GEOMETRY AND PHYSICS

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Geometry and Numbers

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- **Since very ancient times the shapes and sizes of material objects interested our ancestors. The vast spaces on Earth and majesty of stars and planets were no less fascinating.**
- **To describe them, they used numbers, first the simplest ones, then their more and more sophisticated versions. Quantitative relationships between the observable phenomena were given a formal mathematical expression.**

Geometry and Numbers

- **Babylonians have found particular relationship between the lenghts of sides of certain rectangular triangles. They noted them as the so called** *babylonian triplets*, **e.g.** $(3, 4, 5)$, $(6, 8, 10)$, $(5, 12, 13)$, $(7, 24, 25)$, **, etc.**

Geometry and Numbers

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- **But they did not generalize this to any numbers known to them, like fractions**

Geometry and Numbers

Figure: Babylonian tablet with triplets defining right triangles.

Geometry in China

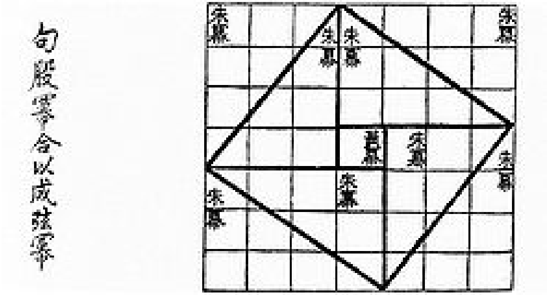
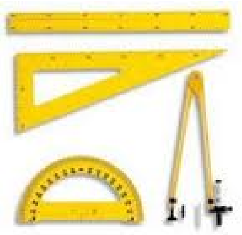


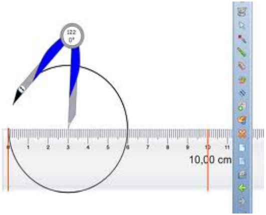
Figure: A Chinese way of proving the Pythagorean relationship - but only on one particular example.

Geometry in Ancient Greece

Ancient Greece, the cradle of modern geometry.

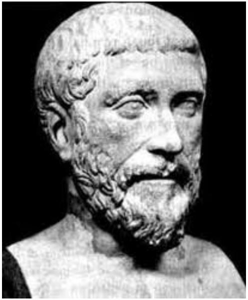
Geometry in Ancient Greece





Euclid ($E\upsilon\kappa\lambda\iota\delta\eta\sigma$, $-340 \sim -275$) has epitomized the geometrical knowledge accumulated during previous generations of Greek and Egyptian scientists. Euclidean geometry is based on definition of straight lines and right angles.

Geometry and Numbers



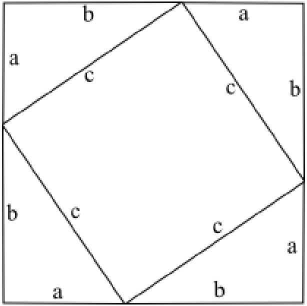
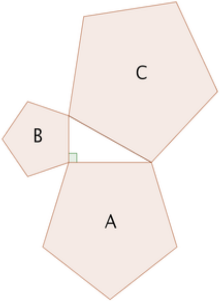
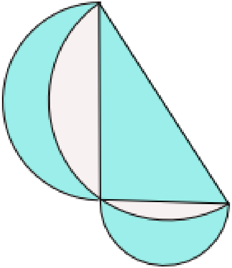


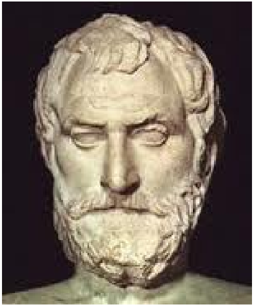
Figure: Pythagoras $(\Pi\upsilon\theta\gamma o\rho\alpha\sigma,\ \ -580 \sim -495)$ was perhaps the first to generalize the theorem. Since then, more than 60 different proofs were invented. One of them is given in the figure above.

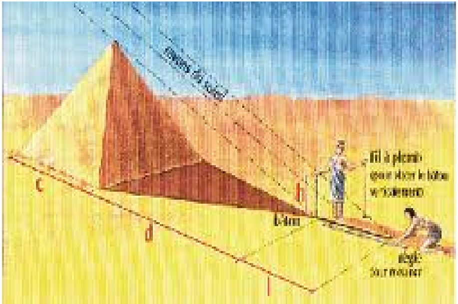




The Pythagorean theorem can be generalized for arbitrary shapes, as well as to three dimensions, and even more than three. The Fourier series and Parceval's theorem are just the application of Pythagorean theorem to infinite dimension.

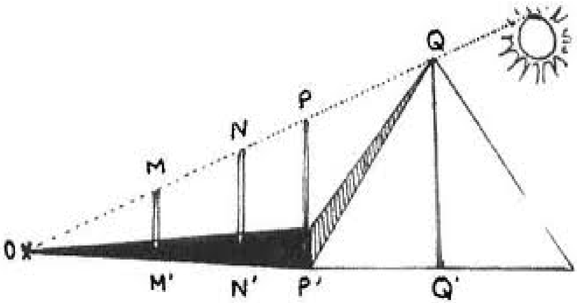
Geometry, Matter and Light





Thales of Miletus ($\Theta\alpha\lambda\eta\sigma$, $-625 \sim -547$) has become famous for his pediction of solar eclipse of -585, and of his ability to evaluate dimensions of objects at a distance, by comparing their shadows.

Geometry, Matter and Light



MM' : OM = QQ' : OQ

Geometry and Reality

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Geometry and Reality

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- **It has become such a commonplace, that the physical aspects of this fundamental experiment are rarely considered in a more detailed manner.**

Geometry and Reality

- **In fact, Thales has performed an important physical experiment relating different definitions of geometry**

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- Let us analyze the premices and hypotheses that enabled Thales to draw his conclusions and to state the theorem of parallel lines cutting an angle formed by two non-parallel straight lines.

Geometry and Reality

- **The first two assumptions are that the line OQ' on the ground is indeed a straight line, and that the two segments, the height of the Pyramid QQ' and the stick MM' are also straight, and form the same angle with the line of the ground $OM'Q'$ (in this case, the right angle of 90^o).**

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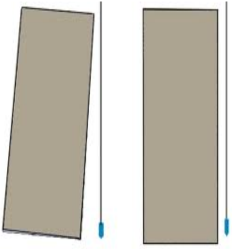
- This is a physical statement, and the fact that the two objects are straight and vertical was checked using of the well known instruments based on the exploitation of gravity.

Geometry and Material World



Two instruments used for checking whether a straight line or a plane is vertical or horizontal. Both are based on the use of terrestrial gravity defining local vertical direction and horizontal planes (equipotential surfaces).

Geometry, Matter and Light

- **The fact that the two segments are vertical and straight is based on the assumption that the string sustaining a heavy object in the gravitational field on the surface of Earth may serve as a definition of *vertical straight line*. Checking the horizontality of the ground is performed using the same principle.**

Geometry, Matter and Light

- **The fact that the two segments are vertical and straight is based on the assumption that the string sustaining a heavy object in the gravitational field on the surface of Earth may serve as a definition of *vertical straight line*. Checking the horizontality of the ground is performed using the same principle.**
- **The fact that the stick remains straight and stiff is due to the assumption that it is made of a material whose cohesion is sufficient to keep its shape unchanged (a solid body).**

- As seen from our present perspective, this hypothesis is based on the assumption that atoms can form stable structures able to keep unchanged under reasonable conditions (e.g. the ambient temperature not exceeding certain values).

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- Incidentally, the ability of atoms and molecules to form stable periodic structures makes possible an alternative definition of straight lines and right (and not only right) angles. Crystals represented in Fig. (4) show remarkable linear structure as well as apparently perfect angles, 90^o in the case of cubic lattice of $NaCl$, and 60^o and 120^o in the case of quartz (SiO_2).

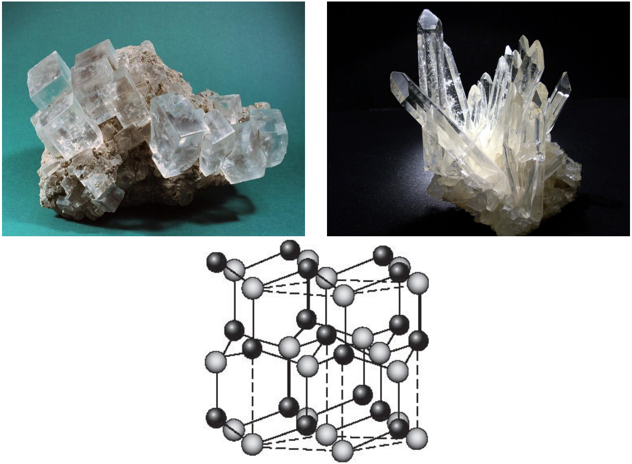

Figure: Crystals of ordinary salt *NaCl*, of quartz SiO_2, and an example of crystalline lattice (SiO_2 - wurtzite). The interatomic forces impose the shapes and the geometry of solid bodies.

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- The geometry based on solid bodies' shapes is independent of gravitational field that determines parallel vertical lines and the horizontal plane in Thales' experiment. From the present point of view, the existence of stable configurations of atoms, as well as that of atoms themselves, can be understood only using the principles of quantum mechanics, until now seemingly independent of gravitational phenomena.

Geometry, Matter and Light

- **The results of Thales' experiment can be interpreted in two ways. In fact, he established the coincidence of three completely different definitions of a straight line. The first came from the natural shape a string with a heavy body attached at its end takes in the gravitational field.**

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- The second definition comes from the material shape of the stick. The third is just a light ray.
- With the height of the pyramid considered as a known quantity, as well as the height of the stick, the result of Thales can be interpreted as a proof that the light rays follow straight lines, too.

Geometry and Reality

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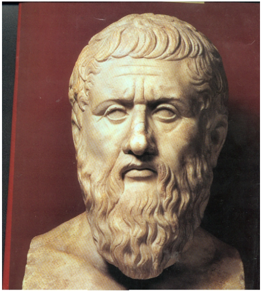
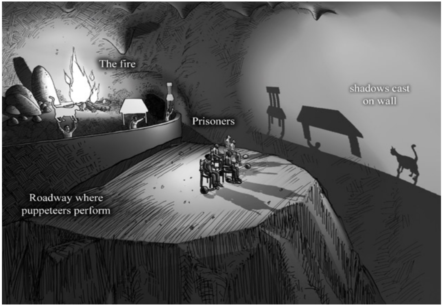
- **Another astounding Thales' discovery was the possibility of replacing real objects by their shadows, or images, which may serve as a model for measurements. The proportions remain the same.**
- **This interpretation must have impressed Plato, who generalized the idea to higher dimensions, including 3. If shadow can replace the pyramid and the stick, it might be that pyramid and stick are themselves "shadows' of some higher dimensional objects ?**

Geometry and Reality



Bust of Plato

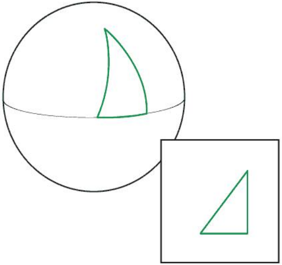


Plato ($\Pi\lambda\alpha\tau o\nu$ $-424 \sim -348$) and his metaphor of the cave (in "The Republic"). In a sense, Plato was a precursor of the Kaluza-Klein theory as well as theory of strings evolving in extra dimensions.

Geometry and Reality

Thales made also an extra tacit assumption, namely, that the properties used for the definition of straight lines, parallelism and right angles were scale independent, i.e. they ware the same for the small stick and for the Great Pyramid.
The extension towards even greater dimensions, including the Skies, seemed also obvious.

Geometry and Reality

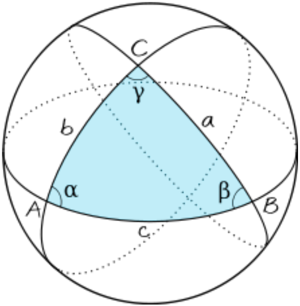


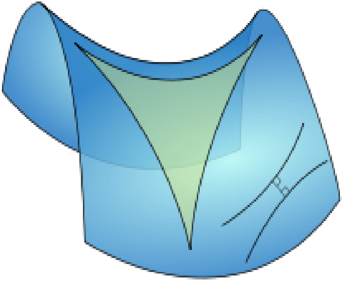
About twenty-two centuries later C.F.Gauss underlined the need for the experimental check of Euclid's fifth postulate.

Geometry and Reality

B. Riemann and N. Lobachevsky have defined two versions of Non-Euclidean geometry..

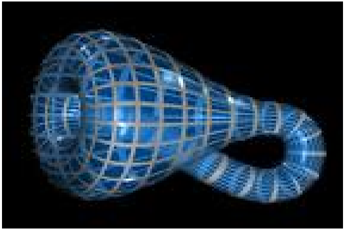
Geometry and Reality





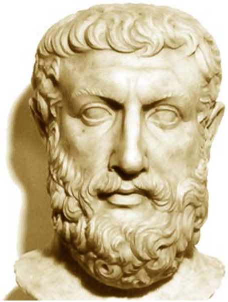
Two non-Euclidean geometries: the spherical (Riemannian) and the hyperbolic (Lobachevsky). In the first, the sum of angles of a triangle presents an excess, in the second a deficit.

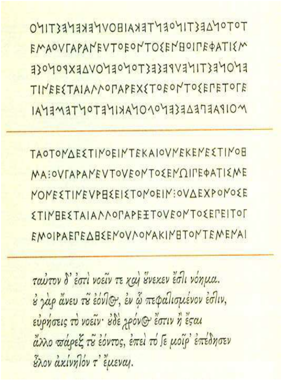
Geometry and Reality



Felix Klein gave the modern definition of geometry identified with the group of transformations under which the relationships between its objects remain invariant.

The Universe of matter





ταὐτὸν δ' ἐστὶ νοεῖν τε καὶ ἕνεκεν ἔστι νόημα.
οὐ γὰρ ἄνευ τοῦ ἐόντος, ἐν ᾧ πεφατισμένον ἐστίν,
εὑρήσεις τὸ νοεῖν· οὐδὲ χρόνος ἔστιν ἢ ἔσται
ἄλλο πάρεξ τοῦ ἐόντος, ἐπεὶ τό γε μοῖρ' ἐπέδησεν
οὖλον ἀκίνητόν τ' ἔμεναι.

The Ancient Greek philosopher Parmenides ($\Pi\alpha\rho\mu\epsilon\nu\eta\delta\eta\sigma$) of Elea

Of matter and space-time

- **In Parmenides' view, the totality of all things that exist can be defined as belonging to what he called *BEING*, and the only thing one can say about the *NON-BEING* is that it does not exist.**

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- Therefore a totally empty space devoid of any being (matter) cannot exist.
- As a corollary, one infers that space (and time) cannot exist by themselves, but only as by-products of matter. In today's fashionable language, they are *EMERGING ENTITIES*.

Of space, matter and motion

Aristotle ($A\rho\iota\sigma\tau o\tau\epsilon\lambda\eta\sigma$), $-390 \sim -320$ was to physics what Euclid was for geometry. He shared Parmenides' view that empty space ("void") cannot exist.

Space, matter and interaction

- **Aristoteles also introduced the notion of action at a distance, a kind of universal attraction, but acting only among similar elements: water attracted to water, air to air, fire to fire and metal (earth) to metal.**

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- **The four Aristotelean interactions prefigured the elementary forces of matter in contemporary physics of elementary particles.**

Of space, matter and motion

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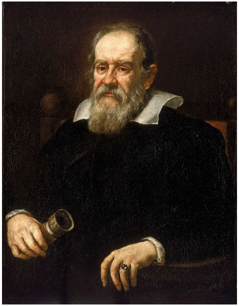
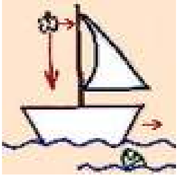
- **According to Aristotelean physics, all motions were composed of two types : the motions along straight lines (rectilinear motion), and along circles (the circular motions)** Which turned out to be true, but only infinitesimally (Steiner's theorem in Mechanics)
- On the Earth, one could observe both types of motions, whereas in the Sky only circular motions were observed. There was no notion of acceleration.

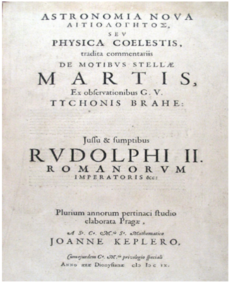
Of space, matter and motion



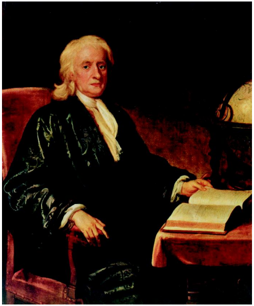


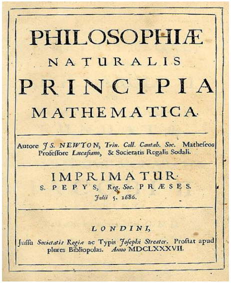
$$t' = t$$
$$x' = x - vt$$
$$y' = y$$
$$z' = z$$

It took about XIX centuries until Galileo shattered the Aristotelean views on the nature of motion. This also has set foundations to the idea of relativity of space.


ASTRONOMIA NOVA
ΑΙΤΙΟΛΟΓΗΤΟΣ,
SEV
PHYSICA COELESTIS,
tradita commentariis
DE MOTIBVS STELLÆ
MARTIS,
Ex obſervationibus G. V.
TYCHONIS BRAHE:
Juſſu & ſumptibus
RVDOLPHI II.
ROMANORVM
IMPERATORIS &c:
Plurium annorum pertinaci ſtudio
elaborata Pragæ,
JOANNE KEPLERO,

The trajectories of planets calculated by Johannes Kepler confirmed the validity of Thales' and Pythagoras' theorems inside the Solar System.





PHILOSOPHIÆ
NATURALIS
PRINCIPIA
MATHEMATICA

Autore J S. NEWTON, Trin. Coll. Cantab. Soc. Matheſeos Profeſſore Lucaſiano, & Societatis Regalis Sodali.

IMPRIMATUR
S. PEPYS, Reg. Soc. PRÆSES.
Julii 5. 1686.

LONDINI,
Juſſu Societatis Regiæ ac Typis Joſephi Streater. Proſtat apud plures Bibliopolas. Anno MDCLXXXVII.

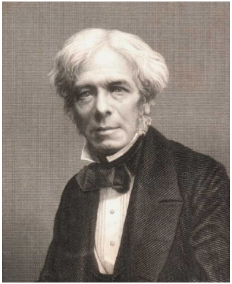
Isaac Newton's physics was based on the Euclidean geometry and the Galileo's and Kepler's laws of motion. Newton introduced the notion of absolute space and time, deprived of any material content besides the compact bodies, interacting at a distance. The light was composed of tiny elastic particles obeying the same rules of mechanics.

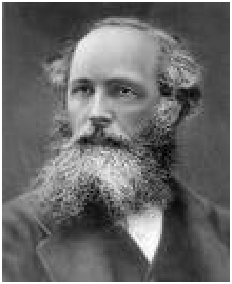
The controversy concerning the nature of light led to deep differences in the interpretation of space. For Huygens, who proved the wave-like propagation of light, space must be filled with some medium enabling the propagation.

Two diametrally opposite views on the status of space and motion prevailed since then. The Newtonian view was reinforced by Immanuel Kant, who raised the status of space to the independent and absolute category, like the starry sky and the "moral imperative".

For Berkeley, only relative motion of material bodies could be observable, and the notions of space and time were only corollaries of the behavior of material bodies.

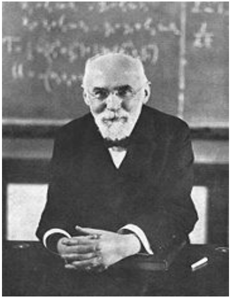
Ernst Mach extended Berkeley's reasoning as to include the very property of inertia of material bodies, interpreted as the result of gravitational interaction with the totality of remote stars and galaxies.





M. Faraday and J.C. Maxwell introduced a revolutionary, anti-Aristotelean and anti-Newtonian point of view according to which no interaction at a distance is possible. All forces are transmitted by a medium; the space is filled with it. It can be called "aether", and the fields of forces become a new physical realm.

The Lorentz-Poincaré group



H.A. Lorentz and H. Poincaré established the group transformations for the electromagnetic field.

Of matter and space-time

- **The Lorentz and Poincaré groups were established as symmetries of the observable macroscopic world.**

Of matter and space-time

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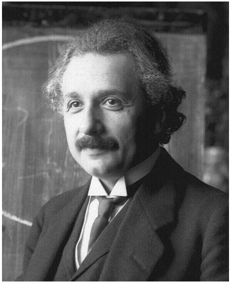
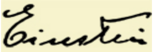
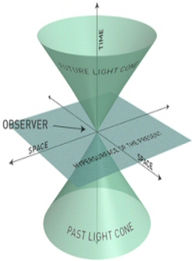
- More precisely, they were conceived in order to take into account the relations between electric and magnetic fields as seen by different Galilean observers.
- Only later on Einstein extended the Lorentz transformations to space and time coordinates, giving them a universal meaning. As a result, the Lorentz symmetry became perceived as group of invariance of Minkowskian space-time metric.





A. Einstein extended these symmetries to all physical phenomena, including mechanics and gravity.



In the spirit of F. Klein's programme, H. Minkowski defined the hyperbolic geometry of space united with time in a single entity named *"space-time manifold"*. Its geometry was defined by the action of the Lorentz-Poincaré group.

Of matter, time and space

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- **The three realms of Physics**
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- **Space and time**
- **Material bodies**
- **Forces acting between them**

Fundamental relationship

- **Newton's third law of dynamics**

$$\mathbf{a} = \frac{1}{m}\,\mathbf{F}. \tag{1}$$

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- **shows the relation between three different realms which are dominant in our perception and description of physical world: massive bodies (m), force fields responsible for interactions between the bodies ("F") and space-time relations defining the acceleration ("a").**

Three realms

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Three realms

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- **we deliberately wrote**

$$\mathbf{a} = \frac{1}{m}\,\mathbf{F}. \tag{2}$$

in order to separate the directly observable entity (a) from the product of two entities whose definition is much less direct and clear.

Causal relationship

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- **In modern language, the notion of force is generally replaced by that of a field.**

Causal relationship

- Also, by putting the acceleration alone on the left-hand side, we underline the causal relationship between the phenomena: the force is the cause of acceleration, and not vice versa.
- In modern language, the notion of force is generally replaced by that of a field.
- The fact that the three ingredients are related by the equation (1) may suggest that perhaps only two of them are fundamentally independent, the third one being the consequence of the remaining two.

Three aspects

The three aspects of theories of fundamental interactions can be symbolized by three orthogonal axes, as shown in following figure, which displays also three choices of pairs of independent properties from which we are supposed to be able to derive the third one.

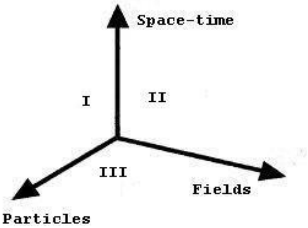

Figure: The three realms of Physics

Three types of theories

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Three types of theories

- The attempts to understand physics with only two realms out of three represented in (49) have a very long history. They may be divided in three categories, labeled *I*, *II* and *III* in the Figure.
- In the category *I* one can easily recognize Newtonian physics, presenting physical world as collection of material bodies (particles) evolving in absolute space and time, interacting at a distance. Newton considered light being made of tiny particles, too; the notion of fields was totally absent.

Three types of theories

- **Theories belonging to the category // assume that physical world can be described uniquely as a collection of fields evolving in space-time manifold. This approach was advocated by Kelvin, Einstein, and later on by Wheeler.**

Three types of theories

- **Theories belonging to the category *II* assume that physical world can be described uniquely as a collection of fields evolving in space-time manifold. This approach was advocated by Kelvin, Einstein, and later on by Wheeler.**
- **As a follower of Maxwell and Faraday, Einstein believed in the primary role of fields and tried to derive the equations of motion as characteristic behavior of singularities of the fields, or the singularities of the space-time curvature.**

Three types of theories

- **The category *III* represents an alternative point of view supposing that the existence of matter is primary with respect to that of the space-time, which becomes an "emergent" realm - an euphemism for "illusion". Such an approach was advocated recently by N. Seiberg and E. Verlinde.**

Three types of theories

- **The category *III* represents an alternative point of view supposing that the existence of matter is primary with respect to that of the space-time, which becomes an "emergent" realm - an euphemism for "illusion". Such an approach was advocated recently by N. Seiberg and E. Verlinde.**
- **It is true that space-time coordinates cannot be treated on the same footing as conserved quantities such as energy and momentum; we often forget that they exist rather as bookkeeping devices, and treating them as real objects is a "bad habit", as pointed out by D. Mermin.**

Three types of theories

- **Seen under this angle, the idea to derive the geometric properties of space-time, and perhaps its very existence, from fundamental symmetries and interactions proper to matter's most fundamental building blocks seems quite natural.**

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- **Many of those properties do not require any mention of space and time on the quantum mechanical level, as was demonstrated by Born and Heisenberg in their version of matrix mechanics, or by von Neumann's formulation of quantum theory in terms of the C^* algebras.**
- **The non-commutative geometry is another example of formulation of space-time relationships in purely algebraic terms.**

Three types of theories

- **In what follows, we shall choose the last point of view, according to which the space-time relations are a consequence of fundamental *discrete symmetries* which characterize the behavior of matter on the quantum level**

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- **In what follows, we shall choose the last point of view, according to which the space-time relations are a consequence of fundamental** *discrete symmetries* **which characterize the behavior of matter on the quantum level**
- **In other words, the Lorentz symmetry observed on the macroscopic level, acting on what we perceive as space-time variables, is an averaged version of the symmetry group acting in the Hilbert space of quantum states of fundamental particle systems.**

Of matter and space-time

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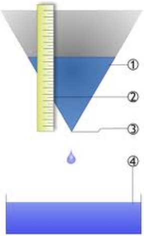
- **Extending the Lorentz transformations to space and time coordinates modified also Newtonian mechanics so that it could become invariant under the Lorentz instead of the Galilei group.**
- **In the textbooks introducing the Lorentz and Poincaré groups the accent is put on the transformation properties of space and time coordinates, and the invariance of the Minkowskian metric tensor $g_{\mu\nu} = diag(+,-,-,-)$.**
- **But neither the components of $g_{\mu\nu}$, nor the space-time coordinates of an observed event can be given an intrinsic physical meaning; they are not related to any conserved or directly observable quantities.**

Time and energy




The measuring of time is based on energy variations.

The Lorentz covariance

- **Under a closer scrutiny, it turns out that only TIME - the proper time of the observer - can be measured directly. The notion of space variables results from the convenient description of experiments and observations concerning the propagation of photons, and the existence of the universal constant c.**

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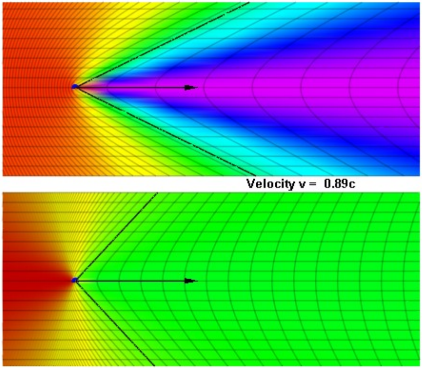
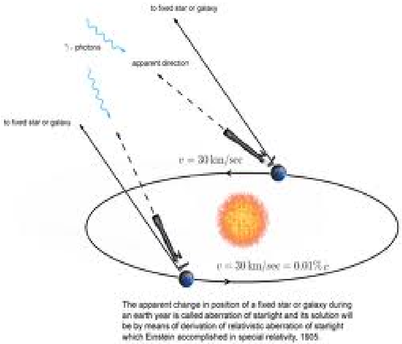
- Under a closer scrutiny, it turns out that only **TIME** - the proper time of the observer - can be measured directly. The notion of space variables results from the convenient description of experiments and observations concerning the propagation of photons, and the existence of the universal constant c.
- Consequently, with high enough precision one can infer that the Doppler effect is relativistic, i.e. the frequency ω and the wave vector $\mathbf{k}$ form an entity that is seen differently by different inertial observers, and passing from $\frac{\omega}{c}, \mathbf{k}$ to $\frac{\omega'}{c}, \mathbf{k}'$ is the Lorentz transformation.

The Lorentz covariance



Relativistic versus Galilean Doppler effect.

The Lorentz covariance



Aberration of light from stars. (Bradley, 1729)

The Lorentz covariance

Both effects, proving the relativistic formulae

$$\omega' = \frac{\omega - Vk}{\sqrt{1 - \frac{V^2}{c^2}}}, \quad k' = \frac{k - \frac{V}{c^2}\omega}{\sqrt{1 - \frac{V^2}{c^2}}},$$

have been checked experimentally by Ives and Stilwell in 1937.

The Lorentz covariance

- **Reliable experimental confirmations of the validity of Lorentz transformations concern measurable quantities such as charges, currents, energies (frequencies) and momenta (wave vectors) much more than the less intrinsic quantities which are the *differentials* of the space-time variables.**

The Lorentz covariance

- **Reliable experimental confirmations of the validity of Lorentz transformations concern measurable quantities such as charges, currents, energies (frequencies) and momenta (wave vectors) much more than the less intrinsic quantities which are the *differentials* of the space-time variables.**
- **In principle, the Lorentz transformations could have been established by very precise observations of the Doppler effect alone.**

The Lorentz covariance

- **It should be stressed that had we only the light at our disposal, i.e. massless photons propagating with the same velocity c, we would infer that the general symmetry of physical phenomena is the** *Conformal Group*, **and not the Poincaré group.**

The Lorentz covariance

- **It should be stressed that had we only the light at our disposal, i.e. massless photons propagating with the same velocity c, we would infer that the general symmetry of physical phenomena is the *Conformal Group*, and not the Poincaré group.**
- **To the observations of light must be added the *the principle of inertia*, i.e. the existence of massive bodies moving with speeds lower than c, and constant if not sollicited by external influence.**

First principles

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- **The distinctive feature of such particles is their** *inertial mass*, **equivalent with their energy at rest, which can be measured classically via Newton's law:**
- $$\mathbf{a} = \frac{1}{m}\mathbf{F}. \tag{3}$$

First principles

The fundamental equation

$$\mathbf{a} = \frac{1}{m}\mathbf{F}. \tag{4}$$

relates the only **observable** (using clocks and light rays as measuring rods) quantity, the acceleration **a**, with a combination of less evidently defined quantities, *mass* and *force*, which is interpreted as a **causality relation**, the force being the cause, and acceleration the effect.

First principles

- **It turned out soon that the force $\mathbf{F}$ may symbolize the action of quite different physical phenomena like gravitation, electricity or inertia, and is not a primary cause, but rather a manner of intermediate bookkeeping.**

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- It turned out soon that the force **F** may symbolize the action of quite different physical phenomena like gravitation, electricity or inertia, and is not a primary cause, but rather a manner of intermediate bookkeeping.
- The more realistic sources of acceleration - or rather of the variation of energy and momenta - are the intensities of electric, magnetic or gravitational fields.

The atomistic hypothesis

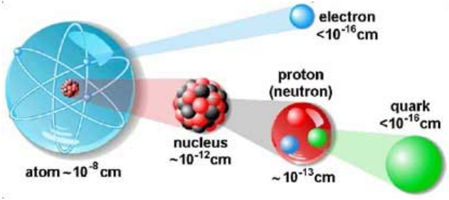
Democritos of Abdera ($\Delta\epsilon\mu o\kappa\rho\iota\tau\eta\sigma$, $-460 \sim -390$), disciple of Leucippos, defended the point of view according to which all things were composed of tiny elementary particles called *atoms*, dancing in the void.

The atomistic hypothesis

The atomistic theory set forth by Democritos is one of the longest-living successful theories. It has no parallel in other ancient civilizations.
It took more than twenty-three centuries before it was accepted, and finally proved beyond any doubt in twentieth century.

Of Matter and Space-Time

According to present knowledge, the ultimate undivisible and undestructible constituents of matter, called atoms by ancient Greeks, are in fact the so-called QUARKS, carrying fractional electric charges and baryonic numbers, the two features that appear to be undestructible and conserved under any circumstances.



The ultimate constituents of matter

- **All baryons, e.g. protons and neutrons, are composed of quarks, which cannot be isolated and unobservable in a free and unbound state. Also the mesons which convey the strong interactions between the baryons, are composed of quark-antiquark pairs.**

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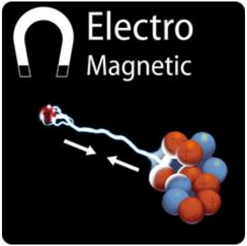
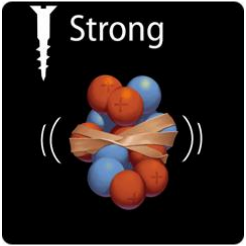
- As for today, we do not know more fundamental constituents of matter than quarks. It is also amazing that the deep inelastic scattering experiments reveal point-like objects inside the proton. It seems reasonable to think that they are at least as stable as the proton itself.

Introduction: Fundamental interactions

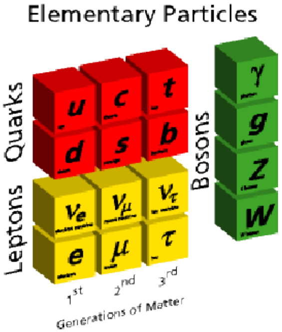
There are three fundamental gauge fields in nature:

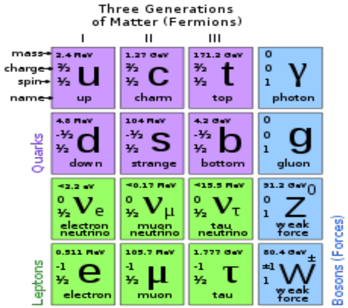




Introduction: Quarks and Leptons

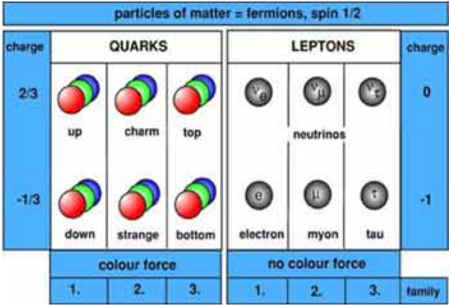
The carriers of elementary charges also go by packs of three: three families of quarks, and three types of leptons.



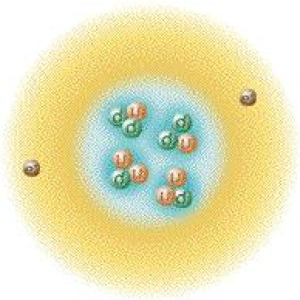


Introduction: Three colors

There is an additional 3-symmetry: the three colors.



The three colors are needed to combine three quarks into a hadron



A schematic representation of a Helium atom.
Each nucleon contains three quarks, of three different colors.

First principles

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- Our questioning about the cause of measurable effects should not stop at the stage of *forces*, which are but expressions of effects of countless fundamental interactions, just like the thermodynamical pressure is in fact an averaged result of countless atomic collisions.
- On a classical level, when theory permits, the symbolical force can be replaced by a more explicit expression in which fields responsible for the forces do appear, like in the case of the Lorentz force.

First principles

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- But the fields acting on a test particle are usually generated by more or less distant charges and currents, according to the formula giving the retarded four-potential $A_\mu(x^\lambda)$:
- $$A_\mu(\mathbf{r},t) = \frac{1}{4\pi c}\int\int\int \frac{j_\mu(\mathbf{r}', t - \frac{|\mathbf{r}-\mathbf{r}'|}{c})}{|\mathbf{r}-\mathbf{r}'|} d^3\mathbf{r}'. \qquad (5)$$

then we get the field tensor given by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

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- **with** $\psi^\dagger = \bar{\psi}^T \gamma^5, \quad \text{where}$

$$\gamma^5 = \gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}, \quad I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

- **In fact, the four-component complex function ψ is composed of two two-component spinors,** ξ_α and $\chi_{\dot{\beta}}$,

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- In fact, the four-component complex function ψ is composed of two two-component spinors, ξ_α and $\chi_{\dot\beta}$,

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- which are supposed to transform under two non-equivalent representations of the $SL(2, \mathbf{C})$ group:

$$\xi_{\alpha'} = S^{\alpha}_{\alpha'} \xi_\alpha, \quad \chi_{\dot\beta'} = S^{\dot\beta}_{\dot\beta'} \chi_{\dot\beta'}, \tag{7}$$

- **The electric charge conservation is equivalent to the annulation of the four-divergence of j^μ:**

$$\partial_\mu j^\mu = \left(\partial_\mu \psi^\dagger \gamma^\mu\right) \psi + \psi^\dagger \left(\gamma^\mu \partial_\mu \psi\right) = 0, \qquad (8)$$

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- from which we infer that this condition will be satisfied if we have

$$\partial_\mu \psi^\dagger \gamma^\mu = -m\psi^\dagger \ \text{ and } \ \gamma^\mu \partial_\mu \psi = m\psi, \qquad (9)$$

which is the Dirac equation.

- **In terms of the spinorial components ξ and χ the Dirac equation can be seen as a pair of two coupled equations which can be written in terms of Pauli's σ-matrices:**

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- $$\left(-i\hbar\frac{1}{c}\frac{\partial}{\partial t} - mc\right)\chi = i\hbar\sigma\cdot\nabla\xi. \tag{10}$$

- The relativistic invariance imposed on this equation is usually presented as follows: under a Lorentz transformation Λ the 4-current j^μ undergoes the following change:

$$j^\mu \to j^{\mu'} = \Lambda^{\mu'}_{\mu} j^\mu. \tag{11}$$

- This means that the matrices γ^{μ} must transform as components of a 4-vector, too. Parallelly, the components of the bi-spinor ψ must be transformed in a way such as to leave the form of the equations (9) unchanged: writing symbolically the transformation of $\mid \psi >$ as $\mid \psi' >= S \mid \psi >$, and $< \psi' \mid=< \psi \mid S^{-1}$, we should have

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$$j^{\mu'} =< \psi' \mid \gamma^{\mu'} \mid \psi' >=< \psi \mid S^{-1}\gamma^{\mu'} S \mid \psi >= \Lambda^{\mu'}_{\mu} j^{\mu} = \Lambda^{\mu'}_{\mu} < \psi \mid \gamma^{\mu} \mid \psi > \tag{12}$$

from which we infer the transformation rules for gamma-matrices:

$$S^{-1}\gamma^{\mu'} S = \Lambda^{\mu'}_{\mu} \gamma^{\mu}. \tag{13}$$

Quantum covariance

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Quantum covariance

- **Quantum Mechanics started as a non-relativistic theory, but very soon its relativistic generalization was created.**
- **As a result, the wave functions in the Schroedinger picture were required to belong to one of the linear representations of the Lorentz group, which means that they must satisfy the following covariance principle:**

$$\tilde{\psi}(\tilde{x}) = \tilde{\psi}(\Lambda(x)) = S(\Lambda)\,\psi(x).$$

Quantum covariance

- **The nature of the representation $S(\Lambda)$ determines the character of the field considered: spinorial, vectorial, tensorial...**

Quantum covariance

- The nature of the representation $S(\Lambda)$ determines the character of the field considered: spinorial, vectorial, tensorial...
- As in many other fundamental relations, the seemingly simple equation

$$\tilde{\psi}(\tilde{x}) = \tilde{\psi}(\Lambda(x)) = S(\Lambda)\,\psi(x).$$

creates a bridge between two totally different realms: the space-time accessible via classical macroscopic observations, and the Hilbert space of quantum states. It can be interpreted in two opposite ways, depending on which side we consider as the cause, and which one as the consequence.

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- **or maybe it is already present as symmetry of quantum states, and then implemented and extended to the macroscopic world in classical limit ? In such a case, the covariance principle should be written as follows:**
- $$\Lambda^{\mu'}_{\mu}(S) j^{\mu} = j^{\mu'}(\psi') = j^{\mu'}(S(\psi)),$$
  In the above formula
  $$j^{\mu} = \bar{\psi}\gamma^{\mu}\psi$$
  is the Dirac current, ψ is the electron wave function.

- **In view of the analysis of the causal chain, it seems more appropriate to write the same transformations with Λ depending on S:**

$$\psi'(x^{\mu'}) = \psi'(\Lambda^{\mu'}_{\nu}(S)x^{\nu}) = S\psi(x^{\nu}) \tag{14}$$

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$$\psi'(x^{\mu'}) = \psi'(\Lambda_{\nu}^{\mu'}(S)x^{\nu}) = S\psi(x^{\nu}) \qquad (14)$$

- This form of the same relation suggests that the transition from one quantum state to another, represented by the unitary transformation S is the primary cause that implies the transformation of observed quantities such as the electric 4-current, and as a final consequence, the apparent transformations of time and space intervals measured with classical physical devices.

Quantum covariance

- **Although mathematically the two formulations are equivalent, it seems more plausible that the Lorentz group resulting from the averaging of the action of the $SL(2, \mathbf{C})$ in the Hilbert space of states contains less information than the original double-valued representation which is a consequence of the particle-anti-particle symmetry, than the other way round.**

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- **In what follows, we shall draw physical consequences from this approach, concerning the strong interactions in the first place.**

Pauli's exclusion principle

Wolfgang Pauli, who introduced the exclusion principle for fermions.

Pauli's exclusion principle

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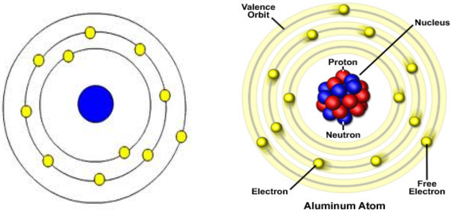
- **Not only does it explain the structure of atoms and the periodic table of elements, but it also guarantees the stability of matter preventing its collapse. as suggested by Ehrenfest, and proved later by Dyson.**
- **The link between the exclusion principle and particle's spin, known as the "spin-and-statistic theorem", is one of the deepest results in quantum field theory.**

Pauli's exclusion principle

Because fermionic operators must satisfy anti-commutation relations $\Psi^a\Psi^b = -\Psi^b\Psi^a$, two electrons (or other fermions) cannot coexist in the same state.



For the principal quantum number n there are only $2 \times n^2$ electrons in different states.

Pauli's exclusion principle

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Pauli's exclusion principle

- **In purely algebraical terms Pauli's exclusion principle amounts to the anti-symmetry of wave functions describing two coexisting particle states.**
- **The easiest way to see how the principle works is to apply Dirac's formalism in which wave functions of particles in given state are obtained as products between the "bra" and "ket" vectors.**

Pauli's exclusion principle

Consider the wave function of a particle in the state $\mid x >$,

$$\Phi(x) = < \psi \mid x > . \tag{15}$$

A two-particle state of $(\mid x >, \mid y)$ is a tensor product

$$\mid \psi > = \sum \Phi(x, y) \, (\mid x > \otimes \mid y >). \tag{16}$$

If the wave function $\Phi(x, y)$ is anti-symmetric, i.e. if it satisfies

$$\Phi(x, y) = -\Phi(y, x), \tag{17}$$

then $\Phi(x, x) = 0$ and such states have vanishing probability.

Pauli's exclusion principle

Conversely, suppose that $\Phi(x, x)$ does vanish. This remains valid in any basis provided the new basis $| x' >$, $| y' >$ was obtained from the former one via unitary transformation. Let us form an arbitrary state being a linear combination of $| x >$ and $| y >$,

$$| z >= \alpha | x > +\beta | y >, \quad \alpha, \beta \in \mathbf{C},$$

and let us form the wave function of a tensor product of such a state with itself:

$$\Phi(z, z) =< \psi | (\alpha | x > +\beta | y >) \otimes (\alpha | x > +\beta | y >), \quad (18)$$

Pauli's exclusion principle

- **which develops as follows:**

$$\alpha^2 < \psi \mid x, x > + \alpha\beta < \psi \mid x, y >$$

$$+ \beta\alpha < \psi \mid y, x > + \beta^2 < \psi \mid y, y > =$$

$$= \alpha^2 \, \Phi(x, x) + \alpha\beta \, \Phi(x, y) + \beta\alpha \, \Phi(y, x) + \beta^2 \, \Phi(y, y). \quad (19)$$

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$$+ \beta\alpha < \psi \mid y, x > + \beta^2 < \psi \mid y, y > =$$

$$= \alpha^2 \, \Phi(x, x) + \alpha\beta \, \Phi(x, y) + \beta\alpha \, \Phi(y, x) + \beta^2 \, \Phi(y, y). \quad (19)$$

- Now, as $\Phi(x, x) = 0$ and $\Phi(y, y) = 0$, the sum of remaining two terms will vanish if and only if (17) is satisfied, i.e. if $\Phi(x, y)$ is anti-symmetric in its two arguments.

Pauli's exclusion principle

After second quantization, when the states are obtained with creation and annihilation operators acting on the vacuum, the anti-symmetry is encoded in the anti-commutation relations

$$\psi(x)\psi(y) + \psi(y)\psi(x) = 0 \tag{20}$$

where $\psi(x) \mid 0 >=\mid x >$.

The invariant antisymmetric form

For a dychotomic variable (spin $\frac{1}{2}$) the exclusion principle imposes the antisymmetric form

$$\epsilon_{12} = -\epsilon_{21} = 1, \quad \epsilon_{11} = 0, \ \ \epsilon_{22} = 0.$$

In a new basis $\epsilon_{\alpha'\beta'} = S^{\alpha}_{\alpha'} S^{\beta}_{\beta'} \, \epsilon_{\alpha\beta}$.
The condition of invariance:

$$\epsilon_{1'2'} = -\epsilon_{2'1'} = 1, \ \ \epsilon_{1'1'} = 0, \ \ \epsilon_{2'2'} = 0$$

leads to $S^1_{1'} \, S^2_{2'} - S^1_{2'} \, S^2_{1'} = 1$,
which defines the $SL(2, C)$ group.

Quarks and Leptons

- **In Quantum Chromodynamics quarks are considered as fermions, endowed with spin $\frac{1}{2}$.**

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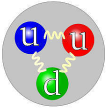
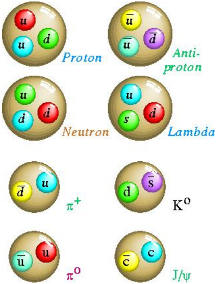
- **In Quantum Chromodynamics quarks are considered as fermions, endowed with spin $\frac{1}{2}$.**
- **Only *three* quarks or anti-quarks can coexist inside a fermionic baryon (respectively, anti-baryon), and a pair quark-antiquark can form a meson with integer spin.**
- **Besides, they must belong to different *colors*, also a three-valued set. There are two quarks in the first generation, u and d ("up" and "down"), which may be considered as two states of a more general object, just like proton and neutron in $SU(2)$ symmetry are two isospin components of a nucleon doublet.**

Quarks and Leptons





Baryons (hadrons) are composed of quarks,
which cannot be observed in a free (unbound) state.

Ternary exclusion principle

This suggests that a convenient generalization of Pauli's exclusion principle would be that no *three* quarks in the same state can be present in a nucleon.
Let us require then the vanishing of wave functions representing the tensor product of *three* (but not necessarily two) identical states. That is, we require that $\Phi(x, x, x) = 0$ for any state $\mid x >$. **As in the former case, consider an arbitrary superposition of three different states,** $\mid x >$, $\mid y >$ and $\mid z >$,

$$\mid w >= \alpha \mid x > + \beta \mid y > + \gamma \mid z >$$

and apply the same criterion, $\Phi(w, w, w) = 0$.

Ternary exclusion principle

We get then, after developing the tensor products,

$$\Phi(w,w,w) = \alpha^3\Phi(x,x,x) + \beta^3\Phi(y,y,y) + \gamma^3\Phi(z,z,z)$$

$$+\alpha^2\beta[\Phi(x,x,y)+\Phi(x,y,x)+\Phi(y,x,x)]+\gamma\alpha^2[\Phi(x,x,z)+\Phi(x,z,x)+\Phi(z,x,x)]$$

$$+\alpha\beta^2[\Phi(y,y,x)+\Phi(y,x,y)+\Phi(x,y,y)]+\beta^2\gamma[\Phi(y,y,z)+\Phi(y,z,y)+\Phi(z,y,y)]$$

$$+\beta\gamma^2[\Phi(y,z,z)+\Phi(z,z,y)+\Phi(z,y,z)]+\gamma^2\alpha[\Phi(z,z,x)+\Phi(z,x,z)+\Phi(x,z,z)]$$

$$+\alpha\beta\gamma[\Phi(x,y,z)+\Phi(y,z,x)+\Phi(z,x,y)+\Phi(z,y,x)+\Phi(y,x,z)+\Phi(x,z,y)] = 0.$$

- The terms $\Phi(x,x,x)$, $\Phi(y,y,y)$ and $\Phi(z,z,z)$ do vanish by virtue of the original assumption; in what remains, combinations preceded by various powers of independent numerical coefficients α, β and γ, must vanish separately.

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- **This is achieved if the following Z_3 symmetry is imposed on our wave functions:**

$$\Phi(x,y,z) = j\,\Phi(y,z,x) = j^2\,\Phi(z,x,y)$$

,

with $\quad j = e^{\frac{2\pi i}{3}}, \;\; j^3 = 1, \;\; j + j^2 + 1 = 0.$

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- Note that the complex conjugates of functions $\Phi(x,y,z)$ transform under cyclic permutations of their arguments with $j^2 = \bar{j}$ replacing j in the above formula

$$\Psi(x,y,z) = j^2\,\Psi(y,z,x) = j\,\Psi(z,x,y).$$

Basic definitions and properties

- Let us introduce N generators spanning a linear space over complex numbers, satisfying the following cubic relations:

$$\theta^A\theta^B\theta^C = j\,\theta^B\theta^C\theta^A = j^2\,\theta^C\theta^A\theta^B, \tag{21}$$

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- with $j = e^{2i\pi/3}$, the primitive root of 1. We have $1 + j + j^2 = 0$ and $\bar{j} = j^2$.

Basic definitions and properties

We shall also introduce a similar set of *conjugate* **generators,** $\bar{\theta}^{\dot{A}}$**,** $\dot{A}, \dot{B}, ... = 1, 2, ..., N$**, satisfying similar condition with** j^2 **replacing** j**:**

$$\bar{\theta}^{\dot{A}}\bar{\theta}^{\dot{B}}\bar{\theta}^{\dot{C}} = j^2\,\bar{\theta}^{\dot{B}}\bar{\theta}^{\dot{C}}\bar{\theta}^{\dot{A}} = j\,\bar{\theta}^{\dot{C}}\bar{\theta}^{\dot{A}}\bar{\theta}^{\dot{B}}, \tag{22}$$

Let us denote this algebra by $\mathcal{A}$.

The Z_3 graded algebra $\mathcal{A}$

- **Let us denote the algebra spanned by the θ^A generators by $\mathcal{A}$. We shall endow it with a natural Z_3 grading, considering the generators θ^A as grade 1 elements, and their conjugates $\bar{\theta}^{\dot{A}}$ being of grade 2.**

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- **The grades add up modulo 3, so that the products $\theta^A\theta^B$ span a linear subspace of grade 2, and the cubic products $\theta^A\theta^B\theta^C$ being of grade 0.**

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- **The grades add up modulo 3, so that the products $\theta^A\theta^B$ span a linear subspace of grade 2, and the cubic products $\theta^A\theta^B\theta^C$ being of grade 0.**
- **Similarly, all quadratic expressions in conjugate generators, $\bar{\theta}^{\dot{A}}\bar{\theta}^{\dot{B}}$ are of grade $2+2=4_{mod\,3}=1$, whereas their cubic products are again of grade 0, like the cubic products od θ^A's.**

The Z_3 graded algebra $\mathcal{A}$

Combined with the associativity, these cubic relations impose finite dimension on the algebra generated by the Z_3 graded generators. As a matter of fact, cubic expressions are the highest order that does not vanish identically. The proof is immediate:

$$\theta^A\theta^B\theta^C\theta^D = j\,\theta^B\theta^C\theta^A\theta^D = j^2\,\theta^B\theta^A\theta^D\theta^C =$$

$$= j^3\,\theta^A\theta^D\theta^B\theta^C = j^4\,\theta^A\theta^B\theta^C\theta^D, \qquad (23)$$

and because $j^4 = j \neq 1$, the only solution is

$$\theta^A\theta^B\theta^C\theta^D = 0. \qquad (24)$$

The simplest case: two generators

Let us consider the simplest case of cubic algebra with two generators, $A, B, ... = 1, 2$. Its grade 1 component contains just these two elements, θ^1 and θ^2; its grade 2 component contains four independent products,

$$\theta^1\theta^1,\ \theta^1\theta^2,\ \theta^2\theta^1,\quad \text{and}\quad \theta^2\theta^2.$$

Finally, its grade 0 component (which is a subalgebra) contains the unit element 1 and the two linearly independent cubic products,

$$\theta^1\theta^2\theta^1 = j\,\theta^2\theta^1\theta^1 = j^2\,\theta^1\theta^1\theta^2,$$

and

$$\theta^2\theta^1\theta^2 = j\,\theta^1\theta^2\theta^2 = j^2\,\theta^2\theta^2\theta^1.$$

General definition of invariant forms

Let us consider multilinear forms defined on the algebra $\mathcal{A} \otimes \bar{\mathcal{A}}$. Because only cubic relations are imposed on products in $\mathcal{A}$ and in $\bar{\mathcal{A}}$, and the binary relations on the products of ordinary and conjugate elements, we shall fix our attention on tri-linear and bi-linear forms.

Consider a tri-linear form ρ^{α}_{ABC}. We shall call this form Z_3-invariant if we can write, by virtue of (21).:

$$\rho^{\alpha}_{ABC}\,\theta^A\theta^B\theta^C = \frac{1}{3}\left[\rho^{\alpha}_{ABC}\,\theta^A\theta^B\theta^C + \rho^{\alpha}_{BCA}\,\theta^B\theta^C\theta^A + \rho^{\alpha}_{CAB}\,\theta^C\theta^A\theta^B\right] =$$

$$= \frac{1}{3}\left[\rho^{\alpha}_{ABC}\,\theta^A\theta^B\theta^C + \rho^{\alpha}_{BCA}\,(j^2\,\theta^A\theta^B\theta^C) + \rho^{\alpha}_{CAB}\,j\,(\theta^A\theta^B\theta^C)\right],$$

General definition of invariant forms

From this it follows that we should have

$$\rho^{\alpha}_{ABC}\ \theta^A\theta^B\theta^C = \frac{1}{3}\left[\rho^{\alpha}_{ABC} + j^2\,\rho^{\alpha}_{BCA} + j\,\rho^{\alpha}_{CAB}\right]\theta^A\theta^B\theta^C, \qquad (25)$$

from which we get the following properties of the ρ-cubic matrices:

$$\rho^{\alpha}_{ABC} = j^2\,\rho^{\alpha}_{BCA} = j\,\rho^{\alpha}_{CAB}. \qquad (26)$$

General definition of invariant forms

Even in this minimal and discrete case, there are covariant and contravariant indices: the lower and the upper indices display the inverse transformation property. If a given cyclic permutation is represented by a multiplication by j for the upper indices, the same permutation performed on the lower indices is represented by multiplication by the inverse, i.e. j^2, so that they compensate each other.

Similar reasoning leads to the definition of the conjugate forms $\bar{\rho}^{\dot{\alpha}}_{\dot{C}\dot{B}\dot{A}}$ satisfying the relations similar to (26) with j replaced be its conjugate, j^2:

$$\bar{\rho}^{\dot{\alpha}}_{\dot{A}\dot{B}\dot{C}} = j\,\bar{\rho}^{\dot{\alpha}}_{\dot{B}\dot{C}\dot{A}} = j^2\,\bar{\rho}^{\dot{\alpha}}_{\dot{C}\dot{A}\dot{B}} \tag{27}$$

Invariant forms: the two-generator case

- **In the simplest case of two generators, the j-skew-invariant forms have only two independent components:**

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- **In the simplest case of two generators, the j-skew-invariant forms have only two independent components:**
- $$\rho^1_{121} = j\,\rho^1_{211} = j^2\,\rho^1_{112},$$
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Invariant forms: the two-generator case

- **In the simplest case of two generators, the j-skew-invariant forms have only two independent components:**
- $$\rho^1_{121} = j\,\rho^1_{211} = j^2\,\rho^1_{112},$$
  $$\rho^2_{212} = j\,\rho^2_{122} = j^2\,\rho^2_{221},$$
- and we can set

$$\rho^1_{121} = 1,\ \rho^1_{211} = j^2,\ \rho^1_{112} = j,$$
$$\rho^2_{212} = 1,\ \rho^2_{122} = j^2,\ \rho^2_{221} = j.$$

The invariance group of cubic matrices

- The constitutive cubic relations between the generators of the Z_3 graded algebra can be considered as intrinsic if they are conserved after linear transformations with commuting (pure number) coefficients, i.e. if they are independent of the choice of the basis.

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- Let $U^{A'}_{A}$ denote a non-singular $N \times N$ matrix, transforming the generators θ^A into another set of generators, $\theta^{B'} = U^{B'}_{B}\, \theta^B$.

The invariance group of cubic matrices

- The constitutive cubic relations between the generators of the Z_3 graded algebra can be considered as intrinsic if they are conserved after linear transformations with commuting (pure number) coefficients, i.e. if they are independent of the choice of the basis.
- Let $U^{A'}_A$ denote a non-singular $N \times N$ matrix, transforming the generators θ^A into another set of generators, $\theta^{B'} = U^{B'}_B \, \theta^B$.
- We are looking for the solution of the covariance condition for the ρ-matrices:

$$S^{\alpha'}_{\beta} \, \rho^{\beta}_{ABC} = U^{A'}_A \, U^{B'}_B \, U^{C'}_C \, \rho^{\alpha'}_{A'B'C'}. \tag{28}$$

The invariance group of cubic matrices

- Now, $\rho^1_{121} = 1$, and we have two equations corresponding to the choice of values of the index α' equal to 1 or 2. For $\alpha' = 1'$ the ρ-matrix on the right-hand side is $\rho^{1'}_{A'B'C'}$, which has only three components,

$$\rho^{1'}_{1'2'1'} = 1, \quad \rho^{1'}_{2'1'1'} = j^2, \quad \rho^{1'}_{1'1'2'} = j,$$

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$$\rho^{1'}_{1'2'1'} = 1, \quad \rho^{1'}_{2'1'1'} = j^2, \quad \rho^{1'}_{1'1'2'} = j,$$

- which leads to the following equation:

$$S^{1'}_1 = U^{1'}_1 \, U^{2'}_2 \, U^{1'}_1 + j^2 \, U^{2'}_1 \, U^{1'}_2 \, U^{1'}_1 + j \, U^{1'}_1 \, U^{1'}_2 \, U^{2'}_1 = U^{1'}_1 \, (U^{2'}_2 \, U^{1'}_1 - U^{2'}_1 \, U^{1'}_2),$$

because $j^2 + j = -1$.

The invariance group of cubic matrices

For the alternative choice $\alpha' = 2'$ the ρ-matrix on the right-hand side is $\rho^{2'}_{A'B'C'}$, whose three non-vanishing components are

$$\rho^{2'}_{2'1'2'} = 1, \quad \rho^{2'}_{1'2'2'} = j^2, \quad \rho^{2'}_{2'2'1'} = j.$$

The corresponding equation becomes now:

$$S^{2'}_1 = U^{2'}_1 \, U^{1'}_2 \, U^{2'}_1 + j^2 \, U^{1'}_1 \, U^{2'}_2 \, U^{2'}_1 + j \, U^{2'}_1 \, U^{2'}_2 \, U^{1'}_1 = U^{2'}_1 \, (U^{1'}_2 \, U^{2'}_1 - U^{1'}_1 \, U^{2'}_2),$$

The invariance group of cubic matrices

The remaining two equations are obtained in a similar manner. We choose now the three lower indices on the left-hand side equal to another independent combination, (212). Then the ρ-matrix on the left hand side must be ρ^2 whose component ρ^2_{212} is equal to 1. This leads to the following equation when $\alpha' = 1'$:

$$S_2^{1'} = U_2^{1'} U_1^{2'} U_2^{1'} + j^2 U_2^{2'} U_1^{1'} U_2^{1'} + j U_2^{1'} U_1^{1'} U_2^{2'} = U_2^{1'} (U_2^{1'} U_1^{2'} - U_1^{1'} U_2^{2'}),$$

and the fourth equation corresponding to $\alpha' = 2'$ is:

$$S_2^{2'} = U_2^{2'} U_1^{1'} U_2^{2'} + j^2 U_2^{1'} U_1^{2'} U_2^{2'} + j U_2^{2'} U_1^{2'} U_2^{1'} = U_2^{2'} (U_1^{1'} U_2^{2'} - U_1^{2'} U_2^{1'}).$$

The invariance group of cubic matrices

The determinant of the 2×2 complex matrix $U^{A'}_{B}$ appears everywhere on the right-hand side.

$$S^{2'}_{1} = -U^{2'}_{1}\,[det(U)], \tag{29}$$

The remaining two equations are obtained in a similar manner, resulting in the following:

$$S^{1'}_{2} = -U^{1'}_{2}\,[det(U)], \qquad S^{2'}_{2} = U^{2'}_{2}\,[det(U)]. \tag{30}$$

The determinant of the 2×2 complex matrix $U^{A'}_{B}$ appears everywhere on the right-hand side. Taking the determinant of the matrix $S^{\alpha'}_{\beta}$ one gets immediately

$$det\,(S) = [det\,(U)]^3. \tag{31}$$

The invariance group of cubic matrices

However, the U-matrices on the right-hand side are defined only up to the phase, which due to the cubic character of the covariance relations and they can take on three different values: 1, j or j^2,

i.e. the matrices $j\, U^{A'}_{B}$ or $j^2\, U^{A'}_{B}$ satisfy the same relations as the matrices $U^{A'}_{B}$ defined above.

The determinant of U can take on the values 1, j or j^2 if $det(\Lambda) = 1$

But for the time being, we have no reason yet to impose the unitarity condition. It can be derived from the conditions imposed on the invariance and duality.

Duality and covariance

In the Hilbert space of spinors the $SL(2,\mathbf{C})$ action conserved naturally two anti-symmetric tensors,

$$\varepsilon_{\alpha\beta} \quad \text{and} \quad \varepsilon_{\dot{\alpha}\dot{\beta}}.$$

and their duals,

$$\varepsilon^{\alpha\beta} \quad \text{and} \quad \varepsilon^{\dot{\alpha}\dot{\beta}}.$$

Spinorial indeces thus can be raised or lowered using these fundamental $SL(2,\mathbf{C})$ tensors:

$$\psi_\beta = \epsilon_{\alpha\beta}\,\psi^\alpha, \quad \psi^{\dot{\delta}} = \varepsilon^{\dot{\delta}\dot{\beta}}\,\psi_{\dot{\beta}}.$$

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- Supposing that our cubic combinations of quark states behave like fermions, there is no choice left: if we want to define the duals of cubic forms ρ^{α}_{ABC} displaying the same symmetry properties, we must impose the covariance principle as follows:

$$\epsilon_{\alpha\beta}\ \rho^{\alpha}_{ABC} = \varepsilon_{AD}\varepsilon_{BE}\varepsilon_{CG}\ \rho^{DEG}_{\beta}.$$

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$$\epsilon_{\alpha\beta}\ \rho^{\alpha}_{ABC} = \varepsilon_{AD}\varepsilon_{BE}\varepsilon_{CG}\ \rho^{DEG}_{\beta}.$$

- The requirement of the invariance of tensor ε_{AB}, $A, B = 1, 2$ with respect to the change of basis of quark states leads to the condition $\det U = 1$, i.e. again to the $SL(2, \mathbf{C})$ group.

The vector representation

A similar covariance requirement can be formulated with respect to the set of 2-forms mapping the quadratic quark-anti-quark combinations into a four-dimensional linear real space. As we already saw, the symmetry (??) imposed on these expressions reduces their number to four. Let us define two quadratic forms, $\pi^{\mu}_{A\dot{B}}$ **and its conjugate** $\bar{\pi}^{\mu}_{\dot{B}A}$

$$\pi^{\mu}_{A\dot{B}}\,\theta^{A}\bar{\theta}^{\dot{B}} \quad \text{and} \quad \bar{\pi}^{\mu}_{\dot{B}A}\,\bar{\theta}^{\dot{B}}\theta^{A}. \tag{32}$$

The Greek indices $\mu, \nu ...$ take on four values, and we shall label them $0, 1, 2, 3$.

The vector representation

The four tensors $\pi^{\mu}_{A\dot{B}}$ and their hermitina conjugates $\bar{\pi}^{\mu}_{\dot{B}A}$ define a bi-linear mapping from the product of quark and anti-quark cubic algebras into a linear four-dimensional vector space, whose structure is not yet defined.
Let us impose the following invariance condition:

$$\pi^{\mu}_{A\dot{B}}\,\theta^{A}\bar{\theta}^{\dot{B}} = \bar{\pi}^{\mu}_{\dot{B}A}\bar{\theta}^{\dot{B}}\theta^{A}. \tag{33}$$

The vector representation

It follows immediately from (**??**) that

$$\pi^{\mu}_{A\dot{B}} = -j^2\, \bar{\pi}^{\mu}_{\dot{B}A}. \tag{34}$$

Such matrices are non-hermitian, and they can be realized by the following substitution:

$$\pi^{\mu}_{A\dot{B}} = j^2\, i\, \sigma^{\mu}_{A\dot{B}}, \quad \bar{\pi}^{\mu}_{\dot{B}A} = -j\, i\, \sigma^{\mu}_{\dot{B}A} \tag{35}$$

where $\sigma^{\mu}_{A\dot{B}}$ are the unit 2 matrix for $\mu = 0$, and the three hermitian Pauli matrices for $\mu = 1, 2, 3$.

The vector representation

Again, we want to get the same form of these four matrices in another basis. Knowing that the lower indices A and $\dot{B}$ undergo the transformation with matrices $U_B^{A'}$ and $\bar{U}_{\dot{B}}^{\dot{A}'}$, we demand that there exist some 4×4 matrices $\Lambda_\nu^{\mu'}$ representing the transformation of lower indices by the matrices U and $\bar{U}$:

$$\Lambda_\nu^{\mu'} \, \pi^\nu_{A\dot{B}} = U_A^{A'} \, \bar{U}_{\dot{B}}^{\dot{B}'} \pi^{\mu'}_{A'\dot{B}'}, \tag{36}$$

this defines the vector (4×4) representation of the Lorentz group.

The vector representation

The first four equations relating the 4×4 real matrices $\Lambda^{\mu'}_{\nu}$ with the 2×2 complex matrices $U^{A'}_{B}$ and $\bar{U}^{\dot{A}'}_{\dot{B}}$ are as follows:

$$\Lambda^{0'}_{0} + \Lambda^{0'}_{3} = U^{1'}_{1}\,\bar{U}^{\dot{1}'}_{\dot{1}} + U^{2'}_{1}\,\bar{U}^{\dot{2}'}_{\dot{1}}$$

$$\Lambda^{0'}_{0} - \Lambda^{0'}_{3} = U^{1'}_{2}\,\bar{U}^{\dot{1}'}_{\dot{2}} + U^{2'}_{2}\,\bar{U}^{\dot{2}'}_{\dot{2}}$$

$$\Lambda^{0'}_{0} - i\Lambda^{0'}_{2} = U^{1'}_{1}\,\bar{U}^{\dot{1}'}_{\dot{2}} + U^{2'}_{1}\,\bar{U}^{\dot{2}'}_{\dot{2}}$$

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The vector representation

The next four equations relating the 4×4 real matrices $\Lambda^{\mu'}_{\nu}$ with the 2×2 complex matrices $U^{A'}_{B}$ and $\bar{U}^{\dot{A}'}_{\dot{B}}$ are as follows:

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The next four equations relating the 4×4 real matrices $\Lambda^{\mu'}_{\nu}$ with the 2×2 complex matrices $U^{A'}_{B}$ and $\bar{U}^{\dot{A}'}_{\dot{B}}$ are as follows:

$$\Lambda^{2'}_{0} + \Lambda^{2'}_{3} = -i\; U^{1'}_{1}\; \bar{U}^{\dot{2}'}_{\dot{1}} + i\; U^{2'}_{1}\; \bar{U}^{\dot{1}'}_{\dot{1}}$$

$$\Lambda^{2'}_{0} - \Lambda^{2'}_{3} = -i\; U^{1'}_{2}\; \bar{U}^{\dot{2}'}_{\dot{2}} + i\; U^{2'}_{2}\; \bar{U}^{\dot{1}'}_{\dot{2}}$$

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The vector representation

The last four equations relating the 4×4 real matrices $\Lambda_{\nu}^{\mu'}$ with the 2×2 complex matrices $U_{B}^{A'}$ and $\bar{U}_{\dot{B}}^{\dot{A}'}$ are as follows:

$$\Lambda_0^{3'} + \Lambda_3^{3'} = U_1^{1'}\,\bar{U}_{\dot{1}}^{\dot{1}'} - U_1^{2'}\,\bar{U}_{\dot{1}}^{\dot{2}'}$$

$$\Lambda_0^{3'} - \Lambda_3^{3'} = U_2^{1'}\,\bar{U}_{\dot{2}}^{\dot{1}'} - U_2^{2'}\,\bar{U}_{\dot{2}}^{\dot{2}'}$$

$$\Lambda_0^{3'} - i\Lambda_2^{3'} = U_1^{1'}\,\bar{U}_{\dot{2}}^{\dot{1}'} - U_1^{2'}\,\bar{U}_{\dot{2}}^{\dot{2}'}$$

$$\Lambda_0^{3'} + i\Lambda_2^{3'} = U_2^{1'}\,\bar{U}_{\dot{1}}^{\dot{1}'} - U_2^{2'}\,\bar{U}_{\dot{1}}^{\dot{2}'}$$

The metric tensor $g_{\mu\nu}$

With the invariant "spinorial metric" in two complex dimensions, ε^{AB} and $\varepsilon^{\dot{A}\dot{B}}$ such that $\varepsilon^{12} = -\varepsilon^{21} = 1$ and $\varepsilon^{\dot{1}\dot{2}} = -\varepsilon^{\dot{2}\dot{1}}$, we can define the contravariant components $\pi^{\nu\ A\dot{B}}$. It is easy to show that the Minkowskian space-time metric, invariant under the Lorentz transformations, can be defined as

$$g^{\mu\nu} = \frac{1}{2}\left[\pi^{\mu}_{A\dot{B}}\,\pi^{\nu\ A\dot{B}}\right] = diag(+,-,-,-) \tag{37}$$

Together with the anti-commuting spinors ψ^{α} the four real coefficients defining a Lorentz vector, $x_{\mu}\,\pi^{\mu}_{A\dot{B}}$, can generate now the supersymmetry via standard definitions of super-derivations.

The invariance group of cubic matrices

Let us then choose the matrices $\Lambda^{\alpha'}_{\beta}$ to be the usual spinor representation of the $SL(2, \mathbf{C})$ group, while the matrices $U^{A'}_{B}$ will be defined as follows:

$$U^{1'}_{1} = j\Lambda^{1'}_{1}, U^{1'}_{2} = -j\Lambda^{1'}_{2}, U^{2'}_{1} = -j\Lambda^{2'}_{1}, U^{2'}_{2} = j\Lambda^{2'}_{2}, \qquad (38)$$

the determinant of U being equal to j^2.

The invariance group of cubic matrices

Obviously, the same reasoning leads to the conjugate cubic representation of the same symmetry group $SL(2, \mathbf{C})$ if we require the covariance of the conjugate tensor

$$\bar{\rho}^{\dot{\beta}}_{\dot{D}\dot{E}\dot{F}} = j\,\bar{\rho}^{\dot{\beta}}_{\dot{E}\dot{F}\dot{D}} = j^2\,\bar{\rho}^{\dot{\beta}}_{\dot{F}\dot{D}\dot{E}},$$

by imposing the equation similar to (28)

$$\Lambda^{\dot{\alpha}'}_{\dot{\beta}}\;\bar{\rho}^{\dot{\beta}}_{\dot{A}\dot{B}\dot{C}} = \bar{\rho}^{\dot{\alpha}'}_{\dot{A}'\dot{B}'\dot{C}'}\,\bar{U}^{\dot{A}'}_{\dot{A}}\;\bar{U}^{\dot{B}'}_{\dot{B}}\;\bar{U}^{\dot{C}'}_{\dot{C}}. \tag{39}$$

The matrix $\bar{U}$ is the complex conjugate of the matrix U, and its determinant is equal to j.

The vector representation

Moreover, the two-component entities obtained as images of cubic combinations of quarks, $\psi^\alpha = \rho^\alpha_{ABC}\theta^A\theta^B\theta^C$ and $\bar{\psi}^{\dot\beta} = \bar{\rho}^{\dot\beta}_{\dot D\dot E\dot F}\bar{\theta}^{\dot D}\bar{\theta}^{\dot E}\bar{\theta}^{\dot F}$ should anti-commute, because their arguments do so, by virtue of (**??**):

$$(\theta^A\theta^B\theta^C)(\bar{\theta}^{\dot D}\bar{\theta}^{\dot E}\bar{\theta}^{\dot F}) = -(\bar{\theta}^{\dot D}\bar{\theta}^{\dot E}\bar{\theta}^{\dot F})(\theta^A\theta^B\theta^C)$$

- Let us first underline the Z_2 symmetry of Maxwell and Dirac equations, which implies their hyperbolic character, which makes the propagation possible. Maxwell's equations *in vacuo* can be written as follows:

$$\frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} = \nabla \wedge \mathbf{B},$$
$$-\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} = \nabla \wedge \mathbf{E}. \tag{40}$$

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$$-\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} = \nabla \wedge \mathbf{E}. \tag{40}$$

- These equations can be decoupled by applying the time derivation twice, which in vacuum, where $div\mathbf{E} = 0$ and $div\mathbf{B} = 0$ leads to the d'Alembert equation for both components separately:

$$\frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} - \nabla^2 \mathbf{E} = 0, \qquad \frac{1}{c^2}\frac{\partial^2 \mathbf{B}}{\partial t^2} - \nabla^2 \mathbf{B} = 0.$$

Nevertheless, neither of the components of the Maxwell tensor, be it E or B, can propagate separately alone. It is also remarkable that although each of the fields E and B satisfies a second-order propagation equation, due to the coupled system (40) there exists a quadratic combination satisfying the forst-order equation, the Poynting four-vector:

$$P^{\mu} = \left[P^0, \mathbf{P} \right], \quad P^0 = \frac{1}{2} \left(\mathbf{E}^2 + \mathbf{B}^2 \right), \quad \mathbf{P} = \mathbf{E} \wedge \mathbf{B},$$

$$\partial_{\mu} P^{\mu} = 0. \tag{41}$$

The Dirac equation for the electron displays a similar Z_2 symmetry, with two coupled equations which can be put in the following form:

$$i\hbar\frac{\partial}{\partial t}\psi_+ - mc^2\psi_+ = i\hbar\sigma\cdot\nabla\psi_-,$$

$$-i\hbar\frac{\partial}{\partial t}\psi_- - mc^2\psi_- = -i\hbar\sigma\cdot\nabla\psi_+, \tag{42}$$

where ψ_+ and ψ_- are the positive and negative energy components of the Dirac equation; this is visible even better in the momentum representation:

$$\left[E - mc^2\right]\psi_+ = c\sigma\cdot\mathbf{p}\psi_-,$$

$$\left[-E - mc^2\right]\psi_- = -c\sigma\cdot\mathbf{p}\psi_+. \tag{43}$$

The Dirac equation

The same effect (negative energy states) can be obtained by changing the direction of time, and putting the minus sign in front of the time derivative, as suggested by Feynman.

Each of the components satisfies the Klein-Gordon equation, obtained by successive application of the two operators and diagonalization:

$$\left[\frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2 - m^2 \right] \psi_\pm = 0$$

As in the electromagnetic case, neither of the components of this complex entity can propagate by itself; only all the components can.

Generalized Dirac equation

Apparently, the two types of quarks, *u* and *d*, cannot propagate freely, but can form a freely propagating particle perceived as a fermion, only under an extra condition: they must belong to three *different* species called *colors*; short of this they will not form a propagating entity.

- Therefore, quarks should be described by *three fields* satisfying a set of coupled linear equations, with the Z_3-symmetry playing a similar role of the Z_2-symmetry in the case of Maxwell's and Dirac's equations. Instead of the "-" sign multiplying the time derivative, we should use the cubic root of unity j and its complex conjugate j^2 according to the following scheme:

- **Therefore, quarks should be described by *three fields* satisfying a set of coupled linear equations, with the Z_3-symmetry playing a similar role of the Z_2-symmetry in the case of Maxwell's and Dirac's equations. Instead of the "-" sign multiplying the time derivative, we should use the cubic root of unity j and its complex conjugate j^2 according to the following scheme:**
-

$$\frac{\partial}{\partial t} \mid \psi >= \hat{H}_{12} \mid \phi >,$$
$$j \frac{\partial}{\partial t} \mid \phi >= \hat{H}_{23} \mid \chi >,$$
$$j^2 \frac{\partial}{\partial t} \mid \chi >= \hat{H}_{31} \mid \psi >, \tag{44}$$

- We do not specify yet the number of components in each state vector, nor the character of the hamiltonian operators on the right-hand side; the three fields $\mid \psi >$, $\mid \phi >$ and $\mid \chi >$ should represent the three colors, none of which can propagate by itself.

- We do not specify yet the number of components in each state vector, nor the character of the hamiltonian operators on the right-hand side; the three fields $| \psi >$, $| \phi >$ and $| \chi >$ should represent the three colors, none of which can propagate by itself.
- The quarks being endowed with mass, we can suppose that one of the main terms in the hamiltonians is the mass operator $\hat{m}$; and let us suppose that the remaining parts are the same in all three hamiltonians.

- **This will lead to the following three equations:**

$$
\begin{aligned}
\frac{\partial}{\partial t} \mid \psi > - \hat{m} \mid \psi >= \hat{H} \mid \phi >, \\
j \frac{\partial}{\partial t} \mid \phi > - \hat{m} \mid \phi >= \hat{H} \mid \chi >, \\
j^2 \frac{\partial}{\partial t} \mid \chi > - \hat{m} \mid \chi >= \hat{H} \mid \psi >,
\end{aligned} \tag{45}
$$

- **This will lead to the following three equations:**

$$\frac{\partial}{\partial t} \mid \psi > - \hat{m} \mid \psi >= \hat{H} \mid \phi >,$$
$$j \frac{\partial}{\partial t} \mid \phi > - \hat{m} \mid \phi >= \hat{H} \mid \chi >,$$
$$j^2 \frac{\partial}{\partial t} \mid \chi > - \hat{m} \mid \chi >= \hat{H} \mid \psi >, \tag{45}$$

- **Supposing that the mass operator commutes with time derivation, by applying three times the left-hand side operators, each of the components satisfies the same common** *third order* **equation:**

$$\left[\frac{\partial^3}{\partial t^3} - \hat{m}^3 \right] \mid \psi >= \hat{H}^3 \mid \psi > . \tag{46}$$

The anti-quarks should satisfy a similar equation with the negative sign for the Hamiltonian operator. The fact that there exist two types of quarks in each nucleon suggests that the state vectors $|\,\psi>, |\,\phi>$ and $|\,\chi>$ should have two components each. When combined together, the two postulates lead to the conclusion that we must have three two-component functions and their three conjugates:

$$\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}, \begin{pmatrix}\bar{\psi}_{\dot{1}}\\ \bar{\psi}_{\dot{2}}\end{pmatrix}, \quad \begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}, \begin{pmatrix}\bar{\varphi}_{\dot{1}}\\ \bar{\varphi}_{\dot{2}}\end{pmatrix}, \quad \begin{pmatrix}\chi_1\\ \chi_2\end{pmatrix}, \begin{pmatrix}\bar{\chi}_{\dot{1}}\\ \bar{\chi}_{\dot{2}}\end{pmatrix},$$

which may represent three colors, two quark states (e.g. "up" and "down"), and two anti-quark states (with anti-colors, respectively).

Finally, in order to be able to implement the action of the $SL(2, \mathbf{C})$ group via its 2×2 matrix representation defined in the previous section, we choose the Hamiltonian $\hat{H}$ equal to the operator $\sigma \cdot \nabla$, the same as in the usual Dirac equation. The action of the Z_3 symmetry is represented by factors j and j^2, while the Z_2 symmetry between particles and anti-particles is represented by the "-" sign in front of the time derivative.

The differential system that satisfies all these assumptions is as follows:

$$-i\hbar\frac{\partial}{\partial t}\psi = mc^2\psi - i\hbar c\sigma\cdot\nabla\bar{\varphi},$$

$$i\hbar\frac{\partial}{\partial t}\bar{\varphi} = jmc^2\bar{\varphi} - i\hbar c\sigma\cdot\nabla\chi,$$

$$-i\hbar\frac{\partial}{\partial t}\chi = j^2mc^2\chi - i\hbar c\sigma\cdot\nabla\bar{\psi},$$

$$i\hbar\frac{\partial}{\partial t}\bar{\psi} = mc^2\bar{\psi} = -i\hbar c\sigma\cdot\nabla\varphi,$$

$$-i\hbar\frac{\partial}{\partial t}\varphi = j^2mc^2\varphi - i\hbar c\sigma\cdot\nabla\bar{\chi},$$

$$i\hbar\frac{\partial}{\partial t}\bar{\chi} = jmc^2\bar{\chi} - i\hbar c\sigma\cdot\nabla\psi, \tag{47}$$

Here we made a simplifying assumption that the mass operator is just proportional to the identity matrix, and therefore commutes with the operator $\sigma \cdot \nabla$.
The functions ψ, φ and χ are related to their conjugates via the following third-order equations:

$$-i\frac{\partial^3}{\partial t^3}\psi = \left[\frac{m^3c^6}{\hbar^3} - i(\sigma \cdot \nabla)^3\right]\bar{\psi} = \left[\frac{m^3c^6}{\hbar^3} - i\sigma \cdot \nabla\right](\Delta\bar{\psi}),$$

$$i\frac{\partial^3}{\partial t^3}\bar{\psi} = \left[\frac{m^3c^6}{\hbar^3} - i(\sigma \cdot \nabla)^3\right]\psi = \left[\frac{m^3c^6}{\hbar^3} - i\sigma \cdot \nabla\right](\Delta\psi), \quad (48)$$

and the same, of course, for the remaining wave functions φ and χ.

The overall $Z_2 \times Z_3$ symmetry can be grasped much better if we use the matrix notation, encoding the system of linear equations (47) as an operator acting on a single vector composed of all the components. Then the system (47) can be written with the help of the following 6×6 matrices composed of blocks of 3×3 matrices as follows:

$$\Gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad B = \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix}, \quad P = \begin{pmatrix} 0 & Q \\ Q^T & 0 \end{pmatrix}, \tag{49}$$

with I the 3×3 identity matrix, and the 3×3 matrices B_1, B_2 and Q defined as follows:

$$B_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & j & 0 \\ 0 & 0 & j^2 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & j^2 & 0 \\ 0 & 0 & j \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

The matrices B_1 and Q generate the algebra of traceless 3×3 matrices with determinant 1, introduced by Sylvester and Cayley under the name of *nonionalgebra*. With this notation, our set of equations (47) can be written in a very compact way:

$$-i\hbar\Gamma^0 \frac{\partial}{\partial t} \Psi = \left[Bm - i\hbar Q \sigma \cdot \nabla \right] \Psi, \tag{50}$$

Here Ψ is a column vector containing the six fields,

$$[\psi, \varphi, \chi, \ \bar{\psi}, \bar{\varphi}, \bar{\chi}],$$

in this order.

But the same set of equations can be obtained if we dispose the six fields in a 6×6 matrix, on which the operators in (50) act in a natural way:

$$\Psi = \begin{pmatrix} 0 & X_1 \\ X_2 & 0 \end{pmatrix}, \quad \text{with} \quad X_1 = \begin{pmatrix} 0 & \psi & 0 \\ 0 & 0 & \phi \\ \chi & 0 & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & 0 & \bar{\chi} \\ \bar{\psi} & 0 & 0 \\ 0 & \bar{\varphi} & 0 \end{pmatrix} \tag{51}$$

By consecutive application of these operators we can separate the variables and find the common equation of sixth order that is satisfied by each of the components:

$$-\hbar^6 \frac{\partial^6}{\partial t^6} \psi - m^6 c^{12} \psi = -\hbar^6 \Delta^3 \psi. \tag{52}$$

- Identifying quantum operators of energy nand the momentum,

$$-i\hbar\frac{\partial}{\partial t} \to E, \quad -i\hbar\nabla \to \mathbf{p},$$

we can write (56) simply as follows:

$$E^6 - m^6c^{12} = \mid \mathbf{p} \mid^6 c^6. \tag{53}$$

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- This equation can be factorized showing how it was obtained by subsequent action of the operators of the system (47):

$$E^6 - m^6c^{12} = (E^3 - m^3c^6)(E^3 + m^3c^6) =$$

$$(E-mc^2)(jE-mc^2)(j^2E-mc^2)(E+mc^2)(jE+mc^2)(j^2E+mc^2) =\mid \mathbf{p} \mid^6 c^6.$$

The equation (56) can be solved by separation of variables; the time-dependent and the space-dependent factors have the same structure:

$$A_1\, e^{\omega\, t} + A_2\, e^{j\,\omega\, t} + A_3 e^{j^2\,\omega\, t}, \qquad B_1\, e^{\mathbf{k}.\mathbf{r}} + B_2\, e^{j\,\mathbf{k}.\mathbf{r}} + B_3\, e^{j^2\,\mathbf{k}.\mathbf{r}}$$

with ω and $\mathbf{k}$ satisfying the following dispersion relation:

$$\frac{\omega^6}{c^6} = \frac{m^6 c^6}{\hbar^6} + \mid \mathbf{k} \mid^6, \tag{54}$$

where we have identified $E = \hbar\omega$ **and** $\mathbf{p} = \hbar\mathbf{k}$.

The relation

$$\frac{\omega^6}{c^6} = \frac{m^6 c^6}{\hbar^6} + \mid \mathbf{k} \mid^6,$$

is invariant under the action of $Z_2 \times Z_3$ symmetry, because to any solution with given real ω and $\mathbf{k}$ one can add solutions with ω replaced by $j\omega$ or $j^2\omega$, $j\mathbf{k}$ or $j^2\mathbf{k}$, as well as $-\omega$; there is no need to introduce also $-\mathbf{k}$ instead of $\mathbf{k}$ because the vector $\mathbf{k}$ can take on all possible directions covering the unit sphere.

But the same set of equations can be obtained if we dispose the six fields in a 6×6 matrix, on which the operators in (50) act in a natural way:

$$\Psi = \begin{pmatrix} 0 & X_1 \\ X_2 & 0 \end{pmatrix}, \quad \text{with} \quad X_1 = \begin{pmatrix} 0 & \psi & 0 \\ 0 & 0 & \phi \\ \chi & 0 & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & 0 & \bar{\chi} \\ \bar{\psi} & 0 & 0 \\ 0 & \bar{\varphi} & 0 \end{pmatrix} \tag{55}$$

By consecutive application of these operators we can separate the variables and find the common equation of sixth order that is satisfied by each of the components:

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with ω and $\mathbf{k}$ satisfying the following dispersion relation:

$$\frac{\omega^6}{c^6} = \frac{m^6 c^6}{\hbar^6} + \mid \mathbf{k} \mid^6, \tag{58}$$

where we have identified $E = \hbar\omega$ **and** $\mathbf{p} = \hbar\mathbf{k}$.

The relation

$$\frac{\omega^6}{c^6} = \frac{m^6 c^6}{\hbar^6} + \mid \mathbf{k} \mid^6,$$

is invariant under the action of $Z_2 \times Z_3$ symmetry, because to any solution with given real ω and $\mathbf{k}$ one can add solutions with ω replaced by $j\omega$ or $j^2\omega$, $j\mathbf{k}$ or $j^2\mathbf{k}$, as well as $-\omega$; there is no need to introduce also $-\mathbf{k}$ instead of $\mathbf{k}$ because the vector $\mathbf{k}$ can take on all possible directions covering the unit sphere.

The nine complex solutions can be displayed in two 3×3 matrices as follows:

$$\begin{pmatrix} e^{\omega\, t-\mathbf{k}\cdot\mathbf{r}} & e^{\omega\, t-j\mathbf{k}\cdot\mathbf{r}} & e^{\omega\, t-j^2\mathbf{k}\cdot\mathbf{r}} \\ e^{j\omega\, t-\mathbf{k}\cdot\mathbf{r}} & e^{j\omega\, t-j\mathbf{k}\cdot\mathbf{r}} & e^{j\omega\, t-j^2\mathbf{k}\cdot\mathbf{r}} \\ e^{j^2\omega\, t-\mathbf{k}\cdot\mathbf{r}} & e^{j^2\omega\, t-\mathbf{k}\cdot\mathbf{r}} & e^{j^2\omega\, t-j^2\mathbf{k}\cdot\mathbf{r}} \end{pmatrix},$$

$$\begin{pmatrix} e^{-\omega\, t-\mathbf{k}\cdot\mathbf{r}} & e^{-\omega\, t-j\mathbf{k}\cdot\mathbf{r}} & e^{-\omega\, t-j^2\mathbf{k}\cdot\mathbf{r}} \\ e^{-j\omega\, t-\mathbf{k}\cdot\mathbf{r}} & e^{-j\omega\, t-j\mathbf{k}\cdot\mathbf{r}} & e^{-j\omega\, t-j^2\mathbf{k}\cdot\mathbf{r}} \\ e^{-j^2\omega\, t-\mathbf{k}\cdot\mathbf{r}} & e^{-j^2\omega\, t-\mathbf{k}\cdot\mathbf{r}} & e^{-j^2\omega\, t-j^2\mathbf{k}\cdot\mathbf{r}} \end{pmatrix}$$

and their nine independent products can be represented in a basis of real functions as

$$\begin{pmatrix} A_{11}\, e^{\omega\, t - \mathbf{k}\cdot\mathbf{r}} & A_{12}\, e^{\omega\, t + \frac{\mathbf{k}\cdot\mathbf{r}}{2}}\, \cos(\mathbf{k}\cdot\xi) & A_{13}\, e^{\omega\, t + \frac{\mathbf{k}\cdot\mathbf{r}}{2}}\, \sin(\mathbf{k}\cdot\xi) \\ A_{21}\, e^{-\frac{\omega\, t}{2} - \mathbf{k}\cdot\mathbf{r}}\, \cos\omega\tau & A_{22}\, e^{-\frac{\omega\, t}{2} + \frac{\mathbf{k}\cdot\mathbf{r}}{2}}\, \cos(\omega\tau - \mathbf{k}\cdot\xi) & A_{23}\, e^{-\frac{\omega\, t}{2} + \frac{\mathbf{k}\cdot\mathbf{r}}{2}}\, \cos(\omega\tau + \mathbf{k} \\ A_{31}\, e^{-\frac{\omega\, t}{2} - \mathbf{k}\cdot\mathbf{r}}\, \sin\omega\tau & A_{32}\, e^{-\frac{\omega\, t}{2} + \frac{\mathbf{k}\cdot\mathbf{r}}{2}}\, \sin(\omega\tau + \mathbf{k}\cdot\xi) & A_{33}\, e^{-\frac{\omega\, t}{2} + \frac{\mathbf{k}\cdot\mathbf{r}}{2}}\, \sin(\omega\tau - \mathbf{k} \end{pmatrix}$$

where $\tau = \frac{\sqrt{3}}{2}\, t$ and $\xi = \frac{\sqrt{3}}{2}\mathbf{kr}$; the same can be done with the conjugate solutions (with $-\omega$ instead of ω).

Cubic generalization of Dirac's equation

The functions displayed in the matrix do not represent a wave; however, one can produce a propagating solution by forming certain cubic combinations, e.g.

$$e^{\omega\, t - \mathbf{k}\cdot\mathbf{r}}\, e^{-\frac{\omega\, t}{2} + \frac{\mathbf{k}\cdot\mathbf{r}}{2}}\, \cos(\omega\tau - \mathbf{k}\cdot\xi)\, e^{-\frac{\omega\, t}{2} + \frac{\mathbf{k}\cdot\mathbf{r}}{2}}\, \sin(\omega\tau - \mathbf{k}\cdot\xi) = \frac{1}{2}\, \sin(2\omega\tau - 2\mathbf{k}\cdot\xi).$$

Cubic generalization of Dirac's equation

- **What we need now is a multiplication scheme that would define triple products of non-propagating solutions yielding propagating ones, like in the example given above, but under the condition that the factors belong to three distinct subsets b(which can be later on identified as "colors").**

Cubic generalization of Dirac's equation

- **What we need now is a multiplication scheme that would define triple products of non-propagating solutions yielding propagating ones, like in the example given above, but under the condition that the factors belong to three distinct subsets b(which can be later on identified as "colors").**
- **This can be achieved with the 3×3 matrices of three types, containing the solutions displayed in the matrix, distributed in a particular way, each of the three matrices containing the elements of one particular line of the matrix:**

$$[A] = \begin{pmatrix} 0 & A_{12}\, e^{\omega\, t - \mathbf{k}\cdot\mathbf{r}} & 0 \\ 0 & 0 & A_{23}\, e^{\omega\, t + \frac{\mathbf{k}\cdot\mathbf{r}}{2}} \cos \mathbf{k}\cdot\xi \\ A_{31} e^{\omega\, t + \frac{\mathbf{k}\cdot\mathbf{r}}{2}} \sin \mathbf{k}\cdot\xi & 0 & 0 \end{pmatrix} \quad (59)$$

$$[B] = \begin{pmatrix} 0 & B_{12}\, e^{-\frac{\omega}{2}\, t + \frac{\mathbf{k}\cdot\mathbf{r}}{2}} \cos(\tau + \mathbf{k}\cdot\xi) & 0 \\ 0 & 0 & B_{23}\, e^{-\frac{\omega}{2}\, t - \mathbf{k}\cdot\mathbf{r}} \sin\tau \\ B_{31} e^{\omega\, t - \mathbf{k}\cdot\mathbf{r}} \cos\tau & 0 & 0 \end{pmatrix} \quad (60)$$

$$[C] =$$

$$\left(\begin{array}{ccc} 0 & C_{12}\, e^{-\frac{\omega}{2}\, t+\frac{\mathbf{k}\cdot\mathbf{r}}{2}}\, \cos(\tau+\mathbf{k}\cdot\xi) & 0 \\ 0 & 0 & C_{23}\, e^{-\frac{\omega}{2}\, t+\frac{\mathbf{k}\cdot\mathbf{r}}{2}}\, \sin(\tau \\ C_{31}\, e^{-\frac{\omega}{2}\, t+\frac{\mathbf{k}\cdot\mathbf{r}}{2}}\, \cos(\tau+\mathbf{k}\cdot\xi) & 0 & 0 \end{array}\right. \quad (61)$$

Now it is easy to check that in the product of the above three matrices, ABC all real exponentials cancel, leaving the periodic functions of the argument $\tau + \mathbf{k}\cdot\mathbf{r}$. The trace of this triple product is equal to $Tr(ABC) =$

$$[\sin\tau\ \cos(\mathbf{k}\cdot\mathbf{r}) + \cos\tau\ \sin(\mathbf{k}\cdot\mathbf{r})]\ \cos(\tau+\mathbf{k}\cdot\mathbf{r}) + \cos(\tau+\mathbf{k}\cdot\mathbf{r})\ \sin(\tau+\mathbf{k}\cdot\mathbf{r}),$$

representing a plane wave propagating towards $-\mathbf{k}$. Similar solution can be obtained with the opposite direction. From four such solutions one can produce a propagating Dirac spinor.

This model makes free propagation of a single quark impossible, (except for a very short distances due to the damping factor), while three quarks can form a freely propagating state.