Simulation Methods in Financial Mathematics
Computer Lab 2

Goals of the lab:

- To familiarize yourself with Brownian motion and the Black-Scholes model of the stock market
- To program the Black-Scholes call and put option pricing formulas
- To understand that the prices of options can be calculated as expected values

A standard (also called vanilla) option is a contract written by a seller that conveys to the buyer the right – but not the obligation – to buy (in the case of a call option) or to sell (in the case of a put option) in a future a particular asset, such as a piece of property, or shares of stock or some other underlying security, such as, among others, a futures contract, for the price specified in the contract. In return for granting the option, the seller collects a payment (the premium) from the buyer. We will deal with stock options.

For example, an European call option gives the buyer the right on a fixed future date and time (time T) to buy a share of the fixed company for a fixed price E; such a contract is equivalent to the right to receive the amount $p(S(T))$ at time T, where $p(s) = \max(s - E, 0)$. The function p is called the payoff function of the option. A similar put option gives the owner the right to sell a share at some fixed time point with some fixed price; the respective payoff function is $p(s) = \max(E - s, 0)$.

Actually, there are many different types of options and not all of them are related to buying or selling something, but all options can be viewed as contracts giving the owner the right to receive in the future a payment which value is determined by the future price (or prices) of the underlying asset (or assets). So it is important to be able to compute the prices of options with arbitrary payoff functions.

To find the price of an option, we will first need to model the share price. Based on the model we can deduce the rule for calculating the price. One of the most commonly used models is the Black-Scholes model

$$dS(t) = S(t) \cdot (\mu \cdot dt + \sigma \cdot dB(t)),$$

where μ is trend, σ is the volatility of the stock price (quantifies the risk of the instrument) and B is the standard Brownian motion (also known as the Wiener process). In the general case μ and σ can depend on time, stock price and the Brownian motion. In the current lab, however, we deal with constant μ and σ.

Tasks:

1. The standard Bownian motion is defined by the following properties

   - $B(0) = 0$;
   - increments $B(t_2) - B(t_1)$ are normally distributed $N(0, \sqrt{t_2 - t_1})$ and independent for disjoint intervals.

   To generate the paths of a Brownian motion we may split the interval $[0, T]$ into m equal subintervals and generate the values of the Brownian motion at the time instants $t_i = i \cdot \frac{T}{m}$ as $B(t_{i+1}) = B(t_i) + X_i$, where X_i are iid normally distributed random variables with mean 0 and standard deviation $\sigma = \sqrt{\frac{T}{m}}$.

Produce a graph consisting of 10 different paths of a Brownian motion in interval $[0, 0.5]$, by dividing the latter into $m = 100$ subintervals. It is recommended to store the paths in a matrix with dimensions 101×10 (one trajectory in each column).

2. When μ and σ are constant, then it is known that the stock prices corresponding to the Black-Scholes model are given in terms of the standard Brownian motion through the formula

$$S(t) = S(0)e^{(\mu-\sigma^2/2)t+\sigma B(t)}.$$

Thus we have a one-to-one correspondence between the paths of the Brownian motion and the paths of a stock price.

Assume $S(0) = 100$, $\mu = 0.1$, $\sigma = 0.5$ and produce a graph of 10 stock price paths in interval $[0, 0.5]$, by dividing the latter into $m = 100$ subintervals.

3. Assuming the validity of the Black-Scholes market model with a constant risk-free interest rate r, stock dividend percent D and volatility σ, it is possible to derive formulas (called Black-Scholes formulas) for the prices of the European call and put options. Implement the Black-Scholes formulas for European call and put options as functions in R. The respective formulas are

$$C(S, E, T, r, \sigma, D, t) = Se^{-D(T-t)}\Phi(d_1) - Ee^{-r(T-t)}\Phi(d_2),$$

$$P(S, E, T, r, \sigma, D, t) = -Se^{-D(T-t)}\Phi(-d_1) + Ee^{-r(T-t)}\Phi(-d_2),$$

where

$$d_1 = \frac{\ln(\frac{S}{E}) + (r - D + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}, \quad d_2 = d_1 - \sigma\sqrt{T-t},$$

S is the stock price at time t and Φ is the cdf of a standard normal distribution (function `pnorm` in R). Make a graph of call and put option prices at $t = 0$ and the respective payoff functions in the interval $0 \le S \le 200$ when $r = 0.02$, $\sigma = 0.5$, $E = 100$, $T = 1$, $D = 0$, by computing the option prices for integer values of S.

4. Make an experiment to test the fact that in the case of the Black-Scholes market model with constant volatility we can compute the prices of European as an expected value

$$H(t, T, S(t)) = \mathbb{E}[e^{-r(T-t)}p(S(T))],$$

where $S(T)$ is a random variable defined as

$$S(T) = S(t)e^{(r-D-\frac{\sigma^2}{2})(T-t)+\sigma(B(T)-B(t))} \tag{1}$$

and p is the option payoff function. Use the Monte-Carlo method to compute the prices of put and call options at $t = 0$ for $S(0) = 95$ and $S0 = 105$ (using other parameters from the previous task) with accuracy 0.01 at the probability level $\alpha = 0.05$. Output the price obtained by MC method, the exact price according to Black-Scholes formulas and the difference of the results for both put and call option.

Hint: use the function MC2 from the previous lab. Thus one needs to define a generator which for a given n would output n values of $S(T)$ and a suitable function g, the expected value of which needs to be computed. For defining the generator of the stock prices, one can benefit from the fact that $B(T) - B(t)$ has distribution $N(0, T-t)$ (where the second argument of $N()$ is variance).