Computational Finance
Computer Lab 10

The aim of the lab is to learn to use the explicit finite difference method in the case of variable coefficients and to learn to use special solutions of the Black-Scholes equation for constructing boundary conditions. If we have variable coefficient $\alpha(x,t)$ in the transformed Black-Scholes equation, then we can choose m so that the stability constraint $b_{ik} \geq 0$ is satisfied for all i,k if we know the maximal value of $\alpha_{max} = \max_{i,t} \alpha(x_i, t)$. If we want to solve the problem for $t \in [0,T]$, then $\Delta t = \frac{T}{m}$ and

$$b_{ik} = 1 - 2\frac{T}{m\Delta x^2}\alpha\left(x_i, k\frac{T}{m}\right) - r\frac{T}{m} \geq 1 - 2\frac{T}{m\Delta x^2}\alpha_{max} - r\frac{T}{m},$$

so by choosing m that satisfies the condition

$$1 - 2\frac{T}{m\Delta x^2}\alpha_{max} - r\frac{T}{m} \geq 0$$

we make sure that the stability constraint is satisfied. If we want that our program works with all reasonable functions α, we have to write a piece of code that estimates the value α_{max} automatically. A possible way to do this is the following:

1. Choose a number of time steps m_0 that we use for estimating α_{max};
2. Find the maximum of numbers $\alpha(x_i, k\frac{T}{m_0})$;
3. Take α_{max} to be slightly larger than the maximal value found in the previous step. If we have some idea how large the derivative of α with respect to time can be, we can add to the result of the previous step $c \cdot \frac{T}{2m_0}$ (since, according to Lagrange mean value Theorem we have $|\alpha(x_i,t) - \alpha(x_i, k\frac{T}{m_0})| \leq c|t - k\frac{T}{m_0}|$), where c is an upper bound for the absolute value of the derivative. Let us use $c = 10$ in our calculations (which means that our code works for functions α that do not change more than by $10\Delta t$ when time changes by Δt). Note that when we want to solve the problem in the region $t \in [t_0, T]$ for $t_0 \neq 0$, the formulas change slightly (think about what changes and why!)

After estimating α_{max} we can find m from the stability constraint condition and compute the approximate values of the solution of the partial differential equation by the explicit finite difference method (see the handout of the previous lab).

When constructing boundary conditions ϕ_1 and ϕ_2 in the case of option pricing it is often a good idea to use the fact that the solution of the Black-Scholes equation satisfying the final condition $v(s,T) = c_1 s + c_2$ is

$$v(s,t) = c_1 e^{-D(T-t)} s + c_2 e^{-r(T-t)}.$$

Hence, if the payoff function is a linear function starting from some value $s = s_1$, then for large values of s the solution is practically equal to the special solution corresponding to this linear function. And if the payoff function is linear for $s < s_2$, then for small values of s the solution is practically equal to the special solution corresponding to that linear function. This gives us a possibility to define boundary values so that they are not very different from the actual option prices at those boundaries, hence significantly reducing the error caused by introducing x_{min} and x_{max} in option pricing equations. For example, when finding the value of a call option price by solving the untransformed equation, we have that for large values of s the payoff is $p(s) = s - E$. Since for transformed equation $u(x,t) = v(e^x, t)$, a suitable boundary condition for $x = x_{max}$ is

$$\phi_2(x_{max}, t) = e^{-D(T-t)} e^{x_{max}} - E e^{-r(T-t)}.$$

In this Lab we price an European option with $T = 0.5$ and payoff function

$$p(s) = \begin{cases} \frac{195-2\cdot s}{8}, & s < 95, \\ \frac{(s-100)^2}{40}, & 95 \leq s \leq 120, \\ s - 110, & s > 120. \end{cases}$$

We assume that $r = 0.01$, $D = 0.05$ and $S(0) = 97$.

Exercise 1. Derive suitable boundary conditions for the transformed Black-Scholes equation for the payoff function defined above. Define corresponding functions $\phi_1(x_{min}, t)$ and $\phi_2(x_{max}, t)$.

Exercise 2a. Define a function `f_max_2d(f,x,T,t_0=0,m0=100,c=10)` which estimates the maximal value of a function of two variables when the first argument takes the values in the given vector x and the second argument is between t_0 and T, by computing the values at $m_0 + 1$ points in the interval $[t_0, T]$ for each value of of the first argument from the vector x. Check the correctness of your function when $f(x,t) = \frac{10}{(x-1)^2+t+0.1}$, x is a uniformly spaced vector in the interval $[0, 3]$ with 31 values and $T = 1$ (other parameters use default values).

Exercise 2b. Implement the explicit finite difference method described in the previous lab so that it works with non-constant α and β. Test your solver by using the boundary conditions derived in the previous exercise in the case $n = 20$, $\rho = 1.5$ and

$$\sigma(s,t) = 0.5 + 0.2 \cdot e^{-t} \arctan(0.1\, s - 10).$$

The answer depends on how a suitable value of m is determined from the stability condition, but should be close to 10.67.

Exercise 3. Consider the European option with the pay-off function

$$p(s) = \begin{cases} \frac{s}{4}, & 0 \le s < 40, \\ \frac{1}{100}\,(s-40)^3 - \frac{1}{6}\,(s-40)^2 + \frac{s}{4}, & 40 \le s \le 55, \\ 2\,(s-50) & s > 55. \end{cases}$$

Assume $r = 0.01$, $D = 0.05$, $S(0) = 50$, $T = 1$ and that the volatility function is given by

$$\sigma(s,t) = 0.5 + \frac{0.2 \cdot e^{-t}}{1 + 0.01 \cdot (s-40)^2}.$$

For $\rho = 2.5$ and both in the case of the boundary conditions given by appropriate special functions and in the case of using the simple (constant) boundary conditions, determine the limiting value (as $n \to \infty$) of the approximate option price $v(50, 0)$ with the accuracy 0.005 by computing the approximate values by the explicit finite difference method for $n = 10, 20, 40, \ldots$ and by estimating the accuracy of the current result by

$$\frac{1}{3}|\text{current\_result} - \text{previous\_result}|.$$