Simulation Methods in Financial Mathematics
Computer Lab 8

Goal of the lab:

- To learn to use the stratified sampling method for speeding up the numerical option pricing process.

Stratified sampling is based on specifying (disjoint) events $(A_i)_{i=1}^k$ with $P(A_i) = p_i$ that partition the probability space and then invoking the formula

$$E(Y) = \sum_{i=1}^{k} E(Y|A_i)p_i.$$

A straightforward way to use the formula is to calculate conditional expectations on the right-hand side separately by using the MC method and then the weighted sum of the results can be calculated. It can be shown that if the conditional expectations are computed with the error $\frac{\varepsilon}{\sqrt{p_i}}$ in each stratum with a given error probability, then the total error is less than ε at the same probability level. Namely, when we generate n_i random variables $Y_{ij},\ j = 1, \ldots, n_i$ in stratum i then the variance of

$$Z = \sum_{i=1}^{k} p_i \sum_{j=1}^{n_i} \frac{Y_{ij}}{n_i}$$

is equal to

$$Var(Z) = \sum_{i=1}^{k} \frac{p_i^2 \sigma_i^2}{n_i}. \tag{1}$$

Additionally, as Z is the sum of a large number of independent variables, it can be assumed to be normally distributed and therefore (prove it!) we have that with probability α we have

$$|Z - EY| \le -\Phi^{-1}(\frac{\alpha}{2})\sqrt{Var(Z)} \tag{2}$$

If we compute the answer with the error $\frac{\varepsilon}{\sqrt{p_i}}$ at probability level α then (by denoting $c_\alpha = -\Phi^{-1}(\frac{\alpha}{2})$) we have

$$\frac{c_\alpha \sigma_i}{\sqrt{n_i}} \approx \frac{\varepsilon}{\sqrt{p_i}}$$

and therefore

$$Var(Z) \approx \sum_{i=1}^{k} p_i \frac{\varepsilon^2}{c_\alpha^2} = \frac{\varepsilon^2}{c_\alpha^2}.$$

Thus, from (2) we get that $|Z - EY| \le \varepsilon$ with probability $1 - \alpha$.

In this lab we use the system of events

$$A_i = \{B(T) \in (\sqrt{T}\Phi^{-1}(\frac{i-1}{k}), \sqrt{T}\Phi^{-1}(\frac{i}{k})]\},\ i = 1, 2, \ldots, k,$$

thus $P(A_i) = \frac{1}{k},\ i = 1, 2, \ldots, k.$

In the following we will be interested in a call option with $S(0) = 50$, $r = 0.1$, $\sigma = 0.5$, $T = 1$, $t = D = 0$, $E = 100$.

1. Write a procedure, that uses the representation $S(T) = S(0)e^{(r-D-\frac{\sigma^2}{2})T+\sigma B(T)}$ and the number i of the strata to generate stock prices corresponding to the event A_i. Values of the Brownian motion from the conditional distribution can be generated as $\sqrt{T}\Phi^{-1}(X_i)$, where $X_i \sim U(\frac{i-1}{k}, \frac{i}{k})$. Use this procedure to find the price of the call option when using k with values $k = 5$, 10, 20, 100 with precision 0.01 (take the probability of an error as $\alpha = 0.05$). Compare the number of generations (that is the sum of the number of generations for all strata) with the ordinary MC method as well as with the importance sampling method.

The simple approach of using stratified sampling described previously does not usually give the highest possible speedup. It is quite easy to show that maximal variance reduction is achieved when in stratum i the number n_i of generated random variables is proportional to $\sigma_i p_i$ (where σ_i^2 is the conditional variance in stratum i). Thus one possibility for generating the optimal number of Y_i is setting the proportionality constant C to some fixed value and in every stratum generating random variables until $n \geq Cp_i \cdot \sigma_i$ where the value of σ_i is estimated.

Proofs of the previous results can be found, for example, in P. Glassermann, “Monte Carlo Methods in Financial Engineering”, Section 4.3.

Tasks:

2. Write a function for calculating the expected value in a stratum so that the generator for a given stratum, the proportionality constant C and stratum probability p can be given as input and the expected value is calculated as the average of n generated values, where n is the smallest number in whole hundreds such that $n \geq C \cdot p \cdot \sigma_Y$, where σ_Y is the estimate of the standard deviation (in the stratum) of the random variable, whose expected value we are calculating. The output of the function should include the calculated expected value, estimated variance and the number of generated random variables.

3. Program a procedure that calculates the price of the option using stratified sampling so that the conditional expected value in every stratum is calculated using the previously defined function with some given C. Output the overall error estimate (by estimating the variance of the result by (1) and the error by (2)) and the total number of generated random variables in addition to the calculated expected value. Using this procedure, write an additional function for calculating the expected value with a given error for a given α by using the optimal stratified sampling. Use the knowledge that if we multiply C by a certain number x, then the error of the final result is divided approximately by $\sqrt{x}$.