Order structures. A relation $\le$ on a set X is a ***preorder*** if it is

(1) reflexive: $x \le x$,
(2) transitive: $x \le y$ and $y \le z \implies x \le z$.

A preordered set X is ***directed downwards*** if

(3) for all x, y there exists z such that $z \le x, y$

and it is ***directed upwards*** if it is directed downwards for the inverse relation $\ge$.

A preorder is a ***partial order*** if it is

(4) antisymmetric: $x \le y$ and $y \le x \implies x = y$.

A partially ordered set is ***linearly ordered*** if $x \le y$ or $y \le x$ for all x, y. A subset $E \subset X$ is ***bounded from above*** if for some $b \in X$ all $e \in E$ satisfy $e \le x$. An element $m \in E$ is called a ***maximal element*** of E if $e \in E$ and $m \le e$ imply $m = e$.

Zorn's lemma. If every linearly ordered subset of a partially ordered set X is bounded from above, then X contains a maximal element.

Vector spaces. A vector space over a field $\mathbb{K}$ ($\mathbb{K}$ is) is a set X equipped with addition $+ : X \times X \to X$ and scalar multiplication $\cdot : \mathbb{K} \times X \to X$ operations such that $(X, +)$ is an Abelian group:

- associative: $x + (y + z) = (x + y) + z$,
- commutative: $x + y = y + x$,
- there exists $0 \in X$ such that $0 + x = x$,
- every $x \in X$ has the inverse $-x \in X$ such that $-x + x = 0$,

and multiplication is compatible with addition:

- $(\lambda\mu)x = \lambda(\mu x)$,
- $(\lambda + \mu)x = \lambda x + \mu x$,
- $\lambda(x + y) = \lambda x + \lambda y$,
- $1x = x$.

We consider $\mathbb{K}$ to be either $\mathbb{R}$ or $\mathbb{C}$ and then talk about real or complex vector spaces. Given $x \in X$, $\lambda \in \mathbb{K}$, $\Lambda \subset \mathbb{K}$, and $E, G \subset X$, we will use the following notation: $E + G = \{e + g \mid e \in E,\ g \in G\}$, $x + G = \{x\} + G$, $\Lambda E = \{\lambda e \mid \lambda \in \Lambda,\ e \in E\}$, and $\lambda E = \{\lambda\}E$.

Vector subspaces and linear span. A subset $Y \subset X$ in a vector space X is a ***subspace*** if $\mathbb{K}Y + Y \subset Y$. The minimal subspace containing a given subset $E \subset X$ is called a ***linear span*** of E and denoted $\operatorname{span} E$. The linear span of E is just the collection of all linear combinations of elements in E:

$$\operatorname{span} E = \bigcup_{n=1}^{\infty} \sum_{i=1}^{n} \mathbb{K}E = \left\{ \sum_{i=1}^{n} \lambda_i x_i \mid \lambda_i \in \mathbb{K},\ x_i \in E,\ n \in \mathbb{N} \right\}.$$

We denote $\langle x \rangle := \operatorname{span}\{x\}$ for any $x \in X$.

Linear independence and basis. A subset $E \subset X$ is ***linearly independent*** if for any finite set $\{x_1, \dots, x_n\} \subset E$ from $\sum_{i=1}^n \lambda_i x_i = 0$ it follows that $\lambda_i = 0$ for all $i = 1, \dots, n$. A linearly independent set $E \subset X$ such that $\operatorname{span} E = X$ is called a ***basis*** of X.

Linear maps. A map (or an operator) $A : X \to Y$ between vector spaces X and Y is ***linear*** if $A(\lambda x + y) = \lambda A(x) + A(y)$ for all $(\lambda, x, y) \in \mathbb{K} \times X \times X$. If a linear operator A is bijective, then A^{-1} is linear, too. A ***linear functional*** is linear operator from X to $\mathbb{K}$.

Bilinear maps. Given vector spaces X, Y, Z over $\mathbb{K}$, a map $B : X \times Y \to Z$ is ***bilinear*** if the maps $B_x : Y \to \mathbb{K}$, $y \mapsto B(x, y)$, and $B_y : X \to \mathbb{K}$, $y \mapsto B(x, y)$, are linear.

Set-valued maps. In general, given sets X, Y, Z and an operation $* : X \times Y \to Z$ let us denote by $(*) : 2^X \times 2^Y \to 2^Z$ the induced elementwise operation defined on subsets of these sets: $A(*)B = \{a * b \mid a \in A,\ b \in B\}$, with shorthands $a(*)B = \{a\}(*)B$ and $A(*)b = A(*)\{b\}$. Sometimes, when the context is clear (as above with operations in a vector space), we will omit the brackets. A special notation is $\mathscr{F}|_A := \mathscr{F}(\cap)A = \{F \cap A \mid F \in \mathscr{F}\}$ for any $\mathscr{F} \subset 2^X$ and $A \subset X$.

We switch the first two subsections, because the original first section seems to benefit from using filters as well.

1.1. **Filters and nets.** Let X be a set.

A system $\mathscr{B} \subset 2^X$ is a ***prefilter*** or a ***filter base*** if

(1) $\emptyset \notin \mathscr{B}$,
(2) $A, B \in \mathscr{B} \implies \exists C \in \mathscr{B} \quad C \subset A \cap B$ (i.e., the poset $(\mathscr{B}, \subset)$ is directed downwards).

A system $\mathscr{F} \subset 2^X$ is a ***filter*** if

(1) $\emptyset \notin \mathscr{F}$,
(2) $A, B \in \mathscr{F} \implies A \cap B \in \mathscr{F}$,
(3) $A \supset B \in \mathscr{F} \implies A \in \mathscr{F}$.

Denote $\mathscr{A}^{\uparrow} := \{S \subset X \mid \exists A \in \mathscr{A} \ A \subset S\}$ for any $\mathscr{A} \subset 2^X$. Condition (3) means that $\mathscr{F}^{\uparrow} = \mathscr{F}$. Clearly, a filter is exactly a prefilter satisfying this additional condition. And a system $\mathscr{B} \subset 2^X$ is a prefilter if and only if $\mathscr{B}^{\uparrow}$ is a filter (then $\mathscr{B}$ is called a filter base of $\mathscr{B}^{\uparrow}$).

Given two prefilters $\mathscr{B}_1$ and $\mathscr{B}_2$, let us denote $\mathscr{B}_1 \le \mathscr{B}_2$ if $\mathscr{B}_1^{\uparrow} \subset \mathscr{B}_2^{\uparrow}$, this means that for every $A \in \mathscr{B}_1$ there is $B \in \mathscr{B}_2$ such that $B \subset A$.

Definition. A ***net*** $(x_\gamma)_{\gamma \in \Gamma}$ is a mapping $\gamma \mapsto x_\gamma$ from a non-empty directed set $(\Gamma, \succ)$ to X.

Example. Given a net $(x_\gamma)_{\gamma \in \Gamma} \subset X$, its ***tails*** or ***eventuality*** prefilter is $\mathscr{B}_{(x_\gamma)} := \{\{x_\beta : \beta \succ \alpha\} \mid \alpha \in \Gamma\}$ and its ***eventuality*** filter is $\mathscr{F}_{(x_\gamma)} := \mathscr{B}_{(x_\gamma)}{}^{\uparrow}$.

Note that given a filter $\mathscr{F}$, we can always construct a net $(x_F)_{F \in \mathscr{F}} \in \mathscr{F}$ for which it is the eventuality filter. (Just take the set of pairs $\{(F, x) \mid F \in \mathscr{F},\ x \in F\}$ directed by the first coordinate.)

Definition. Let $A \subset X$. A prefilter $\mathscr{B}$ on X is

- ***eventually*** in A if $A \in \mathscr{B}^{\uparrow}$ (i.e., $F \subset A$ for some $F \in \mathscr{B}$),
- ***frequently*** in A if $A \cap F \neq \emptyset$ for all $F \in \mathscr{B}$ (in short, $\emptyset \notin \mathscr{B}|_A$).

The system $\mathscr{F}^{\#}$ of all sets, where $\mathscr{F}$ is frequent is called the ***grill*** of $\mathscr{F}$.

A net is eventually or frequently in A if its eventuality filter is so.

Lemma 1.2. *Let (x_γ) be a net. There exists a system $\mathscr{C} \subset \mathscr{P}(X)$ such that*

(1) *(x_γ) is frequent in all $A \in \mathscr{C}$,*
(2) *$A, B \in \mathscr{C} \implies A \cap B \in C$,*
(3) *for any $A \subset X$ either $A \in \mathscr{C}$ or $X \setminus A \in \mathscr{C}$.*

Q?: How do you translate this lemma to the language of filters?

Definition. A net $(y_\beta)_{\beta \in \Delta} \subset X$ is called a ***subnet*** of $(x_\gamma)_{\gamma \in \Gamma} \subset X$ if $\mathscr{F}_{(x_\gamma)} \subset \mathscr{F}_{(y_\beta)}$. In other words, taking a ***subnet*** corresponds to taking a ***superfilter***.

Q?: Prove that the latter definition is equivalent to

$$\forall \gamma \in \Gamma\ \exists \beta \in \Delta\ \forall \beta' \ge \beta\ \exists \gamma' \ge \gamma\ :\ y_{\beta'} = x_{\gamma'}.$$

Lemma 1.3. *Let (x_γ) be a net and let $\mathscr{A} \subset \mathscr{P}(X)$ be such a system that*

(1) *(x_γ) is frequent in all $A \in \mathscr{A}$,*
(2) *$A, B \in \mathscr{A} \implies \exists C \quad C \subset A \cap B$.*

Then there is exists a subnet (y_β) of (x_γ), which is eventually in A for all $A \in \mathscr{A}$.

Q?: Again, please translate this lemma to the language of filters.

Definition. A filter $\mathscr{F}$ on X is called an ***ultrafilter*** if it is a maximal filter (i.e., it is not contained in any different filter $\mathscr{G}$ on X). In other words, for any $A \subset X$ either $A \in \mathscr{F}$ or $X \setminus A \in \mathscr{F}$.

An ***ultranet*** is a net, whose eventuality filter is an ultrafilter.

The two lemmas above essentially amount to proving the "ultrafilter theorem" (which can also be proven directly).

Proposition 1.4 (Ultrafilter theorem)**.** *Every filter is contained in an ultrafilter.*

Or, in the language of nets: every net contains a subnet, which is an ultranet.

1.2. **Basics of general topology.**

Topological space. Given a set X, a system $\tau \subset 2^X := \{E \mid E \subset X\}$ is a ***topology*** on X if:

(T1) $\emptyset, X \in \tau$,
(T2) $\mathscr{G} \subset \tau \implies \bigcup \mathscr{G} := \bigcup_{G \in \mathscr{G}} G \in \tau$,
(T3) $F, G \in \tau \implies F \cap G \in \tau$.

A set X equipped with a topology is a ***topological space***, the sets in τ are ***open*** sets.

Comparing topologies. Given two topologies τ_1, τ_2 such that $\tau_1 \subset \tau_2$, τ_1 is called ***weaker*** and τ_2 ***stronger***. The weakest topology on X is the antidiscrete topology $\{\emptyset, X\}$, and the strongest is the discrete topology 2^X.

Subspace of a TS. Given a subset $X_0 \subset X$ in a TS X equipped with a topology τ, it becomes a TS itself if equipped with the subspace topology $\tau_0 = \tau|_{X_0} := \{X_0 \cap G \mid G \in \tau\}$, then (X_0, τ_0) is a ***subspace*** of (X, τ).

Interior points and neighbourhoods. If for some $x \in X$ and $E \subset X$ there is $G \in \tau$ such that $x \in G \subset E$, then x is an ***interior point*** of E and E is a ***neigbourhood*** of x.

Neighbourhood filter and bases. The set $\mathscr{N}_x = \mathscr{N}_x^\tau$ of all neighbourhoods of $x \in X$ (for a topology τ) is a filter (called the ***neighbourhood filter*** at x). Any its filter base $\mathscr{B}_x$ is called a ***neighbourhood base*** of x.

Theorem 1.1. *Let X be a set and fix a non-empty system $\mathscr{B}_x \subset 2^X$ for every point $x \in X$. The systems $\mathscr{B}_x$, $x \in X$, are neighbourhood bases for some topology τ on X if and only if*

(B1) *$x \in V$ for all $V \in \mathscr{B}_x$,*
(B2) *$A, B \in \mathscr{B}_x \implies \exists C \in \mathscr{B}_x \quad C \subset A \cap B$ (this together with (B1) means that $\mathscr{B}_x$ is a filter base),*
(B3) *for every $V \in \mathscr{B}_x$ there is $V' \in \mathscr{B}_x$ such that $V' \subset V$ and for all $y \in V'$ there exists $W \in \mathscr{B}_y$ such that $W \subset V'$ (in short, the condition on V' is: $V' \in \mathscr{B}_y^\uparrow$ for every $y \in V'$).*

Fix systems of neighbourhood bases $\{\mathscr{B}_x^\tau\}_{x \in X}$ and $\{\mathscr{B}_x^{\tau'}\}_{x \in X}$ for topologies τ and τ' on X. Then clearly, $\tau \subset \tau'$ if and only if $\mathscr{N}_x^\tau \subset \mathscr{N}_x^{\tau'}$ for all $x \in X$ if and only if $\mathscr{B}_x^\tau \le \mathscr{B}_x^{\tau'}$ for all $x \in X$.

Let (X, τ) be a TS, $Y \subset X$, and $y \in Y$. In the subspace topology $\tau|_Y$ the neighbourhood filter at y is exactly $\mathscr{N}_y^\tau|_Y = \{Y \cap U \mid U \in \mathscr{N}_y^\tau\}$, and $\mathscr{B}_y^\tau|_Y$ is its base whenever $\mathscr{B}_y^\tau$ is a neighbourhood base at y for the topology τ.

Set closure and closed sets. A point $x \in X$ is a ***limit point*** of $E \subset X$ if $\mathscr{N}_x$ (or any its base) is frequently in E: that is, $E \cap U \neq \emptyset$ for all $U \in \mathscr{N}_x$. The ***closure*** of $E \subset X$ is the collection of all its limit points, denoted by $\overline{E}$. Some properties of the closure (prove them!)✠:

(1) $E \subset \overline{E}$, $\overline{\overline{E}} = \overline{E}$,
(2) $E_1 \subset E_2 \implies \overline{E}_1 \subset \overline{E}_2$,
(3) $\overline{E_1 \cup E_2} = \overline{E}_1 \cup \overline{E}_2$.

The set E is called ***closed*** if $E = \overline{E}$. Clearly, E is closed if and only if $X \setminus E$ is open.

Continuous maps. Let (X, τ_X) and (Y, τ_Y) be topological spaces. A map $f : X \to Y$ is ***continuous at*** $x \in X$ if for every $U \in \mathscr{N}_{f(x)}$ there is $V \in \mathscr{N}_x$ such that $f(V) \subset U$ (in short, $\mathscr{N}_{f(x)} \subset f(\mathscr{N}_x)^\uparrow$, where $f(\mathscr{F}) = \{f(F) \mid F \in \mathscr{F}\}$ for any system $\mathscr{F} \subset 2^X$).

The function f is ***continuous*** (that, is continuous at every point $x \in X$) if and only if any of the following holds

(1) $G \in \tau_Y \implies f^{-1}(G) \in \tau_X$,
(2) if F is closed in Y, then so is $f^{-1}(F)$ in X.

A continuous bijective map $f : X \to Y$ having a continuous inverse f^{-1} is called a ***homeomorphism*** or an ***isomorphism***. The topological spaces X and Y are then called ***homeomorphic*** or ***isomorphic***.

Products of topological spaces. If X and Y are topological spaces, one can define a topology (called the ***product topology***) on $X \times Y$ by providing neighbourhood bases for each point $w = (x, y) \in X \times Y$ as follows: $\mathscr{B}_{(x,y)} = \mathscr{B}_x(\times)\mathscr{B}_y = \{U \times V \mid U \in \mathscr{B}_x,\ V \in \mathscr{B}_y\}$, where $\mathscr{B}_x$ and $\mathscr{B}_y$ are some bases of $\mathscr{N}_x$ and $\mathscr{N}_y$, respectively. It is easy to check that this system of neighbourhood bases satisfies conditions (B1)-(B3) of Theorem 1.1.

Clearly, a map $f : X \times Y \to Z$ is continuous at (x, y) if and only if for all $W \in \mathscr{N}_{f(x,y)}$ there are $U \in \mathscr{N}_x$ and $V \in \mathscr{N}_y$ such that $f(U \times V) \subset W$. Note that then the functions $f_x : Y \to Z$, $u \mapsto f(x, u)$, and $f_y : X \to Z$, $v \mapsto f(v, y)$, are continuous at y and x, respectively.

Convergence in topological spaces.

Definition. A filter $\mathscr{F}$ ***converges*** to $x \in X$ or $\mathscr{F} \to x$ if $\mathscr{N}_x \subset \mathscr{F}$. A net (x_γ) ***converges*** to x if so does its eventuality filter $\mathscr{F}_{(x_\gamma)}$:

$$x_\gamma \to x \iff \mathscr{F}_{(x_\gamma)} \to x \iff \mathscr{N}_x \subset \mathscr{F}_{(x_\gamma)}.$$

So a filter (or a net) converge to x if they are eventually in every neighbourhood of x.
Let us also say that a prefilter $\mathscr{B}$ ***converges*** to x and write $\mathscr{B} \to x$ if the generated filter $\mathscr{B}^\uparrow$ does so.

Note that given this definition, it is immediate that a subnet of a converging net converges to the same point.

Cluster points.

Definition. A point $x \in X$ is a ***cluster point*** of a filter $\mathscr{F}$ (or a net (x_γ) having $\mathscr{F}$ as the eventuality filter) when $\mathscr{F}$ is frequently in every neighbourhood of x. It is written $\mathscr{F} \rightsquigarrow x$ (or $(x_\gamma) \rightsquigarrow x$).

In short, $\mathscr{F} \rightsquigarrow x \iff \emptyset \notin \mathscr{N}_x(\cap)\mathscr{F}$. Note that then the latter system is a prefilter. So we have

Proposition 1.5. *A point x is a cluster point of a filter $\mathscr{F}$ (or a net (x_γ)) iff there is a superfilter $\mathscr{G} \supset \mathscr{F}$ (or a subnet (y_β) of (x_γ)) converging to x.*

Proposition 1.6. *An ultrafilter (or an ultranet) converges iff it has a cluster point.*

Hausdorff spaces and other separation axioms. A topological space is ***Hausdorff*** (or ***separated***, or T_2) if for all distinct points $x, y \in X$ $(x \neq y)$ there are $U \in \mathscr{N}_x$ and $V \in \mathscr{N}_y$ such that $U \cap V = \emptyset$.

A bit weaker condition is T_1: a topological space is T_1 if for all distinct points $x, y \in X$ there is $U \in \mathscr{N}_x$ such that $y \notin U$.

Proposition 1.7. *A topological space is Hausdorff if and only if every converging filter (or net) has just one limit.*

Note that a topological space X is T_1 if and only if the latter holds just for all neighbourhood filters. Another equivalent condition is: $\bigcap \mathscr{N}_x = \{x\}$ for all $x \in X$ (here $\bigcap \mathscr{A}$ means $\bigcap_{A \in \mathscr{A}} A$).

Describing closed sets and continuous functions via convergence. Let us say that a filter $\mathscr{F}$ is a filter on a subset $E \subset X$ if $E \in \mathscr{F}$.

Proposition 1.8.

(a) *A point x is a limit point of a set $E \iff$ there is a net $(x_\gamma) \subset E$ such that $x_\gamma \to x \iff$ there is a filter $\mathscr{F}$ such that $E \in \mathscr{F}$ and $\mathscr{F} \to x$.*

(b) *A set E is closed $\iff$ it contains all limits of all its converging nets $\iff$ it contains all limits of all converging filters on it.*

(c) *A map $f : X \to Y$ is continuous if and only if any or each of the following holds:*
- *$x_\gamma \to x \implies f(x_\gamma) \to f(x)$ for all converging nets $(x_\gamma) \subset X$,*
- *$\mathscr{F} \to x \implies f(\mathscr{F}) \to f(x)$ for all converging filters $\mathscr{F}$ on X (note that $f(\mathscr{F}) = \{f(F) \mid F \in \mathscr{F}\}$ is a prefilter when $\mathscr{F}$ is a filter),*
- *$f(\mathscr{N}_x) \to f(x)$ for all $x \in X$.*

1.3. **Compact topological spaces.** A system of subsets of X is called
- ***centered*** if any finite intersection of its sets is non-empty,
- ***cover*** if its union equals the whole X.

A topological space X is called ***compact*** if any its cover consisting of open sets contains a finite subcover.

It is easy to see that X is compact $\iff$ any centered system of closed sets has non-empty intersection.

Theorem 1.9. *The following are equivalent.*

(a) *X is compact,*
(b) *every net (or filter) has a cluster point,*
(c) *every net has a converging subnet (or every filter has a converging superfilter),*
(d) *every ultranet (or ultrafilter) converges.*

2. Topological vector spaces

2.1. **The notion of TVS.** Let $\mathbb{K}$ be either $\mathbb{R}$ or $\mathbb{C}$. A vector space X over the field $\mathbb{K}$, equipped with a topology τ, is called a ***topological vector space*** if the addition $T : X \times X \to X$, $(x, y) \mapsto x + y$, and the multiplication by a scalar $S : \mathbb{K} \times X \to X$, $(\lambda, x) \mapsto \lambda x$, are continuous.

Written out, this means:

- Continuity of addition:

$$\forall x, y \in X \; \forall W \in \mathcal{N}_{x+y} \; \exists U \in \mathcal{N}_x \; \exists V \in \mathcal{N}_y \; : \; U + V \subset W,$$

- Continuity of scalar multiplication:

$$\forall x \in X \; \forall \lambda \in \mathbb{K} \; \forall W \in \mathcal{N}_{\lambda x} \; \exists U \in \mathcal{N}_x \; \exists \delta > 0 \; : \; \overline{B}(\lambda, \delta) \cdot U \subset W,$$

where $\overline{B}(\lambda, \delta) = \{\mu \in \mathbb{K} \mid |\lambda - \mu| \le \delta\}$ is a closed ball in $\mathbb{K}$ (e.g., $\overline{B}(\lambda, \delta) = [\lambda - \delta, \lambda + \delta]$ if $\mathbb{K} = \mathbb{R}$).

Translation and homothety. Fixing a coordinate x_0 in the addition T above we get the ***translation*** mapping $T_{x_0} : X \to X$, $x \mapsto x + x_0$. Clearly, T_{x_0} is a continous linear map. Since its inverse is T_{-x_0}, also a continuous translation, T_{x_0} is a homeomorphism from X to itself. In particular, it maps open sets to open sets and closed sets to closed sets.

Similarly, fixing a scalar $\lambda_0 \neq 0$ in the multiplication S, we get the ***homothety*** mapping $S_{\lambda_0} : X \to X$, $x \mapsto \lambda_0 x$. Again, it is a linear homeomorphism with its inverse being S_{1/λ_0}.

Proposition 2.1. *If G is open (closed) subset of a TVS X, $x \in X$, and $\lambda \in \mathbb{K}$, $\lambda \neq 0$, then $\lambda G + x$ is also open (closed).*

In general, a linear homeomorphism between topological vector spaces is called an ***isomorphism*** of topological vector spaces.

Zero neighbourhood base.

Proposition 2.2. *If $\mathcal{B} := \mathcal{B}_0$ is a neighbourhood base at 0, then $x + \mathcal{B} := \{x + U \mid U \in \mathcal{B}\}$ is a neighbourhood base at x. In short, $(x + \mathcal{B})^{\uparrow} = \mathcal{N}_x$.*

In the following, $\mathcal{N} := \mathcal{N}_0$ will denote the zero neighbourhood filter and $\mathcal{B}$ will denote some of its filter base (that is, a zero neighbourhood base).

The two latter propositions imply several statements:

- $U \in \mathcal{N} \implies \lambda U \in \mathcal{N}$ for all $\lambda \neq 0$,
- $x_\gamma \to x \iff x_\gamma - x \to 0$,
- a linear map between TV spaces is continuous if and only if it is continuous at 0.

2.2. **Topologization of vector spaces.**
Absorbing and balanced sets.

Definition. A set E ***absorbs*** a set A if there is $\delta > 0$ such that $\overline{B}(0, \delta) \cdot A \subset E$. A set E is called ***absorbing*** if it absorbs every point $x \in X$ (that is, every one-point set $\{x\}$).

Definition. A set E is ***balanced*** if $\overline{B}(0, 1) \cdot E \subset E$.

- A non-empty balanced set contains 0 and is symmetric.
- If A and B are balanced, then so is $A + B$.
- If A and B are absorbing, then so is $A \cap B$.
- If A is balanced, then so is λA for any $\lambda \in \mathbb{K}$.
- If A is absorbing, then so is λA for any $\lambda \in \mathbb{K}$ such that $\lambda \neq 0$.
- A balanced set E absorbs a set A if and only if there is $\delta > 0$ such that $\delta A \subset E$.

Proposition 2.3. *In a TVS X:*

(1) *every $U \in \mathcal{N}$ is an absorbing set,*
(2) *every $U \in \mathcal{N}$ contains a balanced $V \in \mathcal{N}$.*

Corollary 2.4. *Any TVS has a zero neighbourhood base consisting of absorbing balanced sets.*

Theorem 2.5. *Every TVS has a zero neighbourhood base $\mathscr{B}$ consisting of absorbing balanced sets and such that (**NB4**): for every $U \in \mathscr{B}$ there is $V \in \mathscr{B}$ with $V + V \subset U$.*

Conversely, in a vector space X, any prefilter $\mathscr{B}$ consisting of absorbing balanced sets and satisfying property (NB4) defines a topology on X when taking a neighbourhood base of any point $x \in X$ to be

$$\mathscr{B}_x := x + \mathscr{B}.$$

With respect to this topology, X becomes a TVS.

Proposition 2.6. *Every TVS has a zero neighbourhood base consisting of closed balanced sets.*

Exercise 2.1. Prove that the closure of a balanced set in a TVS is balanced.

Exercise 2.2. Prove that the closure of a vector subspace in a TVS is a vector subspace.

Proposition 2.7. *A TVS is Hausdorff $\iff \bigcap \mathscr{B} = \{0\}$ for all (or some) zero neighbourhood bases $\mathscr{B}$. (In particular, a TVS is Hausdorff $\iff$ it is T_1.)*

Example 2.1. A normed space X is a TVS having $\{\frac{1}{n} B_X \mid n \in \mathbb{N}\}$ as one of its zero neighbourhood bases.

3. Bounded and compact sets in a TVS

3.1. Bounded and completely bounded sets.

Definition. A set $E \subset X$ is ***bounded*** if it is absorbed by every zero neighbourhood.

Definition. A set $E \subset X$ is ***completely bounded*** or ***precompact*** if for every $U \in \mathscr{N}$ there is a finite set $E_0 \subset X$ such that $E \subset E_0 + U$.

Note that E_0 can be chosen inside of E. For every $e \in E_0$ we can assume $E \cap (e+U) \neq \emptyset$ and choose $b_e \in E \cap (e+U)$. Then $e + U \subset b_e - U + U$. So if U is taken to be balanced and such that $U + U \subset V$ for a given $V \in \mathscr{N}$, then $E \subset \{b_e \mid e \in E_0\} + V$.

Let $\mathscr{S}$ denote the system of all bounded or all completely bounded sets in X. Then (Q?:)

- $A \subset B \in \mathscr{S} \implies A \in \mathscr{S}$,
- $A, B \in \mathscr{S} \implies A \cup B \in \mathscr{S}$,
- every finite set is in $\mathscr{S}$,
- $\lambda \in \mathbb{K}$, $A \in \mathscr{S} \implies \lambda A \in \mathscr{S}$,
- $A, B \in \mathscr{S} \implies A + B \in \mathscr{S}$.

A system $\mathscr{S}$ satisfying the first 3 of the above properties is called an ***ideal*** system of sets or a ***bornology*** on X. This is something that is dual to being a filter. In fact, the system of all complements $\{X \setminus A \mid A \in \mathscr{S}\}$ is a filter on X.

Note that every completely bounded set is bounded, because given a balanced $U \in \mathscr{N}$ and a finite (and thus bounded) set E_0 we can find $\mu > 1$ such that $E_0 \subset \mu U$, so $E_0 + U \subset \mu U + U \subset \mu U + \mu U = \mu(U + U)$.

Proposition 3.1. *If $\mathscr{S}$ is as above, then $A \in \mathscr{S} \implies \overline{A} \in \mathscr{S}$.*

Proof. For boundedness, given a closed and balanced $U \in \mathscr{N}$, μU is closed too. So $E \subset \mu U$ if and only if the same holds for $\overline{E}$.

For complete boundedness, given a closed $U \in \mathscr{N}$, find a finite E_0 such that $E \subset E_0 + U = \bigcup_{e \in E_0} e + U$ and note that the latter set is closed as a finite union of closed sets. □

The definition of bounded sets is similar to some kind of continuity. The next proposition says it precisely.

Proposition 3.2. *A set A is bounded if and only if for every $(\lambda_n) \subset \mathbb{K}$ such that $\lambda_n \to 0$ and every $(x_n) \subset A$, one has $\lambda_n x_n \to 0$.*

Exercise 3.1. Prove that $\lambda \overline{E} \subset \overline{\lambda E}$.

3.2. **Compact sets.**

Definition. A filter $\mathscr{F}$ is ***Cauchy*** if for every $U \in \mathscr{N}$ there exists $F \in \mathscr{F}$ such that $F - F \subset U$. A net is ***Cauchy*** if so is its eventuality filter.

A set $A \subset X$ is ***complete*** if every Cauchy filter on (or net in) A converges to some element of A.

Exercise. Define $\Delta\mathscr{F} = \{F - F \mid F \in \mathscr{F}\}$. Is it a filter or a prefilter? If it were, then we could say that $\mathscr{F}$ is Cauchy if and only if $\Delta\mathscr{F} \to 0$.

Proposition 3.3. *A compact set is completely bounded.*

Proof. Given a compact set A and an open $U \in \mathscr{N}$, note that $A + U = \bigcup_{a \in A} a + U$ is an open cover of A, so it contains a finite subcover $A_0 + U$, with A_0 finite. □

Proposition 3.4. *A compact set is complete.*

Proof. It is enough to show that a Cauchy filter $\mathscr{F}$ that clusters at x, converges to x: $\mathscr{F} \rightsquigarrow x \implies \mathscr{F} \to x$.

Given $V \in \mathscr{N}$, find $U \in \mathscr{N}$ with $U + U \subset V$ and $F \in \mathscr{F}$ with $F - F \subset U$. Since $\mathscr{F} \rightsquigarrow x$, we have that $G := (x + U) \cap F \neq \emptyset$, so $F \subset F - G + G \subset F - F + x + U \subset U + x + U \subset x + V$. □

The induction on the definition of ultrafilters implies

Lemma 3.5. *If $\mathscr{F}$ is an ultrafilter on $X = \bigcup_{i=1}^n X_n$, then $X_k \in \mathscr{F}$ for some $k \in \{1, \dots, n\}$.*

Lemma 3.6. *A set E is completely bounded if and only if for every $U \in \mathscr{N}$ there are $S_1, \dots, S_n$ such that $E = \bigcup_{i=1}^n S_i$ and $S_i - S_i \subset U$ for all i.*

Proof. Given $U \in \mathscr{N}$ find a balanced $V \in \mathscr{N}$ with $V + V \subset U$ and $\{x_1, \dots, x_n\}$ such that $E \subset \bigcup_{i=1}^n x_i + V$. Note that $x_i + V - (x_i + V) = V - V = V + V \subset U$. □

Theorem 3.7. *A set is compact if and only if it is complete and completely bounded.*

Proof of necessity. Observe that the two above lemmas say that every ultrafilter on a completely bounded set is Cauchy. Indeed, take a precompact set E and an ultrafilter $\mathscr{F}$ on E. Given $U \in \mathscr{N}$ and applying the above lemmas we get $S \subset E$ such that $S \in \mathscr{F}$ and $S - S \subset U$. □

4. Metrizable TVS

In a metric space (X, ρ) the sets $B(x, 1/n)$, $n \in \mathbb{N}$, form a neighbourhood base for any point $x \in X$. Thus, in a metrizable topological space every point has a countable neighbourhood base.

4.1. **Metrizable TVS.**

Definition. Let X be a vector space. A mapping $|\cdot| \to \mathbb{R}$, $x \mapsto |x|$, is a ***pseudonorm*** if

(1) $x = 0 \iff |x| = 0$,
(2) $|\lambda x| \le |x|$ if $|\lambda| \le 1$,
(3) $|x + y| \le |x| + |y|$.

Note that every norm is a pseudonorm. In turn, a pseudonorm induces a metric on X defined by $\rho(x, y) = |x - y|$, which is ***translation-invariant***: $d(x + z, y + z) = d(x, y)$.

Proposition 4.1. *If a Hausdorff TVS has a countable zero neighbourhood base, then its topology can be induced by a pseudonorm.*

Proof. We can assume that the zero neighbourhood base $\mathscr{B} = \{V_n \mid n \in \mathbb{N}\}$ consists of balanced sets such that $V_{n+1} + V_{n+1} \subset V_n$. Denote by $\mathbb{Q}_2 := \{\frac{m}{2^n} \mid m, n \in \mathbb{Z}\}$ the set of dyadic rationals and define $f : \mathbb{Q}_2 \cap (0, 1) \to \mathscr{P}(X)$ by

$$f\left(\sum_{k=1}^n \frac{1}{2^{j_k}}\right) = \sum_{k=1}^n V_{j_k}.$$

Note that $f(q_1) + f(q_2) \subset f(q_1 + q_2)$ whenever $q_1, q_2, q_1 + q_2 \in \mathbb{Q}_2 \cap (0, 1)$. Indeed, $f(q_1) + f(2^{-n})$ is either $f(q_1 + 2^- n)$ as needed or $f(q_1 - 2^{-n}) + f(2^{-n}) + f(2^{-n}) \subset f(q_1 - 2^{-n}) + f(2^{-n+1})$ with $q_1 - 2^{-n}$ having fewer primitive addends in it than q_1. Note that, in particular, $f(q_1) \subset f(q_2)$ whenever $0 < q_1 \le q_2 < 1$.

Let us define $|x| := \inf(\{q \in \mathbb{Q}_2 \cap (0, 1) \mid x \in f(q)\} \cup \{1\})$. (Then $|x| \le 1$ for all x.)

(1) Clearly, $|0| = 0$, and $|x| = 0$ means that $x \in f(q)$ with $q < 2^{-n}$ for every $n \in \mathbb{N}$, so that $x \in f(2^{-n}) = V_n$, and therefore $x \in \bigcap \mathscr{B} = \{0\}$.
(2) Note that $f(q)$ is balanced as a finite sum of balanced sets, therefore $\lambda x \in f(q)$ for $|\lambda| \le 1$ whenever $x \in f(q)$.
(3) We only have to consider the case $|x| + |y| < 1$. Note that

$$\{q \mid x \in f(q)\} + \{q \mid y \in f(q)\} = \{q_x + q_y \mid x \in f(q_x),\ y \in f(q_y)\} \subset \{q_x + q_y \mid x + y \in f(q_x + q_y)\} = \{q \mid x + y \in f(q)\},$$

therefore

$$|x| + |y| = \inf\big(\{q \mid x \in f(q)\} + \{q \mid y \in f(q)\}\big) \ge |x + y|.$$

This shows that $|\cdot|$ is indeed a pseudonorm. It remains to observe that

$$V_{n+1} \subset B\left(0, \frac{1}{2^n}\right) = \left\{x : |x| < \frac{1}{2^n}\right\} \subset V_n,$$

and therefore $y + V_{n+1} \subset B\left(y, \frac{1}{2^n}\right) \subset y + V_n$ for any $y \in X$, so that the neighbourhood filters at y are the same for the induced and the original TVS topology. □

Theorem 4.2. *The following are equivalent for a Hausdorff TVS:*
(1) *it has a countable zero neighbourhood base,*
(2) *the topology is induced by a pseudonorm,*
(3) *the topology is induced by a translation-invariant metric,*
(4) *the topology is metrizable.*

Proposition 4.3. *If two translation-invariant metrics on a vector spave both induce the same topology, which makes this vector space a TVS, then these metrics have the same Cauchy sequences and are simultaneously complete or incomplete.*

Proposition 4.4. *A metrizable TVS is complete if and only its topology is induced by a complete translation-invariant metric. In other words, a metrizable TVS is complete if and only if every Cauchy sequence converges.*

Definition. A TVS is ***locally bounded*** if there is a bounded $U \in \mathcal{N}$. A TVS is ***locally compact*** if there is a compact $U \in \mathcal{N}$.

Proposition 4.5. *Every locally bounded Hausdorff TVS is metrizable.*

Exercise 4.1. If $B \in \mathcal{N}$ is bounded, then $\{\frac{1}{n}B \mid n \in \mathbb{N}\}$ is a countable zero neighbourhood base.

4.2. **Finite-dimensional TVS.**

Proposition 4.6. *A Hausdorff TVS of dimension $n < \infty$ is isomorphic to m_n.*

Proof. Let $X = \operatorname{span}\{e_1, \dots, e_n\}$. Define $T : m_n \to X$ by $T((\lambda_i)) = \sum_{i=1}^n \lambda_i e_i$. From linear algebra we know that T is a bijective linear map. Since in m_n the convergence of a net of vectors imply the convergence of their corresponding coordinates, the continuity of addition and scalar multiplication imply that T is continuous.

In order to show the continuity of T^{-1}, it is enough to show it at 0: for every zero neighbourhood U in m_n its preimage $(T^{-1})^{-1}(U) = T(U)$ must be a zero neighbourhood of X. Since $\{\varepsilon B_{m_n} | \varepsilon \in (0,1)\}$ form a zero neighbourhood base in m_n and T is linear, it is enough to show that $T(B_{m_n})$ is a zero neighbourhood in X.

Consider the sphere $S_{m_n} = \{x \in m_n : \|x\| = 1\}$. It is a closed and bounded set in a finite-dimensional Banach space, so it is compact. Then its continuous image $T(S_{m_n}) \subset X$ is compact as well and hence closed. Since $0 \notin S_{m_n}$, so also $0 \notin T(S_{m_n})$. Therefore, $X \setminus T(S_{m_n})$ is an open zero neighbourhood. It contains a balanced zero neighbourhood V. If $V \not\subset T(B_{m_n})$ then there exists $y \in m_n$ such that $\|y\| > 1$ and $Ty \in V$. But then $\frac{1}{\|Ty\|} Ty \in V \cap T(S_{m_n})$, a contradiction. So $V \subset T(B_{m_n})$, as needed. □

Corollary 4.7. *A finite-dimensional subspace of a Hausdorff TVS is closed.*

Proof. The above proposition (recall that m_n is complete) and the exercises below give that a finite-dimensional subspace is complete and that a complete set is closed.

□

Exercise. Show that a TVS isomorphic to a complete TVS is complete, too.

Exercise. Show that a subspace of a TVS is complete in the induced topology if and only if it is complete as a subset.

Exercise. Show that a convergent filter is Cauchy.

Proposition 4.8. *A locally compact Hausdorff TVS X is finite-dimensional.*

Proof. Take a compact balanced $V \in \mathscr{N}$. It is bounded, so Exercise 4.1 gives that $\mathscr{N} = \{2^{-n}V \mid n \in \mathbb{N}\}^{\uparrow}$. Since V is completely bounded, $V \subset V_0 + \frac{1}{2}V$ for some finite set $V_0 \subset V$. Consider the finite-dimensional subspace $Y = \operatorname{span} V_0$. We have $V \subset Y + \frac{1}{2}V$. Dividing by 2, this gives $\frac{1}{2}V \subset Y + \frac{1}{4}V$. Adding the two inclusions together, we get $V \subset Y + Y + \frac{1}{4}V \subset Y + \frac{1}{4}V$. Thus by induction we have $V \subset \bigcap_{n\in\mathbb{N}}(Y + 2^{-n}V) = \overline{Y} = Y$. So $\operatorname{span} V \subset Y$. On the other hand $\operatorname{span} V = X$, because V is absorbing. □

4.3. **Examples.**

Example 4.1. Consider the vector space

$$C(\mathbb{C}) := \{x = x(t) \mid x : \mathbb{C} \to \mathbb{K} \text{ is continuous}\}$$

with addition and scalar multiplication defined pointwise: $(\lambda x + y)(t) = \lambda x(t) + y(t)$.

Let us denote $p_n(x) := \max_{|t|\le n} |x(t)|$ and $V_{n,\varepsilon} = \{x \mid p_n(x) < \varepsilon\}$ and show that $\mathscr{B} := \{V_{n,\varepsilon} \mid n \in \mathbb{N},\ \varepsilon > 0\}$ is an additive prefilter consisting of absorbing balanced sets:

(1) $V_{\max(n,m),\min(\varepsilon,\delta)} \subset V_{n,\varepsilon} \cap V_{m,\delta}$, so $\mathscr{B}$ is a prefilter.
(2) $p_n(\lambda x) = |\lambda| p_n(x)$, so $V_{n,\varepsilon}$ is balanced,
(3) for every $x \in X$, $p_n\left(\frac{\varepsilon}{p_n(x)+1}x\right) < \varepsilon$, so $V_{n,\varepsilon}$ is absorbing,
(4) $V_{n,\frac{\varepsilon}{2}} + V_{n,\frac{\varepsilon}{2}} \subset V_{n,\varepsilon}$, so $\mathscr{B}$ is additive.

Thus $\mathscr{B}$ is a zero neighbourhood base for some TVS topology. Since $\bigcap_{\varepsilon>0} V_{n,\varepsilon}$ clearly consists of functions, which are zero on $B_{\mathbb{K}}(0,n)$, we have $\bigcap\mathscr{B} = \{0\}$, so that the induced topology is Hausdorff. Since every $V_{n,\varepsilon}$ contains $V_{n,\frac{1}{i}}$ for some $i \in \mathbb{N}$, our base $\mathscr{B}$ has a countable generating subsystem $\{V_{n,\frac{1}{i}}\}_{n,i\in\mathbb{N}}$, so that the induced topology is metrizable.

Given a Cauchy sequence $(x_k) \subset C(\mathbb{C})$ let us denote $y_k := x_k|_{B_{\mathbb{K}}(0,n)}$ and observe that (y_k) is a Cauchy sequence in the Banach space $C(B_{\mathbb{K}}(0,n))$ equipped with the norm p_n, so it converges to the pointwise limit $y \in C(B_{\mathbb{K}}(0,n))$ in the norm. This means that the pointwise limit x of (x_n) is continuous on every $B_{\mathbb{K}}(0,n)$, hence continous on the whole $\mathbb{C}$, and so clearly $x_n \to x$ in $C(\mathbb{C})$.

So, $C(\mathbb{C})$ is a complete TVS.

Remark. Recall that given a compact Hausdorff topological space K, the space $C(K) \subset \mathbb{K}^K$ of all continuous functions on K is a Banach space, when equipped with the norm $\|f\| := \max_{t\in K}|f(t)|$. The completeness of the norm here follows, e.g., from the Arzelà-Ascoli theorem: a subset $A \subset C(K)$ is relatively compact if and only if it is bounded and equicontinuous.

5. Convex sets and seminorms

5.1. **Convex subsets in a vector space.** Let X be a vector space.....

Definition. Given $x, y \in X$ denote $[x,y] := \{\lambda x + (1-\lambda)y \mid \lambda \in [0,1]\}$. A subset $E \subset X$ is ***convex*** if $[x,y] \subset E$ for any two points $x, y \in E$. A balanced convex set is called ***absolutely convex***.

Proposition 5.1. *A subset $E \subset X$ is absolutely convex if*

$$\left\{\lambda x + \mu y \mid |\lambda| + |\mu| \le 1,\ \lambda, \mu \in \mathbb{K}\right\} \subset E$$

for any two points $x, y \in E$.

If we define the scalar product for vectors $\cdot : \mathbb{K}^n \times X^n \to X$ like this: $(\lambda_k)\cdot(x_k) = \sum_{k=1}^n \lambda_k x_k$, then the convexity and the absolute convexity of E mean respectively that $S^+_{\ell_1^2} \cdot E^2 \subset E$ and $\overline{B}_{\ell_1^2} \cdot E^2 \subset E$, where $\overline{B}_{\ell_1^n} = \{(\lambda_i) \in \mathbb{K}^n \mid \sum|\lambda_i| \le 1\}$ is the closed unit ball of the Banach space ℓ_1^n and $S^+_{\ell_1^n} = \{(\lambda_i) \in \mathbb{R}^n \mid \lambda \ge 0,\ \sum \lambda_i = 1\}$ is the positive part of its sphere.

Some properties:

- if E_α, $\alpha \in \Gamma$, are convex (absolutely convex), then so is $\bigcap_{\alpha\in\Gamma} E_\alpha$,

- given $\delta \in \mathbb{K}$ and a linear map $T : X \to Y$, if $E_1, E_2 \subset X$ are convex (absolutely convex), then so are $E_1 + E_2$, δE_1, and $T(E_1)$.

Lemma 5.2. *Let $E \subset X$.*

(1) *If E is convex then $S^+_{\ell_1^n} \cdot E^n \subset E$ for all $n \in \mathbb{N}$.*
(2) *If E is absolutely convex then $\overline{B}_{\ell_1^n} \cdot E^n \subset E$ for all $n \in \mathbb{N}$.*

Definition. Let $E \subset X$. Then

$$\operatorname{conv} E := \bigcup_{n=1}^{\infty} S^+_{\ell_1^n} \cdot E^n$$

is called the ***convex hull*** of E and

$$\operatorname{absconv} E := \bigcup_{n=1}^{\infty} \overline{B}_{\ell_1^n} \cdot E^n$$

is called the ***absolutely convex hull*** of E.

Example 5.1. Make sure that $E \subset \operatorname{conv} E \subset \operatorname{absconv} E \subset \operatorname{span} E$.

Let us note that $\operatorname{conv} E$ ($\operatorname{absconv} E$) is the minimal (absolutely) convex set containing E.

Example 5.2. Show that if $E \subset X$ is balanced, then so is $\operatorname{conv} E$ and hence $\operatorname{conv} E = \operatorname{absconv} E$.

Proposition 5.3. *Let X be a TVS. If $E \subset X$ is convex, then so is $\overline{E}$.*

Corollary 5.4. *If $E \subset X$ is absolutely convex, then so is $\overline{E}$.*

Let $E_1, \ldots, E_n \subset X$ be absolutely convex. Observe that

$$\operatorname{absconv} \bigcup_{i=1}^{n} E_i = \overline{B}_{\ell_1^n} \cdot (E_1 \times \cdots \times E_n).$$

Proposition 5.5. *If $E_1, \ldots, E_n \subset X$ are compact and absolutely convex sets, then so is* $\operatorname{absconv} \bigcup_{i=1}^{n} E_i$.

5.2. **Seminorms and Minkowski functionals.** Let X be a vector space.

Definition. A functional $p : X \to \mathbb{R}$ is

(1) ***positively homogeneous*** if $p(\lambda x) = \lambda p(x)$ for $\lambda \geq 0$,
(2) ***absolutely homogeneous*** if $p(\lambda x) = |\lambda| p(x)$ for $\lambda \in \mathbb{K}$,
(3) ***subadditive*** if $p(x + y) \leq p(x) + p(y)$,
(4) ***sublinear*** if it is positively homogeneous and subadditive,
(5) ***seminorm*** if it is absolutely homogeneous and subadditive.

Exercise 5.3. Prove that

(1) a sublinear functional p satisfies $p(0) = 0$ and $|p(x) - p(y)| \leq \max\{p(x - y), p(y - x)\}$,
(2) a seminorm p satisfies $p(x) \geq 0$ and $|p(x) - p(y)| \leq p(x - y)$,
(3) if p is a seminorm, then $p^{-1}(0) \subset X$ is a vector subspace of X.

Exercise 5.4. Prove that if p is a seminorm, then both the open and closed unit balls $B_p := p^{-1}([0, 1))$ and $\overline{B}_p := p^{-1}([0, 1])$ are absolutely convex and absorbing.

Definition. Let $U \subset X$ be an absorbing set. The functional $p_U : X \to \mathbb{R}$, $x \mapsto \inf\{\mu > 0 \mid x \in \mu U\}$, is called its ***Minkowski functional*** or ***gauge***.

Exercise 5.5. Prove that $p_U(0) = 0$ and $0 \leq p_U(x) < \infty$.

Proposition 5.6. *Let $U \subset X$ be an absorbing set. Then*

(1) *p_U is positively homogeneous,*
(2) *if U is balanced, then p_U is absolutely homogeneous,*
(3) *if U is convex, then p_U is subadditive and $B_{p_U} \subset U \subset \overline{B}_{p_U}$.*

Corollary 5.7. *The gauge of a convex absorbing set is a positive sublinear functional. The gauge of an absolutely convex absorbing set is a seminorm.*

Proposition 5.8. *Let $p : X \to \mathbb{R}$ be a seminorm and $U := \overline{B}_p$. Then $p_U = p$.*

Let X be a TVS.

Lemma 5.9. *If a sublinear functional $p : X \to \mathbb{R}$ is continuous at 0, then it is continuous everywhere.*

Theorem 5.10. *An absorbing set $U \subset X$ is a zero neighbourhood if and only if p_U is continuous. If U is open, then $U = B_{p_U}$. If U is closed, then $U = \overline{B}_{p_U}$.*

6. Hahn-Banach theorem

6.1. **Hahn-Banach theorem for real vector spaces.** Recall that the ***algebraic dual*** of a vector space X is the vector space X^* of all linear functionals $f : X \to \mathbb{K}$ with vector space operations defined pointwise: $(\lambda f_1 + f_2) = \lambda f_1(x) + f_2(x)$.

Let X be a TVS.

Proposition 6.1. *A linear functional $f \in X^*$ is continuous if and only if there is $U \in \mathcal{N}$ such that $f(U) \subset \mathbb{K}$ is bounded, that is, $f(U) \subset B_{\mathbb{K}}(0, M)$ for some $M > 0$.*

Proof. It is enough to consider continuity at 0.

($\Rightarrow$): By continuity, since $B_{\mathbb{K}}(0,1)$ is a zero neighbourhood in $\mathbb{K}$, there is $U \in \mathcal{N}$ such that $f(U) \subset B_{\mathbb{K}}(0,1)$.

($\Leftarrow$): We have $f(\frac{\varepsilon}{M}U) = \frac{\varepsilon}{M} f(U) \subset B_{\mathbb{K}}(0,\varepsilon)$ for all $\varepsilon > 0$ with the latter sets forming a zero neighbourhood base in $\mathbb{K}$ and $\frac{\varepsilon}{M}U \in \mathcal{N}$ for all $\varepsilon > 0$. □

The ***topological dual*** of X is the vector subspace X' of X^* which consists of continuous functionals.

Let Y be a subspace of a vector space X and let $f \in X^*$. Define $f|_Y \in Y^*$ by $f|_Y(y) = f(y)$ for all $y \in Y$. Then f is called an ***extension*** of $f|_Y$ to X, and $f|_Y$ is called the ***resriction*** of f to Y.

Theorem 6.2 (Hahn-Banach for $\mathbb{K} = \mathbb{R}$)**.** *Let X be a real vector space and $X_0 \subset X$ its subspace. Let $p : X \to \mathbb{R}$ be a sublinear functional. If $f_0 \in X_0^*$ satisfies $f_0(x) \le p(x)$ for all $x \in X_0$, then there exists an extension $f \in X^*$ of f_0 such that $f(x) \le p(x)$ for all $x \in X$.*

Proof. Consider a partially ordered set S of pairs (Y, g) where $Y \subset X$ is a subspace containing X_0 and $g \in Y^*$ is an extension of f_0 to Y such that $g(y) \le p(y)$ for all $y \in Y$, the order given by $(Y_1, y_1) \le (Y_2, y_2)$ if $Y_1 \subset Y_2$ and $g_2|_{Y_1} = g_1$.

Zorn's lemma gives the claim once we show that

1. if $Y \ne X$, then (Y, g) is not maximal,
2. a linearly ordered subset of S has an upper bound.

(1) Take $z \in X \setminus Y$ and note that

$$g(x) + g(y) = g(x + y) \le p(x + y) \le p(x - z) + p(y + z)$$

or

$$g(x) - p(x - z) \le p(y + z) - g(y)$$

for all $x, y \in Y$. So, $A := \sup_{x\in Y}(g(x) - p(z + x)) \le \inf_{x\in Y}(p(x - z) - g(x)) =: B$. Take any $t \in [A, B]$ and define $f : Y + \mathbb{R}z \to \mathbb{R}$ by $f(y + \lambda z) = g(y) + \lambda t$. It is easy to see that $Y + \mathbb{R}z$ is a subspace of X and that f is correctly defined. The inequality $f(y + \lambda z) \le p(y + \lambda z)$ follows from the choice of t.

(2) Given a linearly ordered set $\{(Y_\alpha, g_\alpha)\}_\alpha \subset S$, define $Y = \bigcup_\alpha Y_\alpha$ and $g : Y \to \mathbb{R}$ by $g(y) = g_\alpha(y)$ for any α such that $y \in Y_\alpha$. Again it is easy to see that g is correctly defined, linear, and $g(y) \le p(y)$ for all $y \in Y$, so that $(Y, g) \in S$ is the required upper bound. □

Corollary 6.3. *Let X be a real vector space. For any sublinear functional $p : X \to \mathbb{R}$ there exists $f \in X^*$ such that $f(x) \le p(x)$ for all $x \in X$.*

Note that the above corollary is non-trivial only in the case, when the sublinear functional has some negative values.

6.2. **Hahn-Banach theorem for complex vector spaces.** Given a vector space X over $\mathbb{C}$, consider the restriction of the scalar product $\cdot : \mathbb{C} \times X \to X$ to $\mathbb{R} \times X$. The set X equipped with this restriction and the unchanged addition becomes a real vector space, which we denote by $X_{\mathbb{R}}$.

Theorem 6.4 (Hahn-Banach for $\mathbb{K} = \mathbb{C}$)**.** *Let X be a vector space over $\mathbb{K}$ ($\mathbb{K}$ is either $\mathbb{R}$ or $\mathbb{C}$) and let $p : X \to \mathbb{R}$ be a seminorm. If a linear functional f_0, defined on a subspace $X_0 \subset X$, satisfies $|f_0(x)| \le p(x)$ for all $x \in X_0$, then it has an extension $f \in X^*$ such that $|f(x)| \le p(x)$ for all $x \in X$.*

Proof. ($\mathbb{K} = \mathbb{R}$): Applying Theorem 6.2 we get an extension $f \in X^*$ such that $f(x) \le p(x)$ for all $x \in X$. The claim follows, because $-f(x) = f(-x) \le p(-x) = p(x)$, too.

($\mathbb{K} = \mathbb{C}$): Denote $\phi_0(x) = \operatorname{Re} f_0(x)$, then $\phi_0 \in (X_0)^*_{\mathbb{R}}$ and $f_0(x) = \phi_0(x) - i\phi_0(ix)$. Also, $|\phi_0(x)| \le |f_0(x)| \le p(x)$. So applying the case $\mathbb{K} = \mathbb{R}$, we get $\phi \in X^*_{\mathbb{R}}$ such that $|\phi(x)| \le p(x)$ for all $x \in X$. Then $f(x) := \phi(x) - i\phi(ix)$ defines a linear functional $f \in X^*$. If $f(x) \ne 0$, then $f\left(\frac{|f(x)|}{f(x)} x\right) = |f(x)| > 0$, so that

$$|f(x)| = f\left(\frac{|f(x)|}{f(x)} x\right) = \phi\left(\frac{|f(x)|}{f(x)} x\right) \le p\left(\frac{|f(x)|}{f(x)} x\right) = \left|\frac{|f(x)|}{f(x)}\right| p(x) = p(x).$$

□

6.3. **Separation theorems.** Let $g : X \to \mathbb{R}$ be a mapping. Let us denote $[g = \alpha] = \{x \in X \mid g(x) = \alpha\}$, $[g \le \alpha] = \{x \in X \mid g(x) \le \alpha\}$, and in the same manner also $[g > \alpha]$, $[g \ge \alpha]$, and so on.

Let X be a real vector space and $f \in X^* \setminus \{0\}$. A set $H := [f = \alpha]$ is called a ***hyperplane***. Note that $H_0 := [f = 0] \ne X$ and so $H = z + H_0$ for any $z \in H$. Every such hyperplane yields corresponding ***half-spaces*** $[f \ge \alpha]$, $[f \le \alpha]$ and ***strict half-spaces*** $[f > \alpha]$, $[f < \alpha]$. Both a hyperplane and a functional defining it are said to ***(strictly) separate*** two subsets of X if these subsets reside in different (strict) half-spaces, corresponding to the hyperplane.

Proposition 6.5. *In a real TVS X a hyperplane $H = [f = \alpha]$ is either closed ($\overline{H} = H$) or dense ($\overline{H} = X$). It is closed if and only if f is continuous.*

Proof. Clearly, H is closed if f is continuous. If f is discontinuous, then (prove it!)✠ there is a net $(x_\alpha) \subset X$ such that $x_\alpha \to 0$ but $f(x_\alpha) = 1$. Now $y_\alpha := x - f(x)x_\alpha \to x$ and $(y_\alpha) \subset H_0 = [f = 0]$ for any $x \in X$. So $\overline{H_0} = X$ but then also $\overline{H} = \overline{z + H_0} = z + \overline{H_0} = X$ given any $z \in H$. □

Let X be a TVS.

Lemma 6.6. *Every $f \in X^* \setminus \{0\}$ is an* **open mapping**, *that is, it maps open sets to open sets.*

Proof. Find $x_0 \in X$ such that $f(x_0) = 1$. Take a non-empty open $G \subset X$ and $x \in G$. Then $G - x \in \mathcal{N}$, so it absorbs x_0, hence there exists $\varepsilon > 0$ such that $B_{\mathbb{K}}(0, \varepsilon) \cdot x_0 \subset G - x$. Applying f, we get $B_{\mathbb{K}}(0, \varepsilon) \subset f(G - x) = f(G) - f(x)$ or $f(x) + B_{\mathbb{K}}(0, \varepsilon) \subset f(G)$, so that $f(x)$ is an interior point of $f(G)$. □

Theorem 6.7. *Let $E, G \subset X$ be convex such that $E \cap G = \emptyset$ and G is open. Then there exist $f \in X'$ and $t \in \mathbb{R}$ such that $\operatorname{Re} f(z) < t \le \operatorname{Re} f(y)$ for all $z \in G$ and $y \in E$ (that is, $G \subset [\operatorname{Re} f < t]$ and $E \subset [\operatorname{Re} f \ge t]$).*

Proof. It is clearly enough to prove the case when $\mathbb{K} = \mathbb{R}$ and $G, E \ne \emptyset$. Fix any $y_0 \in E$ and $z_0 \in G$ and denote $x_0 := y_0 - z_0$ and $C := G - E + x_0$. Then C is open, convex, $0 \in C$, and $x_0 \notin C$. Hence its gauge $p := p_C$ is a continuous positive sublinear functional such that $p(x_0) \ge 1$. Define a linear functional $f_0 : \mathbb{R} \cdot x_0 \to \mathbb{R}$ by $f_0(\lambda x_0) := \lambda$. Then $f_0(\lambda x_0) = \lambda \le p(\lambda x_0)$. Applying Theorem 6.2 we obtain an extension $f \in X^*$ such that $f(x) \le p(x)$ for all $x \in X$.

Note that $f(x) \le p(x) \le 1$ and hence also $f(-x) \ge -1$ for all $x \in C$. Thus $f(C \cap (-C)) \subset [-1, 1]$ with $C \cap (-C) \in \mathcal{N}$, so f is continuous by Proposition 6.1.

For $y \in E$ and $z \in G$ we get $f(z) - f(y) + 1 = f(z - y + x_0) \le p(z - y + x_0) < 1$, because $z - y + x_0 \in C$ and C is open, so that $f(z) < f(y)$ for all $z \in G$ and $y \in E$. Since $f(G) \subset \mathbb{R}$ is open, setting $t := \sup f(G)$ yields the needed inequalities. □

Corollary 6.8. *If X contains non-trivial open convex subsets, then $X' \ne \{0\}$.*

7.1. Describing a locally convex topology via zero neighbourhood bases and via seminorms.

Definition. A TVS X is ***locally convex (LCS)*** if every $U \in \mathcal{N}$ contains a convex $V \in \mathcal{N}$.

Proposition 7.1. *In an LCS X every $U \in \mathcal{N}$ contains a closed absolutely convex $V \in \mathcal{N}$.*

Proof. Take a closed $U \in \mathcal{N}$. There are a convex $V \in \mathcal{N}$ and a balanced $W \in \mathcal{N}$ such that $W \subset V \subset U$. Now $\overline{\operatorname{conv} W}$ is absolutely convex (prove it!)✠ and $W \subset \overline{\operatorname{conv} W} \subset \overline{V} \subset U$. □

Recall that a centered system of sets is such that no finite intersection of its sets is empty. Every centered system generates a filter in a unique way: the smallest filter containing it. Vice versa, every subset of a filter is a centered system. A centered system is also called a ***filter subbase***.

If we denote by $\pi(\mathcal{B})$ the system of all finite intersections of the sets in a centered system $\mathcal{B}$, then $\pi(\mathcal{B})$ is a prefilter.

Consider an arbitrary system $\mathcal{B}_0$ of absolutely convex absorbing sets in a vector space X. It is centered, because $0 \in \bigcap \mathcal{B}_0$. However, the generated filter $\pi(\mathcal{B}_0)^{\uparrow}$ may fail to be a zero neighbourhood filter for some TVS because it may fail to be closed under multiplication by some $\varepsilon > 0$.

Denote $\widehat{\mathcal{A}} := \mathbb{R}_+(\cdot)\mathcal{A} = \{\varepsilon U \mid U \in \mathcal{A},\ \varepsilon > 0\}$ for any system $\mathcal{A} \subset 2^X$. Note that $\widehat{\pi(\mathcal{B}_0)} \subset \pi\big(\widehat{\mathcal{B}_0}\big)$ and both these systems are prefilters, generating the same filter.

Exercise 7.1. Prove that this filter satisfies conditions of Theorem 2.5.

Proposition 7.2. *Every system $\mathcal{B}_0$ of absolutely convex absorbing sets in a vector space X generates an LC topology having $\widehat{\pi(\mathcal{B}_0)}$ as a zero neighbourhood base (and $\widehat{\mathcal{B}_0}$ as its subbase). This topology is Hausdorff if and only if $\bigcap \widehat{\mathcal{B}_0} = \{0\}$.*

The system $\mathcal{B}_0$ is then called a ***prebase*** of the corresponding zero neighbourhood filter $\mathcal{N}$. That is, a system $\mathcal{B}_0$ consisting of absorbing absolutely convex sets is a prebase of $\mathcal{N}$ if $\widehat{\mathcal{B}_0}$ is its subbase. Note that any subsystem $\mathcal{C}_0$ consisting of absolutely convex sets and containing $\mathcal{B}_0$ is also a prebase of $\mathcal{N}$ (in particular, any subsystem $\mathcal{C}_0$ such that $\mathcal{B}_0 \subset \mathcal{C}_0 \subset \pi\big(\widehat{\mathcal{B}_0}\big)$).

Proposition 7.3.
(1) *Every system $\mathcal{P}$ of seminorms on a vector space X defines a locally convex topology τ on it via the prebase $\{\overline{B}_p = p^{-1}([0,1]) \mid p \in \mathcal{P}\}$. The elements of the corresponding zero neighbourhood base are of the form*
$$W_{\varepsilon,p_1,\dots,p_n} := \left\{x \in X \mid \max_i p_i(x) \le \varepsilon\right\},$$
where $\varepsilon > 0$, $n \in \mathbb{N}$, and $p_1,\dots,p_n \in \mathcal{P}$. The topology τ is Hausdorff if and only if $\mathcal{P}$ separates points in X, that is,
$$\forall x \in X\ \exists p \in \mathcal{P}:\ p(x) \neq 0.$$
(2) *Every locally convex topology can be generated by a system of seminorms in this way.*

Proposition 7.4. *Let (X,τ) be an LCS and denote by $\mathcal{P}$ the system of all continuous seminorms on X. Then $\mathcal{P}$ generates τ.*

7.2. Convergence and boundedness in LCS.

Proposition 7.5. *Let the LC topology of (X,τ) be defined by a system of seminorms $\mathcal{P}$. Then*
1. $x_\alpha \to x \iff p(x_\alpha - x) \to 0\ \forall p \in \mathcal{P}$,
2. *$E \subset X$ is bounded $\iff \sup_{x\in E} p(x) < \infty$ (i.e., $p(E)$ is bounded) for all $p \in \mathcal{P}$.*

Proof.
1. (⇒): is because seminorms $p \in \mathcal{P}$ are continuous.
   (⇐): It is enough to prove that $\mathcal{F} \to 0 \impliedby p(\mathcal{F}) \to 0\ \forall p \in \mathcal{P}$ for any prefilter $\mathcal{F}$. Note that $p(\mathcal{F}) \to 0$ means that $\varepsilon\overline{B}_p \in \mathcal{F}^{\uparrow}$ for any $\varepsilon > 0$. So the filter $\mathcal{F}^{\uparrow}$ contains $\{\varepsilon\overline{B}_p \mid \varepsilon > 0,\ p \in \mathcal{P}\}$, a subbase of $\mathcal{N}$, hence also $\mathcal{N}$ itself. That is, $\mathcal{F} \to 0$.
2. Note that $\lambda E \subset \overline{B}_p \iff E \subset \frac{1}{\lambda}\overline{B}_p \iff p(E) \subset [0,\frac{1}{\lambda}]$. That is, $p(E)$ is bounded $\iff E$ is absorbed by $\overline{B}_p$. Since absorption is preserved under finite intersections and multiplication by positive scalar, it is enough to be absorbed by all elements in some prebase of $\mathcal{N}$.

□

Proposition 7.6. *Let X be a LCS.*
(1) *If $E \subset X$ is bounded, then so is* absconv E.
(2) *If $E \subset X$ is completely bounded, then so is* absconv E.

Proof. (1) Since X is LC, given $V \in \mathcal{N}$, there is an absolutely convex $U \in \mathcal{N}$, $U \subset V$. If there is $\lambda > 0$ such that $E \subset \lambda U$, then also absconv $E \subset$ absconv $\lambda U = \lambda U \subset \lambda V$.
(2) Given $V \in \mathcal{N}$ take an absolutely convex $U \in \mathcal{N}$ such that $U + U \subset V$. There is a finite set E_0 such that $E \subset E_0 + U$. Then absconv E = absconv$(E_0 + U) \subset$ absconv E_0 + absconv U = absconv $E_0 + U$. It is enough to prove that $H :=$ absconv E_0 is completely bounded. Observe that for any point $x \in E_0$ the set absconv$\{x\} = \overline{B}_{\mathbb{K}} x$ is compact (because it is a continuous image of $\overline{B}_{\mathbb{K}}$, which is compact). Proposition 5.5 now implies that absconv $\bigcup_{x \in E_0} \overline{B}_{\mathbb{K}} x$ is compact and hence completely bounded. Then so is absconv $E_0 \subset$ absconv $\bigcup_{x \in E_0} \overline{B}_{\mathbb{K}} x$. (Note that (see the original conspect) one can show that absconv E_0 is compact, too.)

□

7.3. **Metrizable and normable LCS.** Given a metrizable LCS, there is a countable base of $\mathcal{N}$ consisting of absolutely convex sets. The corresponding Minkowski functionals form a countable seminorm system, which defines the same topology. On the other hand, given a Hausdorff topology defined by a countable seminorm system $\{p_n\}_{n \in \mathbb{N}}$, one can define a translation invariant metric

$$d(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{p_n(x - y)}{1 + p_n(x - y)}$$

(prove it!)✠. It is clear that (prove it!)✠, $d(x_\alpha, 0) \to 0$ if and only if $p_n(x_\alpha) \to 0$ for all $n \in \mathbb{N}$, so that this metric induces the original topology. We have observed

Proposition 7.7. *A Hausdorff LCS is metrizable if and only if its topology can be defined by a countable (or finite) seminorm system.*

Example 7.1. The space $C(\mathbb{C})$ from Example 4.1 is a metrizable LCS, because the sets $V_{n,i}$ forming a base of $\mathcal{N}$ are absolutely convex.

Normable LCS. It is easy to see that if a Hausdorff LC topology is defined by a finite seminorm system $\{p_1, \dots, p_n\}$, then $p(x) := \max_{1 \le i \le n} p_i(x)$ defines a norm, inducing the same topology. In that case, let us say that the space is ***normable***.

Proposition 7.8 (Kolmogorov theorem)**.** *A Hausdorff TVS is normable if and only if it has bounded convex zero neighbourhoods.*

Corollary 7.9. *A Hausdorff LCS is normable if and only if it has a bounded zero neighbourhood.*

Example 7.2. The space ℓ_p, $0 < p < 1$, is a metrizable TVS with a pseudonorm $|x| = \sum_k |x_k|^p$ but it is not locally convex.

7.4. **The dual of an LCS.**

Definition. A subspace $Y \subset X^*$ ***separates the points of*** X if for all $x, y \in X$ with $x \ne y$ there is $f \in Y$ such that $f(x) \ne f(y)$. Equivalently, if for all $x \ne 0$ there is $f \in Y$ such that $f(x) \ne 0$.

Proposition 7.10. *The algebraic dual X^* of a vector space X separates the points of X.*

Proposition 7.11. *Let X be an LCS. A functional $f \in X^*$ is continuous if and only there exists a continuous seminorm p on X such that $|f(x)| \le p(x)$ for all $x \in X$.*

Proposition 7.12. *Let X be an LCS and let $X_0 \subset X$ be a subspace. For every $f_0 \in X_0'$ there is $f \in X'$ such that $f|_{X_0} = f_0$.*

Theorem 7.13. *If X is a Hausdorff LCS, then X' separates points of X.*

Exercise 7.2. Show that if $\tau_1 \subset \tau_2$, then $(X, \tau_1)' \subset (X, \tau_2)'$.

7.5. **Another two separation theorems.**

Proposition 7.14. *Let X be an LCS and let $E \subset X$ be convex. Then $x \in \overline{E}$ if and only if $f(x) \in \overline{f(E)}$ for all $f \in X'$.*

Theorem 7.15. *Let X be an LCS.*

(a) *If $E \subset X$ is absolutely convex and $x \in X \setminus \overline{E}$, then there is $f \in X'$ such that $f(E) \subset \overline{B}_{\mathbb{K}}$ but $f(x) > 1$.*
(b) *Let $X_0 \subset X$ be a subspace. Then $x \in \overline{X_0}$ if and only if $f(x) = 0$ for all $f \in X'$ such that $f|_{X_0} = 0$.*

Corollary 7.16. *Let $X_0 \subset X$ be a subspace of an LCS X. Then $\overline{X_0} = X$ if and only if $f|_{X_0} \neq 0$ for all $f \in X' \setminus \{0\}$. In other words, $\overline{X_0} = X$ if and only if $f|_{X_0} = 0 \implies f = 0$ for all $f \in X'$.*

Example 7.3. Consider the metrizable TVS $S[a,b]$ from Example 4.2. We can show that $S[a,b]' = \{0\}$.

8. Dual pairs of vector spaces

8.1. **Dual pair.**

Definition. Let X and Y be vector spaces over the same field $\mathbb{K}$. The spaces X and Y form a ***dual pair*** $\langle X, Y\rangle$ if there is fixed a bilinear functional $\langle\cdot,\cdot\rangle : X \times Y \to \mathbb{K}$, which separates the points of both X and Y, that is:

- for every $x \in X \setminus \{0\}$ there is $y \in Y$ such that $\langle x, y\rangle \neq 0$,
- for every $y \in Y \setminus \{0\}$ there is $x \in X$ such that $\langle x, y\rangle \neq 0$.

Of course, if $\langle X, Y\rangle$ is a dual pair, then so is $\langle Y, X\rangle$.

Example 8.1. The spaces ℓ_1 and ℓ_∞ form a dual pair with $\langle x, y\rangle := \sum_{n=1}^{\infty} x_n y_n$.

Example 8.2. Every Hilbert space forms a dual pair with itself, the bilinear functional is just the dot product.

Example 8.3. Given any vector space X, the dual pair $\langle X, X^*\rangle$ can be defined by the functional $\langle x, f\rangle = f(x)$.

The last example can be generalized to a dual pair $\langle X, Y\rangle$, where a subspace $Y \subset X^*$ separates the points of X. In fact, this case essentially encompasses all dual pairs. Given a dual pair $\langle X, Y\rangle$ we can define a linear injection $\pi : Y \to X^*$ by $\pi(y)(x) = \langle x, y\rangle$, so that Y is isomorphic to $\pi(Y) \subset X^*$.

Therefore, given a dual pair $\langle X, Y\rangle$ we can (and will) always assume that Y is a subspace X^*, which separates points of X (or that X is a subspace of Y^*). In that case the bilinear functional is automatically defined.

Exercise 8.1. Denote by $\omega = \{(x_n) \subset \mathbb{K}\}$ the space of all sequences and by $\phi = \{(x_n) \subset \mathbb{K} \mid x_n \neq 0 \text{ for finitely many } n\}$ the space of all finite sequences. Then $\langle \omega, \phi\rangle$ is a dual pair by the functional $\langle x, y\rangle = \sum_{n=1}^{\infty} x_n y_n$.

8.2. **Weak topology.**

Definition. Let $\langle X, Y\rangle$ be a dual pair and let τ be an LC topology on X. If $(X,\tau)' = Y$, then τ is said to be ***consistent*** with the duality $\langle X, Y\rangle$.

Note that given $f \in X^*$, we can define a seminorm $p_f : X \to \mathbb{R}$ by $p_f(x) = |\langle x, f\rangle| = |f(x)|$.

Definition. Let $\langle X, Y\rangle$ be a dual pair. The locally convex topology $\sigma(X,Y)$ on X defined by a family of seminorms $\{p_f \mid f \in Y\}$ is called the ***weak topology*** (defined by the duality $\langle X, Y\rangle$).

Properties of the weak topology $\sigma(X,Y)$:

(1) It is Hausdorff.
(2) We know that a zero neighbourhood base is $\pi(\widehat{\{\overline{B}_{p_f}\}_{f\in Y}})$. However, the linearity allows to drop the epsilons and consider just $\pi(\{\overline{B}_{p_f}\}_{f\in Y})$ as the base. That is, the sets in this base are of the form
$$W_{f_1,\ldots,f_n} := \{x \in X \mid \max_i |f_i(x)| \leq 1\},$$
for all $n \in \mathbb{N}$ and $f_1,\ldots,f_n \in Y$.
(3) The convergence: $x_\alpha \to x \iff f(x_\alpha) \to f(x)$ for all $f \in Y$. The boundedness: $E \subset X$ is bounded if and only if $f(E)$ is bounded for all $y \in Y$.
(4) The topology $\sigma(Y,X)$ on Y can be described symmetrically and has the same properties.

Exercise 8.2. Describe the convergence of a sequence $\left(x^{(n)}\right)$ in the LCS $(\ell_\infty, \sigma(\ell_\infty, \ell_1))$. Does the sequence $\left(y^{(n)}\right)$ converge if $y^{(1)} = (1,0,0,\ldots)$, $y^{(2)} = (1,1,0,0,\ldots)$, $\ldots$, $y^{(n)} = (1,1,\ldots,1,0,0,\ldots)$, $\ldots$?

Exercise 8.3. Does the sequence from the above exercise converge in $(\ell_1, \sigma(\ell_1, \ell_\infty))$?

Exercise 8.4. Show that $x^{(n)} \to x$ in the topology $\sigma(\omega, \phi)$ if and only if $x_k^{(n)} \to x_k$ for all $k \in \mathbb{N}$.

Proposition 8.1. *Let $\langle X, Y \rangle$ be a dual pair. In $(X, \sigma(X, Y))$, every bounded set is completely bounded.*

Lemma 8.2. *Let X be a vector space and let $f, f_1, \dots, f_n \in X^*$. Then $f \in \operatorname{span}\{f_1, \dots, f_n\}$ if and only if*

$$\forall i \in \{1, \dots, n\}\ f_i(x) = 0 \implies f(x) = 0.$$

Note that if we knew (e.g., from Theorem 8.3 below) that $\sigma(X^*, X)$ were consistent with $\langle X^*, X \rangle$, i.e. $(X^*, \sigma(X^*, X))' = X$, then the above lemma would immediately follow from Corollary 7.16 (because a finite-dimensional subspace is always closed in a Hausdorff TVS). Alas, we need the above lemma to prove Theorem 8.3 below, so another proof is needed.

Theorem 8.3. *The topology $\sigma(X, Y)$ is consistent with the duality $\langle X, Y \rangle$, that is, $(X, \sigma(X, Y))' = Y$.*

Theorem 8.4. *The weak topology $\sigma(X, Y)$ is the weakest of all LC topologies consistent with the duality $\langle X, Y \rangle$.*

Corollary 8.5. *If $X = (X, \tau)$ is a Hausdorff LCS, then τ is consistent with $\langle X, X' \rangle$ and $\sigma(X, X') \subset \tau$.*

8.3. **Polars.**

Exercise 8.5. If $\tau_1 \subset \tau_2$, then $\overline{E}^{\tau_2} \subset \overline{E}^{\tau_1}$.

Proposition 8.6. *Let $\langle X, Y \rangle$ be a dual pair. The closure of a convex set $E \subset X$ is the same in all LC topologies consistent with $\langle X, Y \rangle$.*

Theorem 8.7. *The closed convex sets are the same in all LC topologies consistent with a given duality.*

Definition. Let $\langle X, Y \rangle$ be a dual pair and let $E \subset X$. The ***polar*** of E is defined as

$$E^0 := \{f \in Y \mid \forall x \in E\colon\ |f(x)| \le 1\} \subset Y.$$

Show that

Exercise 8.6. $E \subset F \implies F^0 \subset E^0$,

Exercise 8.7. $(\lambda E)^0 = \frac{1}{\lambda} E^0$ if $\lambda \in \mathbb{K} \setminus \{0\}$,

Exercise 8.8. $(\bigcup_{\alpha \in \Gamma} E_\alpha)^0 = \bigcap_{\alpha \in \Gamma} E_\alpha^0$,

Exercise 8.9. E^0 is absolutely convex and $\sigma(Y, X)$-closed,

Exercise 8.10. if $X_0 \subset X$ is a subspace, then $(X_0)^0 = (X_0)^\perp := \{f \in Y \mid f|_{X_0} = 0\}$.

Given a system of sets $\mathscr{A} \subset 2^X$, denote $\mathscr{A}^0 := \{U^0 \mid U \in A\}$.

Proposition 8.8. *Let $\mathscr{B}$ be a zero neighourhood base of an LCS X. Then $X' = \bigcup \mathscr{B}^0$, where the polars are taken with respect to duality $\langle X, X^* \rangle$.*

Let $\langle X, Y \rangle$ and $\langle Y, Z \rangle$ be dual pairs such that $X \subset Z$. Given $E \subset X$, we can define the bipolar $E^{00} = (E^0)^0$ with respect to these dualities.

Exercise 8.11. Show that $E \subset E^{00}$.

Proposition 8.9 (bipolar theorem)**.** *Let $\langle X, Y \rangle$ and $\langle Y, Z \rangle$ be dual pairs such that $X \subset Z$. Given $E \subset X$ the bipolar E^{00} with respect to these dualities satisfies*

$$E^{00} = \overline{\operatorname{absconv} E}^{\sigma(Z, Y)}.$$

When X is a Hausdorff LCS, by the bipolar E^{00} of $E \subset X$ we will usually mean the bipolar with respect to the dualities $\langle X, X' \rangle$ and $\langle X', X \rangle$.

Corollary 8.10. *Let X be a Hausdorff LCS X and $E \subset X$. Then $E^{00} = \overline{\operatorname{absconv} E}$.*

Corollary 8.11. *Let X be a Hausdorff LCS X and $E \subset X$. Then $E^{000} = E^0$.*

9.1. **$\mathfrak{S}$-topologies and equicontinuous sets.**

Proposition 9.1. *Let $\langle X, Y\rangle$ be a dual pair and let $B \subset Y$. Then $B^0 \subset X$ is absorbing if and only if B is $\sigma(Y,X)$-bounded.*

Thus given a system of $\mathfrak{S} \subset 2^Y$ of $\sigma(Y,X)$-bounded sets, the system of their polars $\mathfrak{S}^0 \subset 2^X$ consists of absorbing absolutely convex sets, so it is a prebase of some LC topology on X. Let us denote it by $\mathcal{T}_{\mathfrak{S}}$ and call it the ***polar topology*** defined by $\mathfrak{S}$ or the ***topology of uniform convergence on the sets from*** $\mathfrak{S}$.

Since $\mathfrak{S}^{000} = \mathfrak{S}^0$, we can assume that every element of $\mathfrak{S}$ is absolutely convex and $\sigma(Y,X)$-closed. We can also assume the following:

(PT1) $S_1, S_2 \in \mathfrak{S} \implies \exists S\colon\ S_1 \cup S_2 \subset S$ (i.e., $\mathfrak{S}$ is directed upwards),

(PT2) $S \in \mathfrak{S}$ and $\lambda > 0$ imply $\lambda S \in \mathfrak{S}$ (i.e., $\widehat{\mathfrak{S}} = \mathfrak{S}$).

Moreover, $\mathcal{T}_{\mathfrak{S}}$ is Hausdorff if and only if

(PT3) $\overline{\operatorname{span} \bigcup \mathfrak{S}}^{\sigma(Y,X)} = Y$.

It is clear that $\mathcal{T}_{\mathfrak{S}}$ is defined by the family of seminorms $\{p_{S^0}\}_{S \in \mathfrak{S}}$, where p_{S^0} is the Minkowski functional of S^0. Observe that $p_{S^0}(x) = \sup_{f \in S} |f(x)|$ for all $x \in X$, so the convergence with respect to p_{S^0} is the uniform convergence on S. Let us also denote $p^{(S)} := p_{S^0}$.

Since being absorbing is a necessary condition for being an element of a prebase of an LC topology, we get the strongest possible polar topology on X with respect to duality $\langle X, Y\rangle$ if we consider the system $\mathfrak{S}_b \subset Y$ of all $\sigma(Y,X)$-bounded sets. This polar topology $\beta(X,Y) := \mathcal{T}_{\mathfrak{S}_b}$ is called the ***strong topology***.

The weak topology $\sigma(X,Y)$ is also a polar topology with $\sigma(X,Y) = \mathcal{T}_{\mathfrak{S}_\sigma}$, where σ is either the system of all one-element subsets or of all finite subsets of Y.

Proposition 9.2. *Every Hausdorff LC topology τ on a vector space is a polar topology: $\tau = \mathcal{T}_{\mathscr{B}^0}$, where $\mathscr{B}$ is some zero neighbourhood base of (X,τ).*

Definition. Let (X,τ) be a TVS. Then $S \subset X'$ is called ***equicontinuous*** (or ***τ-equicontinuous***) if

$$\forall \varepsilon > 0\ \exists U \in \mathcal{N}:\ \forall f \in S\ \forall x \in U\ |f(x)| \le \varepsilon.$$

Proposition 9.3. *Let X be a Hausdorff LCS. Then $S \subset X'$ is equicontinuous if and only if $S \subset U^0$ for some $U \in \mathcal{N}$.*

Let us denote the collection of all equicontinuous subsets of X' by $\mathscr{E}$.

Exercise 9.1. Prove that $\widehat{\mathscr{E}} = \mathscr{E}$.

Exercise 9.2. Prove that $S_1, S_2 \in \mathscr{E} \implies S_1 \cup S_2 \in \mathscr{E}$.

Theorem 9.4. *Every Hausdorff LC topology τ on a vector space X is the topology of uniform convergence on τ-equicontinuous sets*

Some properties of equicontinuous sets:

- $\mathscr{E}$ is a bornology (or an ideal),
- $\mathscr{E}^{00} \subset \mathscr{E}$,
- if $S \in \mathscr{E}$, then $\overline{S}^{\sigma(X',X)} \in \mathscr{E}$ and $\operatorname{absconv} S \in \mathscr{E}$,
- every equicontinuous set is $\sigma(X',X)$-bounded.

9.2. **Mackey topology.**

Proposition 9.5. *Let X be a vector space. Then $(X^*, \sigma(X^*, X))$ is a complete LCS.*

Proposition 9.6 (Alaoglu theorem)**.** *Let X be a Hausdorff LCS. If $U \in \mathcal{N}$, then $U^0 \subset X'$ is $\sigma(X',X)$-compact. (Due to Proposition 8.8, the polar U^0 is the same for dualities $\langle X, X'\rangle$ or $\langle X, X^*\rangle$.)*

Let $\langle X, Y\rangle$ be a dual pair. Consider the system $\mathfrak{S}_0 \subset 2^Y$ of all $\sigma(Y,X)$-compact and absolutely convex sets. Note that $\mathfrak{S}_0$ satisfies (PT1) and (PT2).

Definition. The topology $\tau(X,Y) := \mathcal{T}_{\mathfrak{S}_0}$ on X is called the ***Mackey*** topology.

Let us point out that $\sigma(X,Y) \subset \tau(X,Y)$, because $\sigma(X,Y) = \mathscr{T}_{\mathfrak{S}_\sigma} = \mathscr{T}_{\mathfrak{S}_\sigma^{00}}$ and for every $S \in \mathfrak{S}_\sigma$ its polar S^0 is a zero neighbourhood of $\sigma(X,Y)$, so S^{00} is $\sigma(Y,X)$-compact by the Alaoglu theorem and hence $\mathfrak{S}_\sigma^{00} \subset \mathfrak{S}_0$. In particular, this implies that $\tau(X,Y)$ is Hausdorff.

Theorem 9.7 (Mackey–Arens theorem)**.** *A Hausdorff LC topology τ on a vector space X is consistent with a duality $\langle X,Y\rangle$ if and only if $\sigma(X,Y) \subset \tau \subset \tau(X,Y)$. In that case, there is a system $\mathfrak{S} \subset \mathfrak{S}_0$ such that $\tau = \mathscr{T}_{\mathfrak{S}}$.*

Corollary 9.8. *The Mackey topology is the strongest LC topology on X, which is consistent with a duality $\langle X,Y\rangle$.*

Corollary 9.9. *A Hausdorff LC topology τ on X is consistent with a duality $\langle X,Y\rangle$ if and only if $\tau = \mathscr{T}_{\mathfrak{S}}$ for some system $\mathfrak{S}$ of absolutely convex and $\sigma(Y,X)$-compact subsets of Y.*

9.3. **Mackey theorem on bounded sets.**

Theorem 9.10 (Mackey theorem)**.** *Let X be a Hausdorff LCS. Then $E \subset X$ is bounded if and only if it is weakly bounded (that is, $\sigma(X,X')$-bounded).*

Corollary 9.11. *Given a duality $\langle X,Y\rangle$, the bounded sets of X are the same in all LC topologies consistent with this duality.*

For proving the Mackey theorem we need the following proposition and the principle of uniform boundedness from the Banach space theory.

Proposition 9.12. *Let V be a compact and absolutely convex set in a Hausdorff LCS X. Then $p_V : \operatorname{span} V \to \mathbb{R}$, the Minkowski functional of V, is a norm on $X_V := \operatorname{span} V$. Moreover, (X_V, p_V) is a Banach space and the norm topology on it is stronger than the induced topology on X_V.*

Proof. Scheme:

(I) $V \subset X$ is absorbing if and only if $\operatorname{span} V = X$,
(II) (X_V, p_V) is Hausdorff if and only if p_V is a norm,
(III) if V is bounded in X, then p_V is a norm,
(IV) if V is compact in X, then (X_V, p_V) is complete.

□

Let us recall the principle of uniform boundedness: if X and Z are Banach spaces and $\mathscr{A}$ is a system of bounded linear maps $A: X \to Z$ such that $\mathscr{A}$ is pointwise bounded, that is, $\{A(x) \mid A \in \mathscr{A}\}$ is bounded in Z for every $x \in X$, then $\mathscr{A}$ is uniformly bounded, that is, $\{\|A\| \mid A \in \mathscr{A}\} \subset \mathbb{R}$ is bounded.

Proof of the Mackey theorem. Scheme: Take a weakly bounded $E \subset X$. As the set of functionals on X', it is pointwise bounded. Take a nice zero neighbourhood $U \subset X$ and consider the Banach space (X'_{U^0}, p_{U^0}). Consider E as the set of continuous linear functionals on it and apply the principle. □

Proposition 9.13. *A metrizable LC topology is a Mackey topology. That is, if an LCS (X,τ) is metrizable, then $\tau = \tau(X,X')$.*

Proof. ... □

10. Barrelled spaces and F-spaces

10.1. **Strong topology and barrelled space.**

Definition. A ***barrel*** is a closed absolutely convex absorbing set. An LCS is a ***barrelled space*** if every barrel is a zero neighbourhood.

It is easy to check that

- if X is a Hausdorff LCS, then $U \subset X$ is a barrel if and only if $U = U^{00}$ with respect to $\langle X,X'\rangle$,
- if X is a Hausdorff LCS, then $U \subset X$ is a barrel if and only if there is a $\sigma(X',X)$-bounded set $S \subset X'$ such that $U = S^0$,
- all LC topologies on X consistent with a duality $\langle X,Y\rangle$ have the same barrels,
- every LCS has a zero neighbourhood base consisting of barrels.

Proposition 10.1. *Given a Hausdorff LCS (X,τ), the following are equivalent:*

(1) (X,τ) *is a barrelled space,*
(2) *every* $\sigma(X',X)$*-bounded set* $S\subset X'$ *is* τ*-equicontinuous,*
(3) $\tau=\beta(X,X')$,
(4) $\tau=\tau(X,X')=\beta(X,X')$,
(5) $\tau=\tau(X,X')$ *and* $\beta(X,X')$ *is consistent with* $\langle X,X'\rangle$.

Let us extend the definition of equicontinuous sets to subsets of operators.

Definition. A set $A\subset L(X,Y)$ is called ***equicontinuous*** if

$$\forall V\in\mathcal{N}_Y\ \exists U\in\mathcal{N}_X:\ \forall T\in A\ T(U)\subset V.$$

Exercise 10.1. Every equicontinuous set $A\subset L(X,Y)$ is pointwise bounded, that is, $\{T(x)\mid T\in A\}$ is bounded in Y for every $x\in X$.

Theorem 10.2 (principle of uniform boundedness)**.** *If* X *is a barrelled space and* Y *is an LCS, then every pointwise bounded set* $A\subset L(X,Y)$ *is equicontinuous*

Theorem 10.3 (continuity of the limit operator)**.** *Let* X *be a barrelled space and let* Y *be an LCS. Consider a sequence* $(T_n)\subset L(X,Y)$ *such that for every* $x\in X$ *there exists* $T(x):=\lim_n T_n(x)$*. Then* $T\in L(X,Y)$.

10.2. **F-spaces. Open mapping theorem.** Recall that an LCS is metrizable if and only if its topology can be induced by an at most countable system of seminorms $\{p_n\}_{n\in\mathbb{N}}$. In that case, the defining translation invariant metric can be chosen to be

$$d(x,y)=\sum_{n=1}^{\infty}\frac{1}{2^n}\frac{p_n(x-y)}{1+p_n(x-y)}.$$

This provides a zero neighbourhood base $\{\frac{1}{n}B\}_{n\in\mathbb{N}}$, where $B=\{x\mid d(x,0)\le 1\}$.

Definition. A complete metrizable LCS is called an ***F-space*** or ***Frechet' space***.

Lemma 10.4. *Let* X *and* Y *be F-spaces and* $T:X\to Y$ *a surjective linear operator. For every barrel* $U\subset X$ *there is* $V\in\mathcal{N}_Y$ *such that* $V\subset\overline{T(U)}$.

Proposition 10.5. *Every F-space is barrelled.*

Theorem 10.6 (open mapping theorem)**.** *Let* X *and* Y *be F-spaces. A surjective* $T\in L(X,Y)$ *(that is, linear and continuous) is open, that is,* $T(G)$ *is open for every open* $G\subset X$.

10.3. **Closed graph theorem.**

Theorem 10.7 (continuity of the inverse)**.** *Let* X *and* Y *be F-spaces. If* $A\in L(X,Y)$ *is bijective, then* $A^{-1}\in L(Y,X)$.

Observe that (prove it!)✠ the topological product $X\times Y$ of F-spaces X and Y is again an F-space, with its topology induced by all seminorms $r_{m,n}$ of the form

$$r_{m,n}(x,y):=p_m(x)+q_n(y),$$

where the systems $\{p_m\}_m$ and $\{q_n\}_n$ induce the topologies of X and Y, respectively.

Definition. Let $T:X\to Y$ be a linear operator. Its ***graph*** is the set $\operatorname{gr}T:=\{(x,Tx)\mid x\in X\}\subset X\times Y$. The operator T is called ***closed*** if $\operatorname{gr}T$ is closed in $X\times Y$.

Theorem 10.8 (closed graph theorem)**.** *Let* X *and* Y *be F-spaces. A closed linear operator* $T:X\to Y$ *is continuous.*

11. Projective limits

11.1. **Projective limit topology.** Fix a vector space X, locally convex spaces X_γ and linear operators $v_\gamma:X\to X_\gamma$ for all $\gamma\in\Gamma$.

Definition. The weakest LC topology on X such that all operators v_γ are continuous is called the ***projective limit topology*** (induced by pairs $\{(X_\gamma,v_\gamma)\mid\gamma\in\Gamma\}$). It is denoted by τ_{proj}. The space (X,τ_{proj}) is called the ***projective limit*** of these pairs.

Given prebases $\mathcal{P}_\gamma$ of $\mathcal{N}_{X_\gamma}$ consisting of absolutely convex sets, the zero neighbourhood filter $\mathcal{N}_X$ of (X,τ_{proj}) is generated, e.g., by the prebase $\bigcup_{\gamma\in\Gamma}v_\gamma^{-1}(\mathcal{P}_{X_\gamma})=\{v_\gamma^{-1}(U)\mid U\in\mathcal{P}_\gamma,\ \gamma\in\Gamma\}$ or by the subbase $\bigcup_{\gamma\in\Gamma}v_\gamma^{-1}(\mathcal{N}_{X_\gamma})$.

Proposition 11.1. *If all X_γ are Hausdorff, then (X, τ_{proj}) is Hausdorff if and only if*

$$\bigcap_\gamma v_\gamma^{-1}(\{0\}) = \{0\}.$$

Proof. Proposition 2.7 says that X is Hausdorff if and only if $\bigcap \mathcal{N}_X = \{0\}$. Note that, in general, given a filter $\mathscr{F}$ and its subbase $\mathscr{B}$, one has $\bigcap \mathscr{F} = \bigcap \mathscr{B}$. So

$$\bigcap \mathcal{N}_X = \bigcap_{\gamma \in \Gamma} \bigcap_{U \in \mathcal{N}_{X_\gamma}} v_\gamma^{-1}(U) = \bigcap_{\gamma \in \Gamma} v_\gamma^{-1}\left(\bigcap \mathcal{N}_{X_\gamma}\right) = \bigcap_\gamma v_\gamma^{-1}(\{0\}).$$

□

Proposition 11.2. *Let Y be an LCS. A linear operator $T : Y \to X$ is continuous if and only if so are all compositions $v_\gamma \circ T : Y \to X_\gamma$.*

Proof. Sufficiency is clear. For the necessity, note that T is continuous if and only if $T^{-1}(U) \in \mathcal{N}_Y$ for all U in some prebase $\mathscr{P}_X$ of $\mathcal{N}_X$. So it is enough to check if $T^{-1}(v_\gamma^{-1}(U)) = (v_\gamma \circ T)^{-1}(U) \in \mathcal{N}_Y$ for all U in some prebase of $\mathcal{N}_{X_\gamma}$ for all γ. This is clearly equivalent to the continuity of $v_\gamma \circ T$ for all γ. □

Proposition 11.3. *A subset $E \subset X$ is τ_{proj}-bounded if and only if $v_\gamma(E)$ is bounded for all γ.*

Proof. As above, again it is enough to check if the set is absorbed by elements of some prebase of $\mathcal{N}_X$. □

To check the validity of the first two examples, just check the equality of prebases for the projective limit and the usual definition.

Example 11.1. Let $X_0 \subset X$. The induced subspace topology on X_0 is a projective limit of the pair $\{(X, i)\}$, where $i : X_0 \to X$ is the injection map.

Example 11.2. Let $\langle X, Y\rangle$ be a dual pair. The weak topology $\sigma(X, Y)$ is the projective limit of $\{(\mathbb{K}, f) \mid f \in Y\}$.

Example 11.3. Given a collection of topologies $\{\tau_\gamma\}_\gamma$ on X, the projective limit $\{((X, \tau_\gamma), I_X) \mid \gamma \in \Gamma\}$ is the weakest LC topology generated by $\bigcup_\gamma \tau_\gamma$, where $I_X : X \to X$ is the identity.

Example 11.4. Any LC topology on X defined by a system of seminorms $\{p_\gamma\}_\gamma$ is the weakest topology such that all the seminorms p_γ are continuous. Thus it is the projective limit of seminormed spaces (X, p_γ), more precisely, of pairs $\{((X, p_\gamma), I_X)\}_\gamma$.

Example 11.5. Any LC topology on X is a projective limit of normed spaces. Take the seminormed spaces (X, p_γ) as above and consider vector spaces $X_p := X/p^{-1}(0)$ equipped with the norm $\|x + p^{-1}(0)\| := p(x)$ for every $p = p_\gamma$. Consider the operator $k_p : X \to X_p$ defined by $k_p(x) = x + p^{-1}(0)$. Note that $k_p^{-1}(B_{X_p}) = p^{-1}([0,1])$, so that the projective limit of $\{(X_p, k_p) \mid p \in \{p_\gamma\}_\gamma\}$ is exactly the original topology on X.

11.2. **Product of locally convex spaces.** Let X as a vector space be equal to the product $\prod_\gamma X_\gamma = \{(x_\gamma) \mid x_\gamma \in X_\gamma\}$. Denote the projections by $\pi_\gamma : X \to X_\gamma$ and the injections by $j_\gamma : X_\gamma \to X$ defined by $\pi_{\gamma_0}((x_\gamma)) = x_{\gamma_0}$ and

$$j_{\gamma_0}(x)_\gamma = \begin{cases} x, \text{ if } \gamma = \gamma_0, \\ 0, \text{ otherwise.} \end{cases}$$

Note that

$$\bigcap_\gamma \pi_\gamma^{-1}(\{0\}) = \{0\}.$$

Exercise 11.1. Prove that j_γ is linear and injective.

Exercise 11.2. Check that $\pi_\gamma \circ j_\gamma = I_{X_\gamma}$ (identity) and $\pi_\nu \circ j_\gamma = 0$ if $\nu \neq \gamma$.

Exercise 11.3. Check that $\pi_\gamma|_{j_\gamma(X_\gamma)} = j_\gamma^{-1}$.

Consider the projective limit topology on X defined by pairs $\{(X_\gamma, \pi_\gamma)\}_\gamma$. The next proposition observes the fact that it coincides with the usual product topology.

Proposition 11.4. *A net $(x^\alpha) \subset X$ converges to $x \in X$ if and only if $\pi_\gamma(x^\alpha) \to_\alpha \pi_\gamma(x)$ for all γ.*

Proof. It is enough to show that for a filter $\mathscr{F}$ on X one has $\mathscr{F} \to 0$ if and only if $\pi_\gamma(\mathscr{F}) \to 0$ for all γ. This is true, because the former means that $\mathscr{N}_X \subset \mathscr{F}$ or, equivalently, $\mathscr{P}_X \subset \mathscr{F}$ for any prebase of $\mathscr{N}_X$, while the latter means $\mathscr{N}_{X_\gamma} \subset \pi_\gamma(\mathscr{F})^\uparrow$ or $\pi_\gamma^{-1}(\mathscr{N}_{X_\gamma}) \subset \mathscr{F}$. □

Proposition 11.5. *A net $(x^\alpha) \subset X$ is Cauchy if and only if the net $(\pi_\gamma(x^\alpha))_\alpha$ is Cauchy for all γ.*

Proof. This follows from the above by noticing that $T(\mathscr{F} - \mathscr{F}) = T(\mathscr{F}) - T(\mathscr{F})$ for any prefilter $\mathscr{F}$ and a linear operator T. □

The next two observations are just special cases of their projective limit versions.

Proposition 11.6. *Let Y be an LCS. A linear operator $T : Y \to X$ is continuous if and only if so are all $\pi_\gamma \circ T$, $\gamma \in \Gamma$.*

Proposition 11.7. *A subset $E \subset X$ is bounded if and only if so are all $\pi_\gamma(E) \subset X_\gamma$, $\gamma \in \Gamma$.*

Proposition 11.8. *X is Hausdorff if and only if so are all X_γ, $\gamma \in \Gamma$.*

Proof. Proposition 11.1 gives the sufficiency. For the necessity note that j_γ is injective and

$$\bigcap \mathscr{N}_X = \bigcap_{\gamma\in\Gamma} \pi_\gamma^{-1}\left(\bigcap \mathscr{N}_{X_\gamma}\right) \supset j_\gamma\left(\bigcap \mathscr{N}_{X_\gamma}\right),$$

because $j_\gamma(X_\gamma) \subset \pi_\beta^{-1}(0)$ if $\gamma \neq \beta$. □

Proposition 11.9. *A closed subset $E \subset X$ is complete if and only if $\pi_\gamma(E) \subset X_\gamma$ is complete for all $\gamma \in \Gamma$.*

Proof. Sufficiency. Using Propositions 11.6 and 11.5 for any Cauchy filter $\mathscr{F}$ on E it is straightforward to obtain $x \in X$ such that $\mathscr{F} \to x$. Since E is closed, $x \in E$. Necessity. Given a Cauchy net $(x_\alpha) \subset \pi_\gamma(E)$, note that $(j_\gamma(x_\alpha)) \subset E$ is Cauchy by Proposition 11.6, because $\pi_\beta \circ j_\gamma = 0$ if $\beta \neq \gamma$, so that $(j_\gamma(x_\alpha))$ converges to $x \in E$ and hence also $((\pi_\gamma \circ j_\gamma)x_\alpha) = (x_\alpha)$ to $\pi_\gamma(x)$. □

Corollary 11.10. *The product of complete LC spaces is a complete LCS.*

Proposition 11.11. *A projective limit $(X_0, \tau_{\text{proj}})$ of $\{(X_\gamma, v_\gamma)\}_{\gamma\in\Gamma}$, such that $\bigcap_\gamma v_\gamma^{-1}(\{0\}) = \{0\}$, is isomorphic to a subspace of $\prod_\gamma X_\gamma$.*

Proof. Define $T : X_0 \to X := \prod_\gamma X_\gamma$ by $Tx = (v_\gamma x)_\gamma$. Then $T^{-1}(0) = \bigcap_\gamma v_\gamma^{-1}(\{0\})$, so that T is injective and we get a linear $T^{-1} : T(X_0) \to X_0$. Note that $v_\gamma = \pi_\gamma \circ T$ and hence $\pi_\gamma = v_\gamma \circ T^{-1}$, so that both T and T^{-1} are continuous by Proposition 11.2. □

Corollary 11.12. *Every Hausdorff LCS is isomorphic to a subspace of a product of normed spaces.*

Proof. This follows from Example 11.5, Proposition 11.1, and the above. □

Proposition 11.13. *Let $X = \prod_\gamma X_\gamma$. For every γ, $j_\gamma : X_\gamma \to j_\gamma(X_\gamma) \subset X$ is an isomorphism. If all X_γ are Hausdorff, then $j_\gamma(X_\gamma)$ is closed in X.*

Proof. Exercises 11.1 and 11.2, together with Proposition 11.2, give that j_γ is linear and continuous. Exercise 11.3 gives that so is j_γ^{-1}, too. Note that $j_\gamma(X_\gamma) = \bigcap_{\beta\neq\gamma} \pi_\beta^{-1}(0)$ and that $\pi_\beta^{-1}(0)$ is closed if X_β is Hausdorff. □

12. Inductive limits. Bornological spaces.

12.1. **Inductive limit topology.** Fix a vector space X, for all $\gamma \in \Gamma$ fix locally convex spaces X_γ and linear operators $u_\gamma : X_\gamma \to X$ such that

$$X = \operatorname{span} \bigcup_\gamma u_\gamma(X_\gamma).$$

Definition. The strongest LC topology on X such that all operators u_γ are continuous is called the ***inductive limit topology*** (induced by pairs $\{(X_\gamma, u_\gamma) \mid \gamma \in \Gamma\}$). It is denoted by τ_{ind}. The space (X, τ_{ind}) is called the ***inductive limit*** of these pairs.

Exercise 12.1. Prove that the topology τ_{ind} really exists.

Proposition 12.1. *An absolutely convex absorbing set $U \subset X$ is a zero neighbourhood of τ_{ind} if and only if $u_\gamma^{-1}(U)$ is in $\mathscr{N}_{X_\gamma}$ for all $\gamma \in \Gamma$.*

Proposition 12.2. *A zero neighbourhood base of τ_{ind} is*

$$\mathscr{B} := \{\text{absconv} \bigcup_{\gamma} u_\gamma(V_\gamma) \mid V_\gamma \in \mathscr{N}_{X_\gamma}\}.$$

Proposition 12.3. *An inductive limit of barrelled spaces is a barrelled space.*

Proposition 12.4. *Let Y be an LCS. A linear operator $T : X \to Y$ is continuous if and only if so are all $T \circ u_\gamma : X_\gamma \to Y$, $\gamma \in \Gamma$. A set $A \subset L(X,Y)$ is equicontinuous if so are all $\{T \circ u_\gamma \mid T \in A\}$, $\gamma \in \Gamma$.*

Let $\langle X, X'\rangle$ and $\langle Y, Y'\rangle$ be dual pairs. The ***adjoint*** of a linear operator $T : X \to Y$ is the operator $T' : Y' \to X^*$ defined by $T'(g) = g \circ T$. Note that (prove it!)✠ $T'(Y') \subset X'$ if and only if T is weakly continuous, that is, $T : (X, \sigma(X, X')) \to (Y, \sigma(Y, Y'))$ is continuous. If X and Y are Hausdorff LCS and $T : X \to Y$ is continuous, then (prove it!)✠ T is weakly continuous, that is, $T'(Y') \subset X'$.

The inductive limit topology is a polar topology.

Proposition 12.5. *If all X_γ and (X, τ_{ind}) are Hausdorff LCS, then $\tau_{\text{ind}} = \mathscr{T}_{\mathfrak{S}}$, where $\mathfrak{S}$ is the system of all sets $S \subset (X, \tau_{\text{ind}})'$ such that $u'_\gamma(S) \subset X'_\gamma$ is τ_γ-equicontinuous for all γ.*

The inductive limit and the projective limit are dual in some sense.

Proposition 12.6. *Let all X_γ and (X, τ_{ind}) be Hausdorff LCS. Assume that for all γ, there is a certain polar topology $\mathscr{T}_{\mathfrak{S}_\gamma}$ on X'_γ, where $\mathfrak{S}_\gamma$ is some system of $\sigma(X_\gamma, X'_\gamma)$-bounded sets.*

Denote by $\mathfrak{S}$ the system of all finite unions of sets of the form $u_\gamma(S_\gamma)$, where $S_\gamma \in \mathfrak{S}_\gamma$. Then the polar topology $\mathscr{T}_{\mathfrak{S}}$ on $(X, \tau_{\text{ind}})'$ is the projective limit of $\{((X'_\gamma, \mathscr{T}_{\mathfrak{S}_\gamma}), u'_\gamma)\}_\gamma$.

12.2. **Bornological spaces.** Note that (prove it!)✠ a linear continuous operator T between LC spaces X and Y is always ***bounded***, that is, it maps bounded sets to bounded sets. Recall that for linear operators between normed spaces, we have the reverse: a linear operator is bounded if and only if it is continuous.

Definition. An LCS X is a ***bornological space*** if every absolutely convex set $U \subset X$ that absorbs every bounded set, is a zero neighbourhood.

Proposition 12.7. *Let X be an LCS. The following are equivalent:*

(a) *X is a bornological space,*
(b) *for every LCS Y every bounded linear operator $T : X \to Y$ is continuous.*

Proof. Schema for (b) $\Longrightarrow$ (a): Take $U \subset X$ as in the definition of bornological spaces, then it is absorbing and hence its Minkowski functional p_U is a seminorm. Note that the identity $i : X \to (X, p_U)$ is bounded and hence continuous. This implies that U is a zero neighbourhood of X. □

Note that (prove it!)✠ if (X, τ) is a Hausdorff bornological space, then τ is the Mackey topology $\tau(X, X')$. Recall that we had the similar claim for the metrizable spaces.

Proposition 12.8. *Every metrizable LCS is bornological.*

Proof. Adapt the proof of 9.13. □

Proposition 12.9. *An inductive limit of bornological spaces is bornological.*

Proof. Use Prop. 12.7 together with Prop. 12.4. □

The above two propositions imply that an inductive limit of metrizable LC spaces is bornological. This statement can be reversed in the following sense.

Theorem 12.10. *A Hausdorff LCS is bornological if and only if it is an inductive limit of normed spaces. A complete Hausdorff LCS is bornological if and only if it is an inductive limit of Banach spaces.*

The theorem above follows from

Lemma 12.11. *Let (X, τ) be a Hausdorff LCS. On X, there exists the strongest LC topology τ' with the same bounded sets as τ. Moreover, (X, τ') is a bornological space and it is an inductive limit of vector subspaces of X equipped with some norms. The topologies τ and τ' coincide if and only if (X, τ) is bornological. If (X, τ) is complete, then (X, τ') is an inductive limit of Banach spaces.*

Schema of the proof. Denote by S the collection of all closed bounded absolutely convex sets in (X,τ). For every $A \in S$, its Minkowski functional p_A is a norm on $X_A := \operatorname{span} A$, the injections $u_A : (X_A, p_A) \to (X,\tau)$ are continuous (see the proof of Prop. 9.12), and $X = \bigcup_{A\in S} u_A(X_A)$. Define τ as the inductive limit of $\{(X_A, u_A) \mid A \in S\}$. Derive the rest of the claims. □

Remarks:

- In the proof of the above lemma, we can replace the system of all closed bounded absolutely convex sets S with any of its fundamental systems S_0, that is, such that for every $A \in S$ there is $B \in S_0$ with $A \subset B$. This implies that if the space (X,τ) admits a countable fundamental system of bounded sets, then it can be represented as an inductive limit of countably many normed spaces.
- In general, the classes of bornological spaces and barrelled spaces are incomparable. However, using the above propositions we can prove (prove it!)✠ that every complete Hausdorff bornological space is barrelled.

13. Special cases of inductive limits: quotient, direct sum, strict inductive limit

13.1. **Quotient space.** Let X be a vector space and let $M \subset X$ be its subspace. The set $X/M := \{x + M \mid x \in X\} \subset 2^X$ becomes a vector space with M being the zero element and addition and scalar multplication defined pointwise with the exception that $0\cdot(x+M) := M$. (prove it!)✠

This vector space X/M is called the ***quotient*** of X with respect to M. The ***canonical projection*** $k : X \to X/M$, $k : x \mapsto x + M$, is linear and surjective. (prove it!)✠

If X is a TVS, then the prefilter $k(\mathscr{N})$ defines a linear topology on X/M, called the ***quotient topology***. It is LC if X is LC. Note that k is continuous with respect to these topologies. (prove it!)✠

Proposition 13.1. *Let X be an LCS. The quotient topology on X/M is Hausdorff if and only if M is closed.*

Proof. If X/M is Hausdorff, then $\{M\} \subset X/M$ is closed, so that $k^{-1}(\{M\}) = M$ is closed. On the other hand, note that $k^{-1}k(A) = A + M$ for all $A \subset X$, so that $k^{-1}k(\mathscr{N}) = M + \mathscr{N}$ and

$$\bigcap k(\mathscr{N}) = k\left(\bigcap k^{-1}k(\mathscr{N})\right) = k\left(\bigcap (M+\mathscr{N})\right) = k\left(\overline{M}\right) = \{M\},$$

where the last equality holds if M is closed.

□

Note that the quotient topology can be induced by seminorms $p_{k(U)}$ with absolutely convex $U \in \mathscr{B}$ forming a base of $\mathscr{N}$. Observe that $p_{k(U)}(x+M) = \inf\{p_U(y) \mid y \in x + M\}$.

It is important to note that the quotient topology is the inductive limit of $\{(X,k)\}$. (prove it!)✠The above proposition then shows that the inductive limit of Hausdorff spaces may fail to be Hausdorff.

A special case of Proposition 12.3 is

Proposition 13.2. *A quotient of a barrelled space is barrelled.*

Proposition 13.3. *Let T be a linear operator between LC spaces X and Y. Then T can be factorized as $T = S \circ k$, where $S : X/\ker T \to Y$ is an injective linear operator and k is the canonical projection onto $X/\ker T$. Moreover, T is continuous if and only if so is S.*

Proposition 13.4. *The topological dual $(X/M)'$ is algebraically isomorphic to $M^\perp := \{f \in X' \mid f(M) = \{0\}\} \subset X'$.*

Proof. The isomorphism is k', the adjoint of k, defined by $g \in (X/M)' \mapsto g \circ k$. □

Let $\langle X, Y\rangle$ be a dual pair and let a subspace $M \subset X$ separate points of Y. Then $\langle M, Y/M^\perp\rangle$ is a dual pair and

Proposition 13.5. $\sigma(M, Y/M^\perp) = \sigma(M,Y) = \sigma(X,Y)|_M$.