Computational Finance
Computer Lab 9

If we consider an European option with exercise time T and payoff function p and assume the validity of Black-Scholes market model, then the option price at time t is given by $v(S(t),t) = u(\ln(S(t)),t)$, where u is the solution of the problem

$$\frac{\partial u}{\partial t}(x,t) + \alpha(x,t)\frac{\partial^2 u}{\partial x^2}(x,t) + \beta(x,t)\frac{\partial u}{\partial x}(x,t) - r\,u(x,t) = 0,\ x \in \mathbf{R}, 0 \le t < T \tag{1}$$

satisfying the final condition

$$u(x,T) = p(e^x),\ x \in \mathbf{R}.$$

Here

$$\alpha(x,t) = \frac{\sigma^2(e^x,t)}{2},$$
$$\beta(x,t) = r - D - \frac{\sigma^2(e^x,t)}{2}.$$

For solving the equation for u numerically, we introduce two boundaries x_{min} and x_{max} and specify boundary conditions $u(x_{min},t) = \phi_1(x_{min},t)$, $u(x_{max},t) = \phi_2(x_{max},t)$ at those points. Next, we introduce the points $x_i = x_{min} + i\Delta x,\ i = 0,\dots,n$ and $t_k = k\Delta t,\ k = 0,\dots,m$ and denote by U_{ik} the approximate values of $u(x_i,t_k)$. Here $\Delta x = \frac{x_{max}-x_{min}}{n}$ and $\Delta t = \frac{T}{m}$. In the case of the explicit finite difference method we compute the values U_{ik} as follows:

$$U_{im} = p(e^{x_i}),\ i = 0,\dots,n,$$
$$U_{0,k-1} = \phi_1(x_{min},t_{k-1}),\ U_{n,k-1} = \phi_2(x_{max},t_{k-1}),\ k = m, m-1,\dots,1,$$
$$U_{i,k-1} = a_{ik}U_{i-1,k} + b_{ik}U_{ik} + c_{ik}U_{i+1,k},\ i = 1,\dots,n-1,\ k = m, m-1,\dots,1,$$

where

$$a_{ik} = \frac{\Delta t}{\Delta x^2}\left(\alpha_{ik} - \frac{\beta_{ik}}{2}\Delta x\right),$$
$$b_{ik} = 1 - 2\frac{\Delta t}{\Delta x^2}\alpha_{ik} - r\Delta t,$$
$$c_{ik} = \frac{\Delta t}{\Delta x^2}\left(\alpha_{ik} + \frac{\beta_{ik}}{2}\Delta x\right).$$

If σ is a constant, then the coefficients a, b and c are also constants and the numerical scheme simplifies to

$$U_{i,k-1} = a\,U_{i-1,k} + b\,U_{ik} + c\,U_{i+1,k},\ i = 1,\dots,n-1,\ k = m, m-1,\dots,1.$$

The stability condition for this scheme is $b \ge 0$.

Exercise 1. Derive a formula for the smallest integer m in terms of n for which the condition $b \ge 0$ is satisfied in the case of constant volatility.

Exercise 2. Write a function that, for given values of n, $\rho > 1$, r, D, $S0$, T, σ and for given functions p, ϕ_1 and ϕ_2, takes m to be equal to the smallest integer satisfying the stability constraint and returns the values $U_{i0},\ i = 0,\dots,n$ of the approximate solution (option prices) obtained by explicit finite difference method in the case $x_{min} = \ln\frac{S0}{\rho}$, $x_{max} = \ln(\rho\,S0)$. Test the correctness of your code by comparing the results to the exact values obtained by Black-Scholes formula in the case $r = 0.03$, $\sigma = 0.5$, $D = 0.05$, $T = 0.5$, $E = 97$, $S0 = 100$, $p(s) = \max(s - E, 0)$, $\phi_1(x_{min},t) = p(e^{x_{min}})$, $\phi_2(x_{max},t) = p(e^{x_{max}})$.

Exercise 3. Let $r = 0.02$, $\sigma = 0.6$, $D = 0.03$, $T = 0.5$, $E = 99$, $S0 = 100$, $p(s) = \max(s - E, 0)$. If we use the explicit method of previous exercise, then even if we let n go to infinity there is going to be a finite error between the exact option price at $t = 0, S(0) = S0$ and the corresponding approximate value. This error is caused by introducing artificial boundaries x_{min} and x_{max} and the boundary conditions specified at those boundaries. Use the boundary conditions $\phi_1(x_{min}, t) = p(e^{x_{min}}),\ \phi_2(x_{max}, t) = p(e^{x_{max}})$ and determine the value of the resulting error for $\rho = 1.5, 2, 2.5$. In order to see the resulting error you should do several computations with fixed ρ and increasing values of n (assuming m is determined from the stability condition, n should be increased by multiplying it by 2 each time). Use the knowledge that for large enough n the part of the error depending on the choice of n behaves approximately like $\frac{const.}{n^2}$ (so the difference of the last two computations divided by 3 is an estimate of this part of the error for the last computation) for determining how far your last computation is from the limiting value.

It is important to understand that a single computation by a finite difference method can be used for computing prices of the same option at different time moments for any reasonable value of the current stock price at the time moments of interest. Of course our numerical method does not give us a formula for the function v, but only a table of values corresponding to certain stock prices and time moments. So to find a value of the function for a given value of s and for given time moment, we should determine the closest values of of stock prices and time moments for which we have results and to use them to compute the value we need. One commonly used approach is **bilinear interpolation**, which works as follows:

Suppose that we know the values of a function $u(x, t)$ at the corners of a rectangle $[x_1, x_2] \times [t_1, t_2]$, denote the known values by $u(x_1, t_1) = U_{11},\ u(x_1, t_2) = U_{12},\ u(x_2, t_1) = U_{21},\ u(x_2, t_2) = U_{22}$. Denote also $\Delta x = x_2 - x_1, \Delta t = t_2 - t_1$. Then the the value of u at the point (x, t), where $x_1 \le x \le x_2,\ t_1 \le t \le t_2$ can be computed approximately by the bilinear interpolation formula

$$u(x, t) \approx \frac{U_{11}(x_2 - x)(t_2 - t) + U_{12}(x_2 - x)(t - t_1) + U_{21}(x - x_1)(t_2 - t) + U_{22}(x - x_1)(t - t_1)}{\Delta x\, \Delta t}.$$

It is known that if the function u has continuous second derivatives in x and t variables, then the approximation error is bounded by $const. \cdot (\Delta x^2 + \Delta t^2)$.

Homework 4. (deadline April 21, 2024) Modify the solver of Exercise 2 so that it uses the explicit method to find approximate values of the solution of the transformed BS equation at points $x_i = x_{min} + i\frac{x_{max} - x_{min}}{n}$, $t_k = t_{min} + k\frac{T - t_{min}}{m}$ for given values of x_{min},x_{max},t_{min} and T, which should also be among parameters of the solver (solver should work with any values of those parameters). The solver should return the matrix U as the answer. Use the solver in the case $r = 0.02$, $\sigma = 0.45$, $D = 0$, $t_{min} = 0.2$, $T = 0.7$, $p(s) = 15\, e^{-0.02(s-95)^2}$, $\phi_1(x_{min}, t) = p(e^{x_{min}})$, $\phi_2(x_{max}, t) = p(e^{x_{max}})$, $x_{min} = \ln 30$, $x_{max} = \ln 200$, $n = 90$ to compute the matrix U. The matrix should be computed only once when solving the homework problem. Use this matrix to find approximate option prices:

1. at time $t = 0.2$, when $S(0.2) = 85.57$ by using interpolation only with respect to the x variable,
2. Define a function `my_interp(x,t,xmin,xmax,tmin,T,U)` which, for given values of x and t, computes an approximate value of the function u (which is the solution of the transformed BS equation) at (x, t) by using the matrix of values U. Bilinear interpolation should use the grid points which are closest to the point at which we want to compute the value of the function! Use your function to compute approximate option price (the value of pricing function v) at $t = 0.379$, when $S(0.379) = 91.73$.
3. In the `scipy` package there is a function `interpolate.RegularGridInterpolator()` which allows to use different interpolation methods including bilinear interpolation (which is the default method). Compute the approximate option price at the point specified in the previous question with this function as well. Was the result different?

Important! Homework assignments of this course are individual assignments! Please solve the problems yourself without using any code of any other student!