Computational Finance
Computer Lab 8

The aim of the Lab is to derive an explicit finite difference method for solving an initial value problem of the heat equation in a bounded domain.
Let us consider the following problem: find u such that

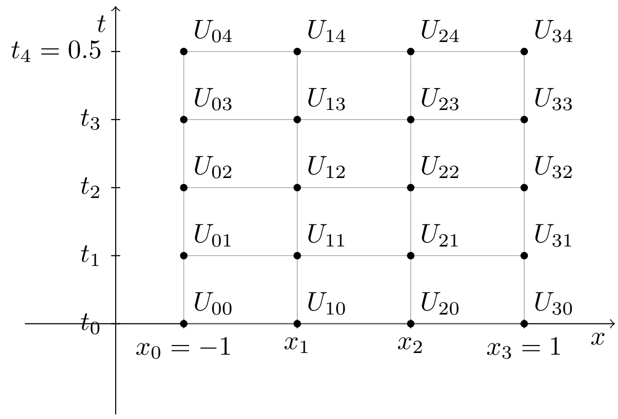
$$\frac{\partial u}{\partial t}(x,t) = \frac{1}{4}\frac{\partial^2 u}{\partial x^2}(x,t),\ x \in [-1,1],\ t \in (0,0.5] \tag{1}$$

$$u(-1,t) = 1, u(1,t) = 0,\ t \in (0,0.5] \tag{2}$$

$$u(x,0) = u_0(x),\ x \in [-1,1] \tag{3}$$

where u_0 is a given function. The procedure for deriving a finite difference approximation for the problem above consists of the following steps.

1. Choose a set of points at which we want to find approximate values of the unknown function. We define this set of points by dividing the interval $[-1,1]$ in x direction into n equal subintervals and the time interval $[0,0.5]$ into m subintervals: we get points (x_i,t_k), where $x_i = -1+i\frac{2}{n},\ i = 0,\ldots,n$, $t_k = k\frac{0.5}{m}$. Since we know the values of the unknown function for $x=-1$, $x=1$ and for $t=0$, we have to determine approximate values $U_{ik} \approx u(x_i,t_k),\ i = 1,\ldots,n-1,\ k = 1,\ldots,m$, thus we have $m\cdot(n-1)$ unknowns (see the picture below for $n=3, m=4$).



2. In order to determine the values for $m\cdot(n-1)$ unknowns, we need $m\cdot(n-1)$ equations. We get those equations by writing down the differential equation at $m\cdot(n-1)$ points and then replacing the derivatives by approximations that use only the function values at points $(x_i,t_k),\ i = 0,\ldots,n,\ k = 0,\ldots,m$.

3. In order to get an explicit finite we should start using the equation at the points where we know the solution. For the current problem this means that we want to use points at the line, where the initial condition is given. So we use the equation at the points $(x_i,t_k),\ i = 1,\ldots,n-1,\ k = 0,\ldots,m-1$ and use the approximations

$$\frac{\partial u}{\partial t}(x_i,t_k) \approx \frac{U_{i,k+1}-U_{ik}}{\Delta t} \text{ (error} \le const.\Delta t),$$
$$\frac{\partial^2 u}{\partial x^2}(x_i,t_k) \approx \frac{U_{i-1,k}-2U_{ik}+U_{i+1,k}}{\Delta x^2} \text{ (error} \le const.\Delta x^2).$$

Using the procedure outlined above, we get a system of equations of the form

$$U_{i,k+1} = a\,U_{i-1,k} + b\,U_{ik} + c\,U_{i+1,k},\ k = 0,\ldots,m-1,\ i = 1,\ldots,n-1,$$

where a, b and c are certain coefficients. Fortunately it is very easy to solve the system of equations: since the values of $U_{i0},\ i = 0,\ldots,n$ are known, we can just compute $U_{i1},\ i = 1,\ldots,n-1$ from the equations, after that we can compute U_{i2} and so on. Since we do not have to solve any systems of equations but can just compute the values of the approximate solutions, the method is called an explicit method.

Exercise 1. Write a function that for given values of m and n and for given function u_0 returns the values $U_{im},\ i = 0, \dots, n$ of the approximate solution obtained by explicit finite difference method. Test the correctness of your function in the case $m = 100, n = 10$ and $u_0(x) = \sin(\pi x) + \frac{1-x}{2}$, when the exact solution is $u(x,t) = e^{-\pi^2 t/4} \sin(\pi x) + \frac{1-x}{2}$.

Exercise 2. The total error caused by replacing exact derivatives with finite difference approximations is $O(\Delta t + \Delta x^2)$, which usually implies that the error of the approximate solution is of the same order. This means, that if we increase m four times and n two times, then the total error should be reduced approximately four times. Verify the convergence rate by computing the errors in the settings of the previous exercise for $m = 4, 16, 64, 256$ and $n = 2, 4, 8, 16$.

Exercise 3. It turns out that explicit methods may be unstable for certain choices of parameters m and n. This means, that if m and n do not satisfy certain condition, the approximate solution may have arbitrarily large errors even when we let m and n to go to infinity. The sufficient condition of stability is that the coefficients a, b and c are all non-negative. Repeat the computations of the previous exercise for $m = 2, 8, 32, 128$ and $n = 10, 20, 40, 80$ and compute the errors.