Computational Finance
Computer Lab 6

The aim of the Lab is to learn to apply Monte-Carlo method for computing option prices.
Often it can be shown (or it is assumed in the case of certain market models) that the price of an European option can be expressed as the expected value

$$V = E[e^{-rT} p(S(T))],$$

where $S(T)$ is generated according to a certain stochastic differential equation. In such a case we can compute V approximately by generating n values of the random variable $S(T)$: $S(T)_1$,$S(T)_2$,...,$S(T)_n$ and computing the arithmetic average of the function under the expectation:

$$V \approx \bar{V}_n = \frac{e^{-rT}}{n} \sum_{i=1}^{n} p(S(T)_i).$$

From Central Limit Theorem it follows that

$$P\left(|V - \bar{V}_n| \le \frac{-\Phi^{-1}(\frac{\alpha}{2})\mathrm{std}(Y)}{\sqrt{n}}\right) \approx 1 - \alpha$$

for large values of n. Here Φ is the cumulative distribution function of the standard normal distribution and $Y = e^{-rT} p(S(T))$.

Exercise 1. If we assume that the Black-Scholes market model with constant volatility holds, then we generate $S(T)$ according to the stochastic differential equation

$$dS(t) = S(t)((r - D)\,dt + \sigma\,dB(t)).$$

From the lecture we know that the solution of this equation is

$$S(t) = S(0)e^{(r-D-\frac{\sigma^2}{2})t+\sigma B(t)},$$

so $S(T) = S(0)e^{(r-D-\frac{\sigma^2}{2})T+\sigma X}$, where $X \sim N(0, \sqrt{T})$. Write a function MC1 that, for given values of $S(0), r, D, \sigma, T, \alpha$ and n, and for given payoff function p, computes an approximate option value and its error estimate holding with the probability $(1 - \alpha)$ by the Monte-Carlo method, using n generated stock prices. Verify the correctness of the function by Black-Scholes formulas for put and call options in the case $S(0) = 100$, $E = 100$, $\sigma = 0.6$, $T = 0.5$, $r = 0.02$, $D = 0.03$, $\alpha = 0.05$ and $n = 10000$. How often is the actual error larger than the error estimate if you use MC1 1000 times?

It is usually not possible to generate $S(T)$ values that correspond exactly to the stochastic differential equation, making the use of approximation methods necessary. One such method is the Euler-Maruyama's method (recall Lab 3), where we divide the interval $[0, T]$ into m equal subintervals and use (in the case of Black-Scholes market model) the approximations

$$S_{i+1} = S_i(1 + (r - D)\,\Delta t + \sigma(S_i, t_i)X_i), i = 0, \ldots, m - 1,$$

where S_i are approximations to $S(i\Delta t)$, $\Delta t = \frac{T}{m}$ and $X_i \sim N(0, \sqrt{\Delta t})$. Now, when using the Monte-Carlo method to find V, instead of $S(T)$ we use S_m to compute an approximate value of $\hat{V}$:

$$\hat{V}_m = E[e^{-rT} p(S_m)].$$

Since S_m for a fixed m does not have exactly the same distribution as $S(T)$, we have in general $\hat{V}_m \neq V$ and therefore the Monte-Carlo method converges to a value that is different from the option price.

Exercise 2. Write a function MC2 that computes approximate option prices so that the stock prices are generated according to Euler's method. Determine how large the difference between $\hat{V}_m$ and the correct option price is in the case of European call option, using the same parameters as in the previous exercise for m = 2, 4, 8, 16. In order to see the difference, large enough value for n should be used (if possible, the corresponding MC error should be at least 5 times smaller than the computed difference).

It is known that if p is continuous and has bounded first derivative (i.e., it is Lipshitz continuous), then

$$\hat{V}_m = V + \frac{C}{m} + o\left(\frac{1}{m}\right),$$

where C is a constant that does not depend on m and $m \cdot o(\frac{1}{m}) \to 0$ as $m \to \infty$. Thus, if we knew the value of the constant C, we could choose a large enough m so that the absolute value of the term $\frac{C}{m}$ is small enough (for example, less than $\frac{\varepsilon}{2}$, where ε is the desired accuracy) and then use the MC method with a large enough n so that the MC error estimate is also small enough (less than $\frac{\varepsilon}{2}$). Unfortunately, we do not usually know the value of C. One possibility to estimate the value of C is by computing V_m approximately (by MC method) for several values of m, and then find the values of V and C such that $V + \frac{C}{m}$ is as close as possible to the obtained results. In statistics, such fitting is called linear regression and there is a function `linregress` in the `stats` subpackage of `scipy` for computing the regression parameters.

Exercise 3. Find an approximate value of C by fitting a linear regression line to the approximate option prices V_2, V_4, V_8, V_{16} computed in the previous exercise. Using the value of C, find a value of m such that $\frac{C}{m} \leq 0.03$.

Exercise 4. With the value of m found in the previous exercise, compute V_m so that MC error is less than 0.03 with probability 0.95. Find also the actual difference of V_m you found and the exact option price. Is the actual difference smaller than 0.06?