Computational Finance, Spring 2022
Computer Lab 11

The aim of the lab is to implement the basic implicit method and Crank-Nicolson method for computing European prices of European options. For this we consider the problem

$$
\begin{aligned}
\frac{\partial u}{\partial t}(x,t) + \alpha(x,t)\frac{\partial^2 u}{\partial x^2}(x,t) + \beta(x,t)\frac{\partial u}{\partial x}(x,t) - r\,u(x,t) = 0,\ x \in (x_{min}, x_{max}), 0 \le t < T,\\
u(x_{min}, t) = \phi_1(x_{min}, t), 0 \le t < T,\\
u(x_{max}, t) = \phi_2(x_m ax, t), 0 \le t < T,\\
u(x, T) = p(e^x),\ x \in (x_{min}, x_{max}).
\end{aligned}
$$

We introduce the points $x_i = x_{min} + i\Delta x,\ i = 0, \dots, n$ and $t_k = k\Delta t,\ k = 0, \dots, m$ and denote by $U_{i,k}$ the approximate values of $u(x_i, t_k)$. Here $\Delta x = \frac{x_{max} - x_{min}}{n}$ and $\Delta t = \frac{T}{m}$. In the case of the basic implicit finite difference method we compute the values U_{ik} as follows: using the final condition we set

$$U_{im} = p(e^{x_i}),\ i = 0, \dots, n$$

and for determining the values of $U_{ik},\ i = 0, \dots, n,\ k = m-1, \dots, 0$ we solve for each value of k (starting with $k = m-1$) a three-diagonal system

$$
\begin{aligned}
U_{0k} &= \phi_1(x_{min}, t_k),\\
a_{ik}U_{i-1,k} + b_{ik}U_{ik} + c_{ik}U_{i+1,k} &= U_{i,k+1},\ i = 1, \dots, n-1,\\
U_{nk} &= \phi_2(x_{max}, t_k)
\end{aligned}
$$

for the unknown values of $U_{ik},\ i = 0, \dots, n$. Here

$$
\begin{aligned}
a_{ik} &= -\frac{\alpha(x_i, \tau_k)\Delta t}{\Delta x^2} + \frac{\beta(x_i, t_k)\Delta t}{2\Delta x},\\
b_{ik} &= 1 + \frac{2\alpha(x_i, t_k)\Delta t}{\Delta x^2} + r\Delta t,\\
c_{ik} &= -\frac{\alpha(x_i, t_k)\Delta t}{\Delta x^2} - \frac{\beta(x_i, t_k)\Delta t}{2\Delta x}.
\end{aligned}
$$

In the case of Crank-Nicolson method we solve at each time step the system

$$
\begin{aligned}
U_{0k} &= \phi_1(x_{min}, t_k),\\
a_{ik}U_{i-1,k} + b_{ik}U_{ik} + c_{ik}U_{i+1,k} &= d_{ik}U_{i-1,k+1} + e_{ik}U_{i,k+1} + f_{ik}U_{i+1,k+1},\ i = 1, \dots, n-1,\\
U_{nk} &= \phi_2(x_{max}, t_k),
\end{aligned}
$$

where

$$
\begin{aligned}
a_{ik} &= -\frac{\alpha(x_i,t_k)\Delta t}{2\Delta x^2} + \frac{\beta(x_i,t_k)\Delta t}{4\Delta x},\\
b_{ik} &= 1 + \frac{\alpha(x_i,t_k)\Delta t}{\Delta x^2} + \frac{r\Delta t}{2},\\
c_{ik} &= -\frac{\alpha(x_i,t_k)\Delta t}{2\Delta x^2} - \frac{\beta(x_i,t_k)\Delta t}{4\Delta x},\\
d_{ik} &= \frac{\alpha(x_i,t_{k+1})\Delta t}{2\Delta x^2} - \frac{\beta(x_i,t_{k+1})\Delta t}{4\Delta x},\\
e_{ik} &= 1 - \frac{\alpha(x_i,t_{k+1})\Delta t}{\Delta x^2} - \frac{r\Delta t}{2},\\
f_{ik} &= \frac{\alpha(x_i,t_{k+1})\Delta t}{2\Delta x^2} + \frac{\beta(x_i,t_{k+1})\Delta t}{4\Delta x}.
\end{aligned}
$$

Exercise 1. Write a function that for given values of m, n, x_{min}, x_{max}, T and for given functions p,σ, ϕ_1 and ϕ_2 returns the values $U_{i0},\ i = 0,\dots,n$ of the approximate solution (option prices) obtained by solving the transformed BS equation by the implicit finite difference method. Use this method for computing approximate values of the option price in the case $r = 0.02$, $\sigma(s,t) = 0.5$, $D = 0.03$, $T = 0.5$, $E = 100$, $S0 = 98$, $p(s) = 2 \cdot |E - s|$, $\rho = 2$, $x_{min} = \ln\frac{S0}{\rho}$, $x_{max} = \ln(\rho\, S0)$, $n = 20$, $m = 100$. Use $\phi_1(x_{min},t) = p(e^{x_{min}})$, $\phi_2(x_{max},t) = p(e^{x_{max}})$

Exercise 2. Repeat the previous exercise in the case of boundary conditions derived from special solutions.

Exercise 3. Implement CN method for solving the untransformed BS equation. Use it for finding the an approximate price at $t = 0$ for the option described in problem 1 in the case of $x_{min} = 0$, $\rho = 3$, $x_{max} = \rho S_0$, $m = 50$, $n = 120$, the exact boundary condition at x_{min} and constant boundary condition at $x = x_{max}$. Also find the actual error of the answer.