Simulation Methods in Financial Mathematics
Computer Lab 9
April 10, 2019

Goal of the lab: To learn to use stratified sampling for speeding the numerical option pricing process in situations where the price of the option is dependent on the path of the stock price.

Remarks: We denote by A' the transpose of a matrix A and by writing $Y \sim N(0, s)$ we mean that Y is normally distributed with mean 0 ans standard deviation s. Recall that Euclidean norm of $\mathbf{x} = (x_1, \dots, x_m)'$ is $\|\mathbf{x}\| = \sqrt{x_1^2 + \ldots + x_m^2} = \sqrt{\mathbf{x}'\mathbf{x}}$.

Lab Theory: Assume $\mathbf{v} = (v_1, \dots, v_m)'$ is a non-zero vector (i.e. $\|\mathbf{v}\| = \sqrt{v_1^2 + \ldots + v_m^2} > 0$),

$$W \sim N(0, a\|\mathbf{v}\|)$$

and

$$Z_i \sim N(0, a), \ i = 1, \dots, m$$

are independent random variables and $a > 0$, then it is easy to check that by defining a vector $\mathbf{X}$ of random variables $X_i, \ i = 1, \dots, m$ as

$$\mathbf{X} = \frac{W}{\|\mathbf{v}\|^2}\mathbf{v} + \mathbf{Z} - \frac{(\mathbf{v}'\mathbf{Z})}{\|\mathbf{v}\|^2}\mathbf{v}$$

we have that the components of $\mathbf{X}$ are independent and have distribution $N(0, a)$, and also

$$\mathbf{v}'\mathbf{X} = W.$$

Indeed, $\mathbf{X}$ is normally distributed since it is a linear combination of (jointly) normally distributed random variables and by a direct calculation we get that the covariance matrix $\mathbb{E}(\mathbf{X}\mathbf{X}')$ is of the form $a\mathbf{I}_m$, where $\mathbf{I}_m$ is the $m \times m$ identity matrix.

If we want to generate at once more than one vector, each corresponding to different value of W, then the former formula can be written as

$$\mathbf{X} = \frac{1}{\|\mathbf{v}\|^2}\mathbf{v}\mathbf{W} + \mathbf{Z} - \frac{(\mathbf{v}\mathbf{v}'\mathbf{Z})}{\|\mathbf{v}\|^2},$$

where $\mathbf{X}$ is now a $m \times n$ matrix with independent normally $N(0, a)$ distributed random variables, $\mathbf{W}$ is a $1 \times n$ matrix (a row vector) of independent $N(0, a\|\mathbf{v}\|)$ distributed random variables and $\mathbf{Z}$ is a $m \times n$ matrix with independent normally $N(0, a)$ distributed random variables. The matrix $\mathbf{X}$ has now the property $\mathbf{v}'\mathbf{X} = \mathbf{W}$ (i.e. each column sums with weights v_i to the value of W_i). Thus we can generate independent normally distributed random variables so that we first generate the value of a linear combination of the variables and then determine the variables itself.

This result allows us to stratify the generation of normally distributed random variables X_i according to any given linear combination (e.g. according to the sum) of the random variables - we just have to generate the values $W \sim N(0, a\|\mathbf{v}\|)$ from a given stratum and to determine X_i by the above formula.

We use the previous result to generate the increments of a Brownian motion $B(t_1) - B(0), B(t_2) - B(t_1), \dots, B(T) - B(t_{m-1})$ so that their sum $B(T)$ would be in a given stratum (i.e., $\mathbf{v} = (1, 1, \dots, 1)'$). To accomplish this we first need to generate the value of W (from the desired stratum) according to the distribution $N(0, \sqrt{T})$ and calculate the vector of Brownian motion increments $\mathbf{X}$ using the formula presented (for intervals that have equal lengths, $a = \sqrt{\frac{T}{m}}$).

Tasks: It is useful to know that in R the matrix multiplication is $\% * \%$ and transposed matrix can be obtained by the function $t()$.

1. Write a procedure that, given the inputs k, m, T, would draw k different Brownian motion paths such that every stratum (based on the value of $B(T)$ and defined as in the previous lab) would include the terminal value of exactly one path.

2. Enhance the stock price generation function that is based on Euler's method and non-constant volatility so that one could use the optimal stratified sampling. Using optimal stratified sampling find the price of an European call with total error 0.01 in the case when $r = 0.06$, $D = 0.03$, $\sigma(s) = \frac{95}{95+s}$, $T = 0.5$, $S(0) = 105$, $E = 100$, using $\alpha = 0.05$.

3. **Homework problem 5** (Deadline April 17, 2019) Use the optimal stratified sampling with 50 stratums to compute with the accuracy 0.0001 (for $\alpha = 0.05$) the expected value of

$$Z = \frac{|X - Y|}{1 + 0.1X^2 + 0.2Y^2}$$

where X and Y are independent random variables from the standard normal distribution. The stratification should be based on the values of the difference $X - Y$. Hint: define the function g so that it works when it's argument is a matrix with two columns, where the values of X are in the first column and the values of Y are in the second column. The generator should generate n pairs of X and Y so that their difference corresponds to the given stratum. The return value of the generator should be a matrix with two columns.