= 30$ and is worthless otherwise. For $\alpha = 0.05$ compute the price with accuracy 0.01.

3. Consider pricing of the same option as in Task 1 but in the case of the variable volatility $\sigma(t, s) = 0.3 + \frac{0.4}{1+0.01\cdot(s-50)^2}$. Use importance sampling together with Euler's method for generating approximate stock prices to find the price of the option with total error less than 0.005 with probability 0.95.

4. **Homework problem 4** (deadline 03.04.2019) Consider pricing the European option with payoff function $p(s) = \max\{22 - |s - 38|, 0\}$ when the stock price follows the BS market model with volatility $\sigma(s) = 0.4 + 0.03 \arctan(0.1 \cdot (s - 78))$. Assume that $T = 0.5$, $r = 0.02$, $D = 0.01$, $S0 = 85$.

   - Compare **the number of generated stock prices** for the usual Euler's method, the control variates method (use the discounted pay-off of the same option in the case of a BS model with a constant volatility as the control variate), and the importance sampling method when computing an approximate price of the option with MC-error 0.02 (with the confidence level $1 - \alpha = 0.95$) by using $m = 10$ time steps.
   - With **the fastest method** compute **the price of the option** with total error less than 0.002 (that is, compute option prices with a given accuracy (see Lab 5, Task 1)).

   Hint 1: The price of the option is related to three call options, namely the call option with the exercise price $E = 20$ (denote it by C_1), the call option with the exercise price $E = 40$ (denoted by C_2) and the call option with the exercise price $E = 60$ (denoted by C_3) according to the formula

$$Price = C_1 - 2C_2 + C_3.$$

   Hint 2: Recall, that when computing option prices with a given accuracy (see Lab 5) the value ϵ_0 is small enough if the change of option prices differ significantly more than by ϵ_0 (say, more than $2\epsilon_0$). If we use too large value of ϵ_0, then the suggested value of m is overestimated and final computations may take too much time.