Computational Finance, Spring 2023
Computer Lab 3

The aim of the Lab is to learn to simulate the paths of solutions of stochastic differential equations corresponding to common stock market models.
Sometimes (especially for applying Monte-Carlo methods) it is important to know how to simulate the stock price trajectories corresponding to a market model. A simple and quite universal (but often not the best) way to generate the trajectories of solutions of stochastic differential equations is Euler-Maruyama method, where differentials are replaced by differences over small time intervals (t_i, t_{i+1}) and all other values on the right hand side are taken at the time moment t_i. The same idea applies if we have one equation or many equations.
For Black-Scholes market model

$$dS(t) = S(t)\big(\,\mu(t)\,dt + \sigma(S(t), t)\,dB(t)\,\big)$$

this leads to an approximation

$$S(t_{i+1}) - S(t_i) \approx S(t_i)\big(\,\mu(t_i)\,\Delta t + \sigma(S(t_i), t_i)\,(B(t_{i+1}) - B(t_i))\,\big)$$

where $0 = t_0 < t_1 < \ldots < t_m = T$ is a partition of the interval $[0, T]$ into (usually equal) subintervals and $h_i = t_{i+1} - t_i$. If the time intervals are equal, we can use the single value $h = \frac{T}{m}$ instead of h_i. Using this approximation, the knowledge that $B(t_{i+1}) - B(t_i) \sim N(0, \sqrt{h_i})$ and a given value of $S_0 = S(0)$ we can compute approximate values $S_1, S_2, \ldots, S_m$ of $S(t_1), S(t_2), \ldots, S(t_m)$ by

$$S_{i+1} = S_i\left(1 + \mu(t_i)\,h_i + \sigma(S_i, t_i)\sqrt{h_i}X_i\right),\ i = 0, \ldots, m-1,$$

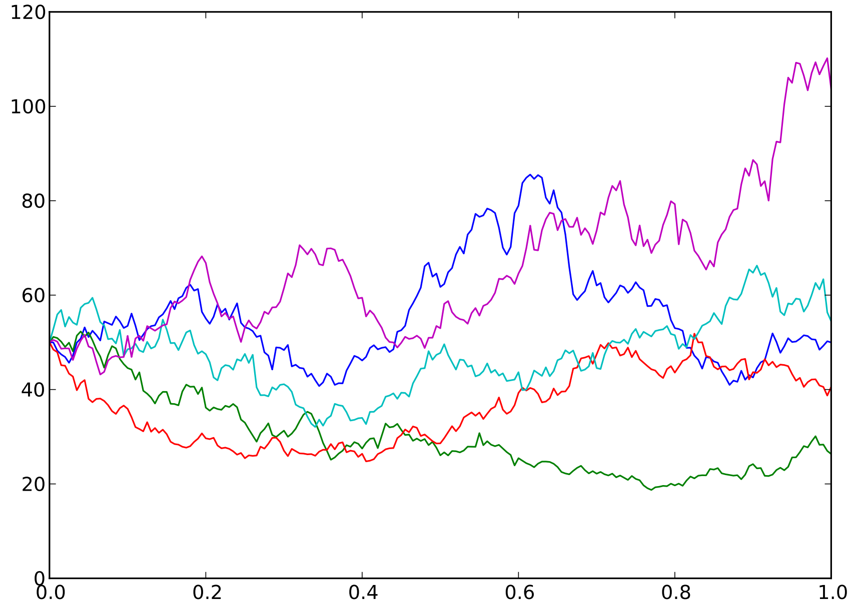
where X_i are independent random variables from the standard normal distribution.

Exercise 1. Write a function `BSgraph(S0,n,m,mu,sigma,T)` that plots the graph of n trajectories of the stock price on the interval $[0, T]$, corresponding to the Black-Scholes market model with constant parameters μ and σ. For computing the values of the stock prices divide the interval $[0, T]$ into m equal subintervals (ie. use the time points $t_i = \frac{i \cdot T}{m},\ i = 0, 1, \ldots, m$) and use the Euler-Maruyama method. For this

1. Define a $(m+1) \times n$ matrix S to store the values of the stock prices, in each column the values of a different trajectory and in i-th row the values corresponding to the time moment $t_i,\ i = 0, 1, 2, \ldots, m$.
2. All the trajectories start from the value S_0 that is given in the parameter $S0$ of the function, so define `S[0,:]=S0`.
3. Now, at each time moment generate a vector of different random numbers for each trajectory by the command `np.random.randn(n)` and compute the $(i+1)$-th row of the matrix with a single line of the code by `S[i+1,:] =S[i,:]*(1+`....
4. Since this function does not have to return a value (it draws the graph instead), there does not have to be any return commands at the end of the function. So the function should end with `plt.plot(...)` and `plt.show()` commands (assuming the package `pyplot` of `matplotlib` has been imported with alias `plt`).

If the code is written correctly, then by entering the command
`BSgraph(S0=50,n=5,m=200,mu=0.1,sigma=0.5,T=1)`

a picture similar to the following shoud be generated:



Exercise 2. In the case of pricing European options by Monte-Carlo methods, we do not want to look at the trajectories of the stock prices but need only to generate values of $S(T)$. Define a function `ST(S0,n,m,mu,sigma,T)` that returns a vector of n randomly generated values of $S(T)$ according to BS market model with constant μ and non-constant volatility given by a function $\sigma(s,t)$. Compute the mean value and standard deviation of 100000 generated stock prices at time $t=T$ in the case $\mu=0.05$, $m=100$, $S_0=10$, $T=1$ and

$$\sigma(s,t)=\frac{e^{-0.1\cdot t}}{1+0.005s^2}.$$

For checking answers: mean and standard deviation should be approximately 10.5 and 6.1.

Exercise 3. Often we have several stochastic processes in a market model. Let us consider a model with stochastic interest rate:

$$\begin{aligned} dS(t) &= S(t)(r(t)\,dt+0.5dB_1(t)),\\ dr(t) &= (0.05-r(t))\,dt+0.02\,dB_2(t), \end{aligned}$$

where B_1 and B_2 are independent Brownian motions. Use Euler-Maruyama method for defining a function that for given n and m outputs n generated values of $S(0.5)$ in the case $S(0)=100,\ r(0)=0.04$.