Simulation Methods in Financial Mathematics
Computer Lab 11

Goal of the lab:

- To learn to use stratified sampling for speeding up the pricing of Asian options

We will consider the case where the average stock price is calculated using the formula

$$A(T) \approx \frac{1}{m}\sum_{i=1}^{m} S_{i-1}\left(1 + (r-D)\frac{T}{2m} + \frac{\sigma(t_{i-1}, S_{i-1})}{2}(B(t_i) - B(t_{i-1}))\right), \tag{1}$$

where $t_i = i\frac{T}{m}$. We already know, how to generate the increments of the Brownian motion $\Delta B = (B(t_1) - B(t_0), B(t_2) - B(t_1), \ldots, B(t_m) - B(t_{m-1}))'$ (here $\mathbf{a}'$ denotes the transpose of $\mathbf{a}$) so that given a vector $\mathbf{v} = (v_1, \ldots, v_m)'$ the stratifying would be based on the values of $v'\Delta B$:

- Generate the value of W from the desired stratum of $N(0, \|v\|\sqrt{\frac{T}{m}})$.
- Generate a random vector $\mathbf{Z} = (Z_1, \ldots, Z_m)'$, where $Z_i \sim N(0, \sqrt{\frac{T}{m}})$
- Calculate
$$\Delta B = \frac{1}{\|v\|^2}W\mathbf{v} + \mathbf{Z} - \frac{1}{\|v\|^2}\mathbf{v}(\mathbf{v}'\mathbf{Z}).$$

When we want to generate a matrix with dimensions $m \times n$ so that each column would represent the increments of the Brownian motion of a corresponding generated value of W (thus in total there are n values of W), then we need to modify the formula as follows: W must be generated as a row vector with n components, Z must be a $m \times n$ matrix and the increment matrix can then be calculated as

$$\Delta B = \frac{1}{\|v\|^2}\mathbf{v}\mathbf{W} + \mathbf{Z} - \frac{1}{\|v\|^2}(\mathbf{v}\mathbf{v}'\mathbf{Z}).$$

Using this formula there are several possibilities for stratification:

- When we want the the strata to be based on the values of $B(T)$, we take $\mathbf{v} = (1, 1, \ldots, 1)'$. This stratification should be appropriate for European options.
- When we want the strata to be based on the average values of the Brownian motion $\frac{1}{T}\int_0^T B(t)\,dt$, we may approximate the integral by
$$\frac{1}{T}\int_0^T B(t)\,dt \approx \sum_{i=0}^{m-1}\frac{B(t_i)}{m} = \sum_{i=1}^{m-1}\sum_{j=1}^{i}\Delta B_j = \sum_{j=1}^{m-1}\frac{m-j}{m}\Delta B_j$$
and hence should use stratification with $\mathbf{v} = (\frac{m-1}{m}, \frac{m-2}{m}, \ldots, \frac{m-m}{m})'$. This stratification should be appropriate for Asian options when the payoff does not depend on the stock price at time T.
- When we want the the strata to be based on differences of $B(T)$ and the average values of the Brownian motion, we take $\mathbf{v} = (\frac{1}{m}, \frac{2}{m}, \ldots, \frac{m}{m})'$. This stratification should be appropriate for average strike options.

- In general, if the payoff is a function of a linear combination of $S(T)$ and $A(T)$, a good choice for stratification vector v is the same linear combination of vectors $\mathbf{v_1} = (1, 1, \ldots, 1)'$ and $\mathbf{v_2} = (\frac{m-1}{m}, \frac{m-2}{m}, \ldots, \frac{m-m}{m})'$

Tasks:

1. Let $m = 20$. Consider the pricing of Asian options with payoff functions $p(s, a) = \max(50 - a, 0)$ and $p(s, a) = \max(s - a, 0)$ using the MC method with MC error 0.01 for $\alpha = 0.05$ in the case when $S(0) = 49$, $r = 0.05$, $D = 0.02$, $T = 0.5$, $\sigma = 0.5$. Compare the number of generations required when not using any variance reduction methods and when using the simple version of stratified sampling (where expected value is computed separately in each stratum with MC error $\sqrt{k} \cdot 0.01$) with the number of strata $k = 40$ for all the stratification possibilities described on the handout (3 different values of v).

2. Use the best stratification method to compute the prices of the options with total error less than 0.05 with probability 0.95.