Simulation Methods in Financial Mathematics
Computer Lab 4

Goal of the lab:

- To learn to use higher order discretization methods for stock price generation and study their error rates

There are many numerical schemes of different strong and weak convergence rates for solving stochastic differential equations. For example, a stochastic differential equation of the form

$$dS(t) = S(t)(\mu(t, S(t))\, dt + \sigma(t, S(t))\, dB(t)) \tag{1}$$

can be solved by using the following methods:

Euler's method

$$S_{i+1} = S_i \cdot (1 + \mu(t_i, S_i)\Delta t + \sigma(t_i, S_i)X_i), \tag{2}$$

Milstein's method

$$S_{i+1} = S_i \cdot (1 + \mu(t_i, S_i)\Delta t + \sigma(t_i, S_i)X_i) + \frac{1}{2}L_2\sigma(t_i, S_i)(X_i^2 - \Delta t) \tag{3}$$

and
A weakly second order method

$$\begin{aligned} S_{i+1} &= S_i \cdot (1 + \mu(t_i, S_i)\Delta t + \sigma(t_i, S_i)X_i) \\ &+ \frac{1}{2}\left(L_1\mu(t_i, S_i)\Delta t^2 + (L_2\mu(t_i, S_i) + L_1\sigma(t_i, S_i))\Delta t\, X_i + L_2\sigma(t_i, S_i)(X_i^2 - \Delta t)\right), \end{aligned} \tag{4}$$

where random variables X_i are independent and have distribution $N(0, \sqrt{\Delta t})$ and the operators L_1 and L_2 are defined by

$$L_1 f(t, s) = s\frac{\partial f}{\partial t}(t, s) + s \cdot \mu(t, s) \cdot (f(t, s) + s\frac{\partial f}{\partial s}(t, s)) + \frac{s^2\sigma(t, s)^2}{2}(2\frac{\partial f}{\partial s}(t, s) + s\frac{\partial^2 f}{\partial s^2}(t, s)),$$

$$L_2 f(t, s) = s\sigma(t, s)(f(t, s) + s\frac{\partial f}{\partial s}(t, s)).$$

L_1 and L_2 are called operators because they act on functions of two variables and the result is also a function of two variables. For example, for each function f the result of applying L_1 to f the outcome is a new function (denoted by $L_1 f$) of two variables. A more concrete example: if $f(t, s) = s^2$ then $\frac{\partial f}{\partial s}(t, s) = 2s$ and therefore the function $L_2 f$ is given by

$$L_2 f(t, s) = s\sigma(t, s)(s^2 + s \cdot 2s) = 3s^3\sigma(t, s).$$

Similarly, if $\mu(t, s) = \mu$ (a constant), then $L_1\mu(t, s) = s \cdot \mu^2$, $L_2\mu(t, s) = s\sigma(t, s) \cdot \mu$.

Tasks:

1. Program methods (3) and (4) for generating solutions of the SDE (1) at time T when the Black-Scholes market model is assumed.Assume that in additions to functions $\mu(t, s)$ and $\sigma(t, s)$ also the functions $L_1\mu(t, s), L_1\sigma(t, s)$, $L_2\mu(t, s)$ and $L_2\sigma(t, s)$ are defined outside of the generators.

2. Experimentally find the strong convergence rates of (3) and (4) using procedure `nls` in the case when $S(0) = 100,\ T = 0.5, \mu = 0.1$ and $\sigma = 0.6$.

3. Find the prices of an European put option with (3) and (4) in the cases $m = 4, 8, 16$ with MC error less than 0.01 (computed for $\alpha = 0.05$) when $E = 55$, $T = 0.5$, $r = 0.05$, $S(0) = 50$, $D = 0$, $\sigma(s) = 0.5 + 0.2 \cdot \sin(0.1s - 5)$. For each method, look at the differences of prices. If MC error is small enough (more than 3 times smaller than the differences), then the differences describe the weak convergence rate - if m is multiplied by 2 for each new computation, the ratios of consecutive differences should be approximately 2^q (if we divide previous difference with the next one), where q is the weak convergence rate (can you figure out, why this is true?). Is the faster convergence rate of the weakly second order method observable?