**Functional Analysis III** (MTMM.00.072)
2021/2022 FALL SEMESTER
**Detailed overview**

## Chapter I: Weak topologies in Banach spaces

**§ 1. Topological spaces. 1.1.** The notion of a topological space; properties of closed sets in a topological space; comparision of topologies (what does it mean for a topology to be stronger or weaker than an other topology); first examples of topological spaces: the discrete topology and the trivial (indiscrete) topology, the topology induced by a metric, the relative topology on a subset of a topological space, the topology generated by a collection of subsets of a given set. **1.2.** Neighbourhoods of a point in a topological space; neighbourhood base (or a local base) at a point. **1.3.** Base for a topology; a necessary and sufficient condition for a collection of sets to be a base for a given topology; a base for the topology generated by a collection of subsets of a given set; pre-base for a topology. **1.4.** Preordered, partially ordered, and directed sets; the notion of a net; convergence of a net in a topological space; the basic construction of a net converging to a given point (with a neighbourhood base at this point being the index set for this net); the notion a subnet; properties of subnets [M; p. 150, Prop. 2.1.31]. **1.5.** The interior, closure, and boundary of a set in a topological vector space; characterisation of interior points in terms of their neighbourhoods; characterisation of closure points in terms of theire neighbourhoods and in terms of convergent nets; characterisation of boundary points in terms of their neighbourhoods; characterisation of closed sets in terms of limits of convergent nets. **1.6.** Continuitiy of a mapping between topological spaces; a necessary and sufficient condition for a mapping to be continuous in terms of originals of sets of a pre-base for the topology of the target space; continuity at a point; *a mapping is continuous if and only if it is continuous at every point of the domain space;* a necessary and sufficient condition for a mapping to be continuous at a point in terms of convergent nets; homemorphism, embedding. **1.7. Separation axioms in topological spaces.** $T_0$, $T_1$, and $T_2$ (or Hausdorff) spaces; characterisation of $T_1$ spaces via closedness of singletons; characterisation of $T_2$ spaces (i.e. Hausdorff spaces) via uniqueness of limits of convergent nets; regular, completely regular, and normal topological spaces; $T_3$, $T_{3\frac{1}{2}}$, and $T_4$ spaces; normality of metric spaces (an exercise); Urysohn's Lemma for normal topological spaces (w.p[1]). **1.8.** The (weak) topology induced by a family of mappings; a necessary and sufficient condition for a mapping to the space equipped with this topology to be continuous; a necessary and sufficient condition for this space to be Hausdorff (in the case where the target spaces of the topologising family are Hausdorff); a necessary and sufficient condition for a net to be convergent in this topology; a sufficient condition for this topology to be metrisable [M; p. 206, Prop. 2.4.8]. **1.9.** the Cartesian product of a family of sets; the product topology on the Cartesian product of a family of topological spaces; a sufficient condition for the product space to be Hausdorff; a necessary and sufficient condition for a mapping to the product space to be continuous; a necessary and sufficient condition for a net in the product space to converge; *the natural mapping from a space, whose topology is induced by a separating topologising family of functions, to the product space of the corresponding target spaces is an embedding* [M; p. 205, Prop. 2.4.7]. **1.10.** Compact topological spaces, compact sets in topological spaces; characterisations of compact topological spaces in terms of centred families of closed sets and existence of convergent subnets; boundedness of a continuous real function on a compact topological space, attainment of its infimum and supremum by a continuous real function on a compact topological space; compactness of closed subsets of compact topological spaces; closedness of compact subsets of Hausdorff topological spaces; separation of compact sets from points by open sets in Hausdorff topological spaces; normality of compact Hausdorff topological spaces; Tichonoff's Theorem on the compactness of the product space of compact topological spaces (w.p.).

**§ 2. Topological vector spaces. 2.1.** The notion of a topological vector space (TVS); *translating by a fixed element and multiplying by a non-zero scalar in a TVS are homeomorphisms, multiplying scalars by a non-zero element of a TVS is an embedding, the translate and multiple by a non-zero scalar of an open set in a TVS are open, the translate and multiple of a closed set are closed* [M; p. 167, Thm. 2.2.9, (c)]; *for an element of a TVS, its neighbourhoods are exactly the translates by this elements of neighbourhoods of zero* [M; p. 167, Thm. 2.2.9, (e)]; *translates of sets of a neighbourhood base at zero in a TVS by a fixed element form a neighbourhood base at this fixed element;* interiors and closures of translates, multiples, and sums of sets in a TVS [M; p. 167, Thm. 2.2.9, (d)]. **2.2. (Topological)**

[1] w.p.—without proof

**separation properties in TVSs.** *Every $T_0$ TVS is a $T_{3\frac{1}{2}}$-space* [M; p. 174, Thm 2.2.14] (w.p.). **2.3.** Convex, absolutely convex, balanced, and absorbing sets in vector spaces; *a set in a vector space is absolutely convex if and only if it is convex and balanced* (an exercise); *the interior and closure of a convex set in a TVS are convex; the closure of a balanced set and, if zero is an interior point of this set, then also its interior are balanced sets* [M; p. 167, Thm. 2.2.9, (i)]; *neighbourhoods of zero in a TVS are absorbing* [M; p. 167, (f)]; existence of a neighbourhood base at zero consisting of balanced sets in a TVS; the notion of a locally convex space (LCS); existence of a neighbourhood base at zero consisting of balanced convex sets in a LCS; existence of a neighbourhood base at zero consisting of closed balanced sets in a TVS; existence of a neighbourhood base at zero consisting of closed balanced convex sets in a LCS. (Ideas for constructing all these desired neighbourhood bases can be taken from the proof of [M; p. 167, (g)]). **2.4. The closed linear span and the closed convex hull of a set in a TVS.** +++ **2.5.** The algebraic dual (space) of a vector space, the topological dual (space) of a TVS; non-existence of non-zero continuous linear functionals on a TVS whose topology is the trivial one (i.e. consists of the empty set and the space itself) (a proof is given in [M: p.178, the paragraph following the proof of Cor. 2.2.22]); necessary and sufficient conditions for a linear functional on a TVS to be continuous [M; p. 175, Thm. 2.2.16]; xxx. **2.6.** Examples of topological vector spaces: a normed space; a subspace of a TVS [M; p. 167, Thm. 2.2.9, (j)]; the space $L_0(\Omega, \Sigma, \mu)$ where the topology is given by a metric with respect to which convergence of sequences is convergence in measure [M; pp. 162–164, Example 2.2.5] (w.p.); the space $L_p(\Omega, \Sigma, \mu)$ where $0 < p < 1$ [M; pp. 164–165, Example 2.2.6] (w.p.) (the latter two examples with $\Omega = [0, 1]$, and $\Sigma$ and $\mu$ being, respectively, the Lebesgue $\sigma$-algebra of $[0, 1]$ and the Lebesgue measure on $[0, 1]$ are examples of topological vector spaces which are not locally convex).

**§ 3. Separation of convex sets (by closed real hyperplanes) in a TVS. 3.1.** The Minkowski functional of an absorbing set, sublinear functional, seminorm; properties of the Minkowski functional [M; p. 80, Prop. 1.9.14]. **3.2.** The vector space version of the Hahn–Banach Extension Theorem [M; p. 73, Thm. 1.9.5]. **3.3.** Mazur's Separation Theorem [M; p. 176, Thm. 2.2.19], its corollaries (separation of a point from a closed subspace in an LCS [M; p. 177, Cor. 2.2.20], continuous extension of functionals froma a subspace of an LCS [M; p. 177, Cor. 2.2.21], totality of the topological dual of an LCS [M; p. 178, Cor. 2.2.22]); Eidelheit's Separation Theorem [M; p. 179, Thm. 2.2.26]; the Tukey–Klee Separation Theorem [M; p. 180, Thm. 2.2.28]; coincidence of the closures of a convex set in two locally convex topology with respect to which the topological duals are the same [M, p. 181, Cor 2.2.29].

**§ 4. Some more basic concepts in topological vector spaces. 4.1. The locally convex topology induced by a family of seminorms.** +++. **4.2.** Topologically Cauchy nets in topological vector spaces (the definition for Abelian topological groups is given in [M; p. 155, Def. 2.1.41]), metrically Cauchy nets in metric spaces (see [M; the first paragraph of p. 155]); (topological) Cauchyness of convergent nets in topological vector spaces (an exercise); coincidence of topologically Cauchy and metrically Cauchy nets in topological vector spaces whose topology is induced by a translation invariant metric (an exercise); convergence of Cauchy nets in Banach spaces (an exercise). **4.3.** Uniqueness of Hausdorff vector topologies on finite dimensional vector spaces [M; p. 181, Thm. 2.2.31] (together with [M; p. 181, Lemma 2.2.30]); closedness of finite-dimensional subspaces of a TVS [M; p. 181, Cor. 2.2.32]; continuity of linear operators from a finite-dimensional Hausdorff TVS to any TVS [M; p. 181, Cor. 2.2.33].

**§ 5. Weak topologies on vector spaces. 5.1.** The weak topology $\sigma(X, \Gamma)$ on a vector space $X$, where $\Gamma$ is a linear subspace of the algebraic dual $X^\sharp$; properties of the TVS $\big(X, \sigma(X, \Gamma)\big)$: *this space is locally convex* [M; p. 208, Thm. 2.4.11, or p. 207, Thm. 2.4.11], its Hausdorffness [???], neighbourhood base at zero in this space [M; p. 208, Prop. 2.4.12]; convergence of nets in this space (via [M; p. ??, ??]), its topological dual space [M; p. 207, Thm. 2.4.11]. **5.2.** The weak topology on a normed space, unboundedness of non-empty weakly open sets [M; p. 214, Cor. 2.5.9], comparison between the weak and the norm topologies [M; p. 212, Thm. 2.5.2, and p. 215, Prop. 2.5.13]; the weak topology of a normed space is induced by a metric if and only if this normed space is finite-dimensional [M; p. 215, Prop. 2.5.14], the relative weak topology of a bounded subset of a normed space whose dual space is separable is induced by a metric (via [M; p. 206, Prop. 2.4.8]); Mazur's Theorem (the weak and norm closure of a convex subset of a normed space are the same) [M; p. 216, Thm. 2.5.16]; the weak and norm closure of a linear subspace of a normed space are the same [M; p. 216, Cor. 2.5.17]; the weak topology of a subspace of a normed space [M, p. 218, Prop. 2.5.22]. **5.3.** The weak$^*$ topology on the dual space of a normed space, its comparison with the norm and weak topologies [M; p. 224; Thm. 2.6.2 and Cor. 2.6.3], unboundedness of non-empty weak$^*$ open sets in duals of infinite dimensional normed spaces [M; p. 226, Prop. 2.6.11]; boundedness of weak$^*$ convergent sequences in duals of Banach spaces

[M; p. 226, Cor. 2.6.10] (as a corollary from the Banach–Steinhaus Uniform Boundedness Principle); the weak$^*$ topology of the dual space of a normed space is induced by a metric if and only if this normed space is finite-dimensional [M; p. 226, Cor. 2.6.12], the relative weak$^*$ topology of a bounded subset of a separable normed space is induced by a metric [M; p. 230, Thm. 2.6.20]; the relative weak$^*$ topology of the dual unit ball of a normed space is induced by a metric if and only if this normed space is separable [M; p. 231, Thm. 2.6.23]; the Banach–Alaoglu Theorem [M; p. 229, Thm. 2.6.18] (together with its immediate corollaries); boundedness of weak$^*$ compact subsets of dual spaces of Banach spaces [M; p. 226, Cor. 2.6.9] (as a corollary from the Banach–Steinhaus Uniform Boundedness Principle); weak$^*$ lower semicontinuity of the norm of the dual space of a normed space [M; p. 227, Thm. 2.6.14]; Goldstine's Theorem [M; p. 232, Thm. 2.6.26]; Helly's Theorem [M; p. 78, Thm. 1.9.12], equivalent dual norms [M; p. 227, Thm. 2.6.15].

**§ 6. Reflexive normed spaces. 6.1.** Properties of annihilators [M; p. 93, Prop. 1.10.15, and p. 225, Prop. 2.6.6]. **6.2.** A Banach space is reflexive if and only if its dual space is [M; p. 104, Cor. 1.11.17]; a closed subspace of a reflexive normed space is reflexive [M; p. 104, Thm. 1.11.16]. **6.3.** Reflexivity of a complex normed space [M; p. 115, Prop. 1.13.1]. **6.4.** A characterisation of reflexivity of a normed space in terms of the existence of weakly convergent subsequences of bounded sequences [M; p. 119, Thm. 1.13.5], and in terms of nested sequences of non-empty closed bounded convex sets [M; p. 119, Thm. 1.13.6]; James's sequential characterisations of reflexivity [M; p. 117, Thm. 1.13.4], [M; p. 120, Thm. 1.13.9]; a Banach space is reflexive if each of its separable closed subspaces is [M; p. 120, Thm. 1.13.8]; James's theorem [M; p. 134, Thm. 1.13.15] (w.p.). **6.5.** Characterisation of weakly compact sets in normed spaces in terms of their canonical embedding into the bidual [M; p. 245, Prop. 2.8.1]; characterisation of reflexive normed spaces via weak compactness of their closed unit ball [M; p. 245, Thm. 2.8.2].

**§ 7. Weakly compact sets in normed spaces. 7.1.** Relatively compactness, countably compactness, relatively countably compactness, sequantially compactness, relatively sequentially compactness, limit point compactness, and relatively limit point compactness in a topological space; The Eberlein–Šmulian Theorem [M; p. 248–249, Thm. 2.8.6] (which of its implications hold in an arbitrary topological space?) together with Day's Lemma [M; p. 247, Lemma. 2.8.5]. **7.2.** James's Weak Compactness Theorem [M; p. 261, Thm. 2.9.3] (w.p.).

**§ 8. The Banach–Dieudonné Theorem.** The Banach–Dieudonné Theorem A.K.A. the Krein–Šmulian Theorem on Weak$^*$ Closed Convex Sets [M; p. 242, Thm. 2.7.11]; its corollaries on the weak$^*$ closedness of a subspace of the dual space of a Banach space [M; p. 242, Thm. 2.7.12], the weak$^*$ closedness of a convex subset of the dual space of a separable Banach space [M; p. 242, Thm. 2.7.12], the continuity of a linear functional on the dual space of a Banach space [M; p. 242, Thm. 2.7.9], and the continuity of a linear functional on the dual space of a separable Banach space [M; p. 242, Thm. 2.7.10].

**§ 9. Compactness of the closed convex hull of a compact set. 9.1.** Mazur's Compactness Theorem [M; p. 254, Thm. 2.8.15] and the Krein–Šmulian Weak Compactness Theorem [M; p. 254, Thm. 2.8.14] as corollaries from their separable versions [M; p. 252, Lemma 2.8.23]); weak$^*$ compactness of the weak$^*$ closed convex hull of a weak$^*$ compact set (an exercise). **9.2.** The Riesz Reprensentation Theorem describing the dual space of the Banach space $C(K)$, where $K$ is a compact Hausdorff topological space (w.p.) [???]. **9.3.** Proof of the separable versions of Mazur's compactness theorem and the Krein–Šmulian Weak Compactness Theorem [M; p. 252, Lemma 2.8.23].

**§ 10. Extreme points of compact convex sets in locally convex Hausdorff spaces. 10.1.** Extreme points, extremal subsets; an example of an extremal subset: the subset of points where the real part of a continuous linear functional attains its maximum on a compact convex set; the Krein–Milman Theorem [M; p. 265, Thm. 2.20.6] or [D; p. 148]. **10.2.** Milman's Converse to the Krein–Milman Theorem [M; p. 268, Thm. 2.10.15] (together with [M; p. 267, Lemma 2.10.12], [M; p. 268, Lemma 2.10.13], and [M; p. 268, Lemma 2.10.14]) (in fact, it is much easier to prove this "Milman's Converse" as a corollary from Bauer's characterisation of extreme points (referred to below) as in [D; p. 151 Cor. 4], or from Choquet's Lemma (also referred to below)). **10.3.** Barycentre of a Borel probability measure, uniqueness of barycentres (an exercise); existence of the barycentre of a regular Borel probability measure on a set whose closed convex hull is compact [D; p. 148, Thm. 1]; a necessary and sufficient condition for a point in a Hausdorff LCS to belong to the closed convex hull of a compact set in terms of the existence of a regular Borel probability measure representing this point [D; p. 149, Thm. 2]; Bauer's characterisation of extreme points [D; p. 150, Thm. 3]. **10.4.** Choquet's Integral Representation Theorem [D; p. 154] (w.p.); measurability of the set of extreme points (the set of extreme points of a metrisable compact convex set in an LCS is a $G_\delta$) [D; p. 154]; Rainwater's Theorem [D; p. 155]. **10.5.** Slices of convex subsets in locally convex Hausdorff spaces; Choquet's Lemma [Ch; p. 107, Prop. 25.13]; Bourgain's Lemma [GGMS; p. 26, Lemma II.1].

## Chapter II. More on bounded linear operators on Banach spaces

**§ 1. The adjoint of a bounded linear operator II. 1.1. Annihilators of kernels and ranges of bounded linear operators** [M; p. 289, Lemma 3.1.16] and [???]. **1.2. Duality theorems for bounded linear operators and their adjoints** [M; p. 290, Thm. 3.1.17], [M; p. 290, Thm. 3.1.18], [M; p. 292, Thm. 3.1.21], [M; p. 293, Thm. 3.1.22].

**§ 2. Continuity criteria for linear operators between normed spaces. Weakly comapact operators. 2.1.** A linear operator between normed spaces is continuous if and only if it is weak-to-weak continuous [M; p. 214, Thm. 2.5.11]; **2.2.** A linear operator between dual spaces of normed spaces is the adjoint of some continuous operator if and only if it is weak$^*$-to-weak$^*$ continuous [M; p. 287, Thm. 3.1.11]. **2.3.** Weakly compact operators, their first properties: compact operator are weakly compact [M; p. 340, Prop. 352], weakly compact operators are bounded [M; p. 340, Prop. 3.5.3], bounded linear operators from or to a reflexive space are weakly compact [M; p. 340, Prop. 3.5.4]; the product of a weakly compact operator and a bounded linear operator is weakly compact [M; p. 342, Prop. 3.5.11]; the Gantmacher–Nakamura Theorem [M; p. 341, Thm. 3.5.8, and p. 343, Thm.-s 3.5.13 and 3.5.14], weakly compact operators form a closed linear subspace in the space of all bounded linear operators [M; p. 342, Prop. 3.5.9 and Cor. 3.5.10].

## Chapter III. Differentiability of functions on subsets of normed spaces

Gâteaux and Fréchet differentiability of an operator on an open subset of a normed space [OO; § VI.1–5]; Derivatives of higher order, Taylor's formula(e) [OO; § VI.8].

## TOPICS FOR THE ORAL PART OF THE EXAM

**1.** Properties of the Minkowski functional. The vector space version of the Hahn–Banach Extension Theorem.

**2.** Separation of convex sets (by closed real hyperplanes) in a TVS: Mazur's, Eidelheit's, and the Tukey–Klee separation theorems.

**3.** The weak topology $\sigma(X, \Gamma)$ on a vector space $X$, where $\Gamma$ is a linear subspace of the algebraic dual $X^\sharp$; THE weak topology of a normed space (without metrisability properties).

**4.** The weak$^*$ topology of the dual space of a normed space (without metrisability properties).

**5.** Metrisability properties of the weak and the weak$^*$ topologies.

**6.** Characterisations of reflexivity (in terms of the existence of weakly convergent subsequences of bounded sequences, and in terms of nested sequences of non-empty closed bounded convex sets; James's sequential characterisations of reflexivity; James's theorem (w.p.)); "separable determinedness" of reflexivity.

**7.** The Eberlein–Šmulian Theorem (together with Day's Lemma).

**8.** Mazur's Compactness Theorem, the Krein–Šmulian Weak Compactness Theorem, and the weak$^*$ compactness of the weak$^*$ closed convex hull of a weak$^*$ compact set.

**9.** The Krein–Milman Theorem, Choquet's Lemma, and Bourgain's Lemma.

**10.** Barycenters of regular Borel probability measures on compact subsets of Hausdorff LCS (the material of subsections I.10.3–4).

**11.** Duality theorems for bounded linear operators and their adjoints.

**12.** Continuity criteria for linear operators between normed spaces. Weakly comapact operators.

**13.** Derivatives of (an operator on an open subset of a normed space) of higher order, Taylor's formula(e).