Contents

1 Curves

1.1 Affine space

Let E^n be an n-dimensional affine space and V^n be its associated n-dimensional vector space. Thus there is a mapping $E^n \times V^n \to E^n$, which assigns to each pair $(p;\vec{v})$, where $p \in E^n$ is a point of affine space E^n and $\vec{v} \in V^n$ is a vector of associated vector space V^n, the point of affine space E^n, which will be denoted by $p+\vec{v}$. This mapping satisfies the conditions:

A$_1$) for each point $p \in E^n$ it holds $p+\vec{0}=p$,

A$_2$) for each pair of points $(p;q) \in E^n \times E^n$ there exists a unique vector $\vec{v} \in V^n$ such that $p+\vec{v}=q$,

A$_3$) for each point $p \in E^n$ and two vectors $\vec{v},\vec{w} \in V^n$ it holds

$$(p+\vec{v})+\vec{w}=p+(\vec{v}+\vec{w}),$$

where $\vec{v}+\vec{w}$ at the right-hand side means the addition of vectors in a vector space V^n.

The vector $\vec{v}$ determined by two points p,q of affine space E^n according to the axiom $\mathbf{A}_2$ will be denoted by $\overrightarrow{pq}$.

Let $\vec{v}_1,\vec{v}_2,\ldots,\vec{v}_m$, where $1 \leq m < n$, be linearly independent vectors of associated vector space V^n and p be a point of affine space E^n. A point of E^n and m linearly independent vectors of V^n determine the m-dimensional plane in E^n. This plane passes through a point p and is defined as follows

$$\mathscr{P}=\{p+\sum_{\alpha=1}^{m} t_\alpha \vec{v}_\alpha : p \in E^n;\ \vec{v}_1,\vec{v}_2,\ldots,\vec{v}_m \in V^n,\ t_1,t_2,\ldots,t_m \in \mathbb{R}\}. \tag{1.1}$$

Particularly, any pair $(p;\vec{v})$, where p is a point and $\vec{v}$ is a non-zero vector, determines the straight line

$$\mathscr{L}=\{p+t\,\vec{v} : p \in E^n,\ \vec{v} \in V^n, t \in \mathbb{R}, \vec{v} \neq \vec{0}\}. \tag{1.2}$$

The straight line $\mathscr{L}$ passes through a point p and $\vec{v}$ is its directional vector.

Let $\{\vec{e}_1,\vec{e}_2,\ldots,\vec{e}_n\}$ be a basis for associated vector space V^n. Any vector $\vec{v}$ of V^n can be written in the form

$$\vec{v}=\sum_{i=1}^{n} v^i\,\vec{e}_i, \tag{1.3}$$

where the real numbers $v^1,v^2,\ldots,v^n$ are called the coordinates of a vector $\vec{v}$. The right-hand side of (1.3) can be written in a more concise form by means of the Einstein summation convention over repeated subscript and superscript. Hence we can write (1.3) in the equivalent form $\vec{v}=v^i\,\vec{e}_i$ and in what follows we will regularly use this convention to write formula more compactly. The formula (1.3) allows us to identify a vector space V^n with the n-dimensional vector space $\mathbb{R}^n$ by identifying a vector $\vec{v}$ with its coordinate vector $(v^1, v^2, \dots, v^n) \in \mathbb{R}^n$. Assuming this identification, we will write $\vec{v} = (v^1, v^2, \dots, v^n)$. Then the addition of vectors and the multiplication by real numbers in V^n can be expressed as follows

$$\begin{aligned} \vec{v} + \vec{w} &= (v^1 + w^1, v^2 + w^2, \dots, v^n + w^n), \\ a\,\vec{v} &= (av^1, av^2, \dots, av^n),\ a \in \mathbb{R}, \end{aligned}$$

where $\vec{v} = (v^1, v^2, \dots, v^n), \vec{w} = (w^1, w^2, \dots, w^n)$.

Let $\mathfrak{R} = \{O; \vec{e}_1, \vec{e}_2, \dots, \vec{e}_n\}$ be an affine frame in the space E^n, where $O \in E^n$ is an origin of an affine frame $\mathfrak{R}$ and $\{\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n\}$ is a basis for the vector space V^n. Affine frame $\mathfrak{R}$ determines the system of coordinates in the space E^n. Indeed given a point $p \in E^n$ we can assign to it the position vector $\vec{r} \in V^n$, where the initial point of $\vec{r}$ is at the origin O and the end point is at the point p. Then we can expand the position vector $\vec{r}$ in terms of a basis

$$\vec{r} = \sum_{i=1}^{n} p^i\,\vec{e}_i,$$

and the coordinates $p^1, p^2, \dots, p^n$ of the position vector are the coordinates of a point p. It is easy to see that if $p^1, p^2, \dots, p^n$ are the coordinates of a point p and $q^1, q^2, \dots, q^n$ are the coordinates of a point q, then the vector $\overrightarrow{pq}$, determined by points p, q, has the coordinates $q^1 - p^1, q^2 - p^2, \dots, q^n - p^n$.

Thus an affine frame $\mathfrak{R}$ defines a coordinate system in affine space E^n and the corresponding coordinates will be denoted by $x^1, x^2, \dots, x^n$. In what follows we will consider coordinates $x^1, x^2, \dots, x^n$ from two different points of view. The first point of view on coordinates $x^1, x^2, \dots, x^n$ is traditional, i.e. we consider $x^1, x^2, \dots, x^n$ as an ordered sequence of n real numbers which determines a single point in an affine space E^n. The second point of view will be used in the framework of an algebra of smooth functions on an affine space E^n and according to this approach we will consider x^i as a function whose domain is an affine space E^n (or its open subset) and the value at a point p is the ith coordinate of a point p, i.e. $x^i(p) = p^i$. In this case, we will call x^i the ith coordinate function. Obviously the coordinate system defined by an affine frame $\mathfrak{R}$ allows us to identify an affine space E^n with the space $\mathbb{R}^n$.

If we choose a different frame $\mathfrak{R}' = \{O'; \vec{e}'_1, \vec{e}'_2, \dots, \vec{e}'_n\}$ in an affine space E^n then we get a different coordinate system whose affine coordinates will be denoted by $\tilde{x}^1, \tilde{x}^2, \dots, \tilde{x}^n$. Let us find how the coordinates $x^1, x^2, \dots, x^n$ determined by an affine frame $\mathfrak{R}$ can be expressed in terms of coordinates $\tilde{x}^1, \tilde{x}^2, \dots, \tilde{x}^n$ determined by an affine frame $\mathfrak{R}'$. Let $A = (A^i_j)$ be a transition matrix from a basis $\{\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n\}$ to a basis $\{\vec{e}'_1, \vec{e}'_2, \dots, \vec{e}'_n\}$, i.e. $\vec{e}'_i = A^j_i\,\vec{e}_j$. The position vector $\vec{r}$ of a point $x \in E^n$ in a frame $\mathfrak{R}$ can be written as $\vec{r} = x^i\,\vec{e}_i$ and, analogously, the position vector $\vec{r}'$ of the same point x in a new frame $\mathfrak{R}'$ can be written as $\vec{r}' = \tilde{x}^i\,\vec{e}'_i$. We have

$$\vec{e}'_j = A^i_j\,\vec{e}_i,\ \ \vec{r} = \vec{r}' + \overrightarrow{OO'},\ \ \overrightarrow{OO'} = a^i\,\vec{e}_i.$$

Hence

$$x^i\,\vec{e}_i = \tilde{x}^j\,\vec{e}'_j + a^i\,\vec{e}_i = A^i_j\,\tilde{x}^j\,\vec{e}_i + a^i\,\vec{e}_i = (A^i_j\,\tilde{x}^j + a^i)\,\vec{e}_i,$$

and

$$x^i = A^i_j\,\tilde{x}^j + a^i. \tag{1.4}$$

Formula (1.4) is called an affine transformation of coordinates. Note that an nth order matrix $A = (A^i_j)$ in (1.4) is a regular matrix, that is, $\mathrm{Det}\,A \neq 0$.

So far, we have considered a space that has only an affine structure. If we want to use the concepts of the length of a vector, a distance between two points, an angle between two vectors, we must assume that an affine space E^n has a Euclidean structure. Let V^n be a Euclidean space. A Euclidean structure in V^n is given by a scalar (or inner) product of vectors which we will denote by $<\vec{v},\vec{w}>$. This inner product is real-valued, symmetric, bilinear and positive-definite. The length of a vector $\vec{v}$ will be denoted by $\|\vec{v}\|$, i.e. $\|\vec{v}\|^2 = <\vec{v},\vec{v}>$. The angle α between two n-dimensional vectors $\vec{v},\vec{w}$ is defined by

$$\cos\alpha = \frac{<\vec{v},\vec{w}>}{\|\vec{v}\|\,\|\vec{w}\|}.$$

In what follows an affine space E^n such that its associated vector space V^n is endowed with a scalar product will be called an *affine Euclidean space*. In affine Euclidean space we can use orthonormal bases. In what follows we will usually assume that a coordinate system in an affine Euclidean space is constructed by means of an orthonormal frame $\mathfrak{R} = \{O;\vec{e}_1,\vec{e}_2,\ldots,\vec{e}_n\}$, where $<\vec{e}_i,\vec{e}_j>=\delta_{ij}$. The coordinate system induced by an orthonormal frame will be called a *Cartesian coordinate system.* An advantage of a Cartesian coordinates is that formulae for a length of a vector, for a distance between two points and for an angle have a simple form. For instance the scalar product of vectors $\vec{v},\vec{w}$, the square of the length of a vector $\vec{v}$ and the cosine of the angle between vectors $\vec{v},\vec{w}$ can be written in Cartesian coordinates as follows

$$<\vec{v},\vec{w}>=\sum_{i=1}^{n} v^i\,w^i, \quad \|\vec{v}\|^2 = \sum_{i=1}^{n}(v^i)^2, \quad \cos\phi = \frac{\sum_{i=1}^{n} v^i\,w^i}{\sqrt{\sum_{i=1}^{n}(v^i)^2}\sqrt{\sum_{j=1}^{n}(w^j)^2}}.$$

The distance between two points p,q of an affine Euclidean space will be denoted by $\|q-p\|$ and $\|q-p\| = \|\overrightarrow{pq}\|$, and the distance between points p,q can be expressed in terms of Cartesian coordinates as follows

$$\|q-p\| = \sqrt{\sum_{i=1}^{n}(q^i-p^i)^2}.$$

The transformation law for Cartesian coordinates has the same form (1.4) as the transformation law for affine coordinates, but in the case of Cartesian coordinates a matrix A in this formula is orthogonal, i.e. $A\in \mathrm{O}(n)$. It should be mentioned that if $A\in \mathrm{SO}(n)$ then an orientation of Cartesian coordinate system will remain the same.

The Euclidean structure of an affine Euclidean space E^n can be used to equip this space with a topology. We briefly recall how this topology in E^n is defined. An *open ball* $B_r(p)$ of radius r and centered at a point p is the following set of points

$$B_r(p) = \{q\in E^n : \|p-q\| < r\}.$$

Now a subset $U\subset E^n$ is said to be an *open subset* in E^n if for any point $q\in U$ there exists an open ball $B_r(q)$ such that $B_r(q)\subset U$. Hence we have the collection $\mathcal{T}$ of open subsets in E^n and it can be proved that this family satisfies all axioms of a topological space. Thus an affine Euclidean space E^n endowed with the collection of open subsets $\mathcal{T}$ is a topological space. It is worth to mention that an affine Euclidean space E^n as topological space is a Hausdorff space with a countable base.

Let $\mathscr{P}$ be an m-dimensional plane (1.1), passing through a point p and determined by linear independent vectors $\vec{v}_1,\vec{v}_2,\ldots,\vec{v}_m$. If $p^1,p^2,\ldots,p^n$ are coordinates of a point p,

$v_k^1, v_k^2, \ldots, v_k^n$ are coordinates of kth vector $\vec{v}_k$, then $\mathscr{P}$ can be described by the parametric equation

$$x^i = p^i + t_1 v_1^i + t_2 v_2^i + \ldots + t_m v_m^i, \;\; i = 1, 2, \ldots, n,$$

where $x^1, x^2, \ldots, x^n$ are coordinates of an arbitrary point of m-dimensional plane $\mathscr{P}$. Particularly, the parametric equation of a straight line $\mathscr{L}$ (1.2) has the form

$$x^i = p^i + t\, v^i, \;\; i = 1, 2, \ldots, n.$$

For the purposes of differential geometry it is useful to introduce a notion of a vector at a point of Euclidean space. A *vector at a point p of Euclidean space E^n* is a pair $(p; \vec{v})$, where $p \in E^n$ and $\vec{v} \in V^n$. A vector at a point $(p; \vec{v})$ will be denoted by $\mathtt{v}_p$, i.e. $\mathtt{v}_p = (p; \vec{v})$. It should be emphasized here that we do not consider a vector at a point $\mathtt{v}_p = (p; \vec{v})$ from the point of view of the affine structure of space E^n, that is, we do not identify a pair $(p; \vec{v})$ with the point $p + \vec{v}$. A vector at a point p will be also called a *tangent vector to Euclidean space* at a point p. The set of all tangent vectors to E^n at a point p will be denoted by T_pE^n. Hence $T_pE^n = \{p\} \times V^n$. If U is a subset of Euclidean space E^n then all vectors at points of U is the direct product $U \times V^n$, which will be denoted by $T(U)$. It is obvious that $T(U) = \cup_{p \in U} T_pE^n$, where $\cup$ stands for disjoint union of all $T_pE^n, p \in U$. We can define a projection $\pi : T(U) \to U$ of this direct product onto subset U by the formula $\pi(\mathtt{v}_p) = p$, where $p \in U$.

The vector space structure of V^n induces the vector space structure in the set of tangent vectors T_pE^n if for two tangent vectors $\mathtt{v}_p = (p; \vec{v}), \mathtt{w}_p = (p; \vec{w})$ we define

$$\mathtt{v}_p + \mathtt{w}_p = (p; \vec{v} + \vec{w}), \;\; a\, \mathtt{v}_p = (p; a\, \vec{v}), \;\; a \in \mathbb{R}.$$

Similarly the Euclidean structure of V^n induces the Euclidean structure in the vector space T_pE^n if we define the inner product of two tangent vectors $\mathtt{v}_p, \mathtt{w}_p$ by the formula

$$< \mathtt{v}_p, \mathtt{w}_p > = < \vec{v}, \vec{w} > .$$

The Euclidean vector space T_pE^n will be called a *tangent space to affine Euclidean space* E^n at a point p. It is worth to mention that a tangent vector $\mathtt{v}_p = (p; \vec{v})$ determines by means of affine structure of E^n the straight line, which passes a point p in the direction of a vector $\vec{v}$ and $p + t\, \vec{v}$ is the parametric equation of this straight line, which in Cartesian coordinates takes the form

$$x^i = p^i + v^i\, t, \; i = 1, 2 \ldots, n,$$

where p^i are the coordinates of a point p, v^i are the coordinates of a vector $\vec{v}$ and t is a parameter that runs through the whole real line. The tangent space T_pE^n at each point p is n-dimensional vector space. We can construct the orthonormal basis for this vector space by means of an orthonormal basis $\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n$ of the Euclidean space V^n if we define $\mathtt{e}_{p,i} = (p; \vec{e}_i), i = 1, 2, \ldots, n$. Then any tangent vector $\mathtt{v}_p = (p; \vec{v})$ can be written

$$\mathtt{v}_p = v^i\, \mathtt{e}_{p,i}$$

1.2 Curvilinear coordinate systems

In the previous section we showed that an affine frame $\mathfrak{R}$ in an affine space E^n defines an affine coordinate system in this space. However an affine coordinate system is a special case of a coordinate system in affine space. Other well-known coordinate systems are polar coordinates in a plane, spherical and cylindrical coordinates in three-dimensional space. Coordinate systems of this kind are called curvilinear coordinates in space. The aim of this section is to give a general definition of a curvilinear coordinate system.

Let E^n be an affine space and $\mathfrak{R}$ be an affine frame in this space. An affine frame $\mathfrak{R}$ induces the coordinate system in E^n whose affine coordinates will be denoted by $x^1, x^2, \ldots, x^n$. An affine coordinate system determines a mapping from affine space E^n to an n-dimensional arithmetic space $\mathbb{R}^n$. This mapping assigns to each point of an affine space an ordered set of n real numbers, which we call coordinates of a point p. Let us denote this mapping by $\phi : E^n \to \mathbb{R}^n$. Then for a point $x \in E^n$ we have $\phi(x) = (x^1, x^2, \ldots, x^n) \in \mathbb{R}^n$, where $x^1, x^2, \ldots, x^n$ are affine coordinates of a point $x \in E^n$ with respect to an affine frame $\mathfrak{R}$. Mapping ϕ has two very important properties which are actually the essence of the concept of a coordinate system. First this mapping is a bijection, that is, a one-to-one (injective) and onto (surjective) mapping. Hence ϕ has the inverse which will be denoted by $\phi^{-1} : \mathbb{R}^n \to E^n$. Second the mapping ϕ and its inverse are continuous mappings, i.e. ϕ is a homeomorphism between topological spaces E^n and $\mathbb{R}^n$. Evidently ϕ depends on a choice of an affine frame $\mathfrak{R}$, but its property to be a homeomorphism is invariant with respect to a choice of an affine frame.

By analogy with an affine coordinate system we can now give a general definition of a coordinate system in an affine space E^n. Let U be an open set in an affine space E^n and ψ be a bijection from a set U to an open subset W of an n-dimensional arithmetic space $\mathbb{R}^n$, that is, $\psi : U \subset E^n \to W \subset \mathbb{R}^n, \psi(U) = W$. Thus a mapping ψ maps each point x of an open subset U to an ordered set of n real numbers $(x^1, x^2, \ldots, x^n)$. It is important that if $x \neq y$ and $\psi(x) = (x^1, x^2, \ldots, x^n), \psi(y) = (y^1, y^2, \ldots, y^n)$ then $(x^1, x^2, \ldots, x^n) \neq (y^1, y^2, \ldots, y^n)$. Hence a mapping ψ defines a one-to-one parametrization of points in an open subset U of affine space E^n and therefore could be considered as a coordinate system in U. However this is not enough for the purposes of differential geometry. Differential geometry uses differential calculus to study curves and surfaces. A central concept here is the concept of a smooth function, that is, a function that has partial derivatives of any order and they are continuous functions. It is important for the purposes of differential geometry to have an invariant concept of a smooth function which does not depend on a choice of coordinate system. Thus we must impose additional conditions on a mapping ψ and for this we need a notion of a diffeomorphism.

Let V, V' be two open subsets of an n-dimensional arithmetic space $\mathbb{R}^n$ and $g : V \to V'$ be a mapping.

Definition 1.1. A mapping $g : V \to V'$ is called a *diffeomorphism* from V to V' if it satisfies the following conditions

1. g is a smooth bijection,
2. $g^{-1} : V' \to V$ is also a smooth mapping.

Let $\phi : U \subset E^n \to V \subset \mathbb{R}^n, \phi(U) = V$ is an affine coordinate system in an open subset U defined with the help of an affine frame.

Definition 1.2. A bijection $\psi : U \subset E^n \to W \subset \mathbb{R}^n$ is called a *coordinate system* in an open subset U of an affine space C^∞-compatible with coordinate system ϕ if the mapping $\chi = \phi \circ \psi^{-1} : W \subset \mathbb{R}^n \to V \subset \mathbb{R}^n$ is a diffeomorphism from W to V.

Obviously χ is a bijection because it is a composition of two bijections ϕ and ψ^{-1}. Hence Definition (1.2) requires a smoothness of f and χ^{-1}. In what follows we will tacitly assume that the smoothness condition of a coordinate system is satisfied and usually omit the word

"C^∞-compatible". Let $f : U \to \mathbb{R}$ be a real-valued function on an open subset $U \subset E^n$ and $\phi : U \to \mathbb{R}^n, \phi(U) = V$ be an affine system of coordinates. We will call a function f a smooth function if the real-valued function of n-variables $\tilde{f} = f \circ \phi^{-1} : V \to \mathbb{R}$ is a smooth function. It is important here that a notion of smoothness does not depend on a choice of coordinate system C^∞-compatible with ϕ. Indeed if $\psi : U \to W \subset \mathbb{R}^n$ is another coordinate system (C^∞-compatible with ϕ) then a function f is a smooth function if and only if it is a smooth function in a coordinate system ψ, i.e. $\tilde{\tilde{f}} = f \circ \psi^{-1} : W \to \mathbb{R}$ is a smooth function. This follows from the relations $\tilde{f} = \tilde{\tilde{f}} \circ \chi^{-1}, \tilde{\tilde{f}} = \tilde{f} \circ \chi$, where $\chi = \phi \circ \psi^{-1}$, and the fact that a composition of smooth mappings is a smooth mapping.

Let us denote by $x^1, x^2, \dots, x^n$ the affine coordinates in U defined by ϕ and by $\tilde{x}^1, \tilde{x}^2, \dots, \tilde{x}^n$ the coordinates in U defined by a coordinate system ψ. Then it is easy to see that a diffeomorphism $\chi : W \to V$ is uniquely determined by a set of n real-valued smooth functions $f^1, f^2, \dots, f^n$ whose domain is $W \subset \mathbb{R}^n$. By other words we can write

$$\chi = (f^1, f^2, \dots, f^n),\ \ \chi : (\tilde{x}^1, \tilde{x}^2, \dots, \tilde{x}^n) \in W \mapsto (x^1, x^2, \dots, x^n) \in V,$$

where

$$x^1 = f^1(\tilde{x}^1, \tilde{x}^2, \dots, \tilde{x}^n),\ x^2 = f^2(\tilde{x}^1, \tilde{x}^2, \dots, \tilde{x}^n), \dots, x^n = f^n(\tilde{x}^1, \tilde{x}^2, \dots, \tilde{x}^n), \tag{1.5}$$

The formulae (1.5) will be referred to as a *transition from an affine coordinate system ϕ to a coordinate system ψ* and we will write them in a more compact and symbolic way $x^i = f^i(\tilde{x}^j)$. If the functions $f^1, f^2, \dots, f^n$ in (1.5) are not linear then a coordinate system ψ will be called a *curvilinear coordinate system*.

In (1.5) $x^1, x^2, \dots, x^n$ are affine coordinates on an open set U. However we can look at this formula from a more general point of view, that is, assume that $x^1, x^2, \dots, x^n$ is a system of curvilinear coordinates. In this case formula (1.5) describes the transition from one coordinate system (not necessarily affine) to another. Therefore we will call (1.5) a *transition from one coordinate system to another*.

We can consider $\chi : V \subset \mathbb{R}^n \to W \subset \mathbb{R}^n$ as a transformation in an n-dimensional arithmetic space $\mathbb{R}^n$. Let us consider the Jacobian $\mathfrak{J}(\chi)$ of a transformation χ

$$\mathfrak{J}(\chi) = \mathrm{Det}\,(\frac{\partial x^i}{\partial \tilde{x}^j}). \tag{1.6}$$

Proposition 1.1. The Jacobian of a transition from one coordinate system to another $\chi : W \to V$ is nonzero at each point of W.

Proof. Due to the fact that $\chi : W \to V$ is a diffeomorphism it is a bijection and hence it is a reversible transformation, that is, we have an inverse transformation $\chi^{-1} : V \to W$. This means that in formula (1.5) we can express coordinates $\tilde{x}^1, \tilde{x}^2, \dots, \tilde{x}^n$ in terms of coordinates $x^1, x^2, \dots, x^n$, i.e.

$$\tilde{x}^1 = g^1(x^1, x^2, \dots, x^n),\ \tilde{x}^2 = g^2(x^1, x^2, \dots, x^n),\ \dots, \tilde{x}^n = g^n(x^1, x^2, \dots, x^n).$$

According to the definition of a diffeomorphism the functions $g^1, g^2, \dots, g^n$ are smooth functions. These functions satisfy

$$f^i\big(g^1(x^1, x^2, \dots, x^n), g^2(x^1, x^2, \dots, x^n), \dots, g^n(x^1, x^2, \dots, x^n)\big) \equiv x^i.$$

Differentiating the both sides of this formula with respect to x^j and taking into account that a function f^i at the left-hand side of this formula is a composite function of variables $x^1, x^2, \dots, x^n$ we get

$$\frac{\partial x^i}{\partial \tilde{x}^k}\,\frac{\partial \tilde{x}^k}{\partial x^j} = \delta^i_j.$$

Recall that we are using the Einstein summation convention and according to this convention there is a summation over k at the left-hand side of the above formula. If we now look at the above formula from the point of view of matrix calculus we see that on the left-hand side of this formula there is a product of two matrices $(\frac{\partial x^i}{\partial \tilde{x}^k}), (\frac{\partial \tilde{x}^k}{\partial x^j})$ and on the right-hand side there is an identity matrix. This implies that matrix $(\frac{\partial \tilde{x}^k}{\partial x^j})$ is the inverse of matrix $(\frac{\partial x^i}{\partial \tilde{x}^k})$, that is, the matrix $(\frac{\partial x^i}{\partial \tilde{x}^k})$ is invertible, which means it is regular, that is, the determinant of this matrix is nonzero. But the determinant of matrix $(\frac{\partial x^i}{\partial \tilde{x}^k})$ is the Jacobian of a transition from one coordinate system to another (1.6). $\square$

Now we will give a few examples of curvilinear coordinates.

Example 1.1. The first example of a curvilinear coordinate system is a *polar coordinate system* on a plane. Let E^2 be a two-dimensional affine Euclidean space which will be referred to as a plane. We construct a coordinate system on a plane E^2 which is different from an affine coordinate system constructed with the help of a frame. Fix a point O in a plane E^2 and a directed straight line L which passes through a point O. A point O will be referred to as a *pole* of a polar coordinate system and L as a *polar axis* of this system. Given a point p of a plane E^2, which is different from a pole O, we define its polar coordinates as follows. The distance between the pole O and a point p be the first polar coordinate of a point p and it will be denoted by r. Now we rotate the polar axis L counterclockwise

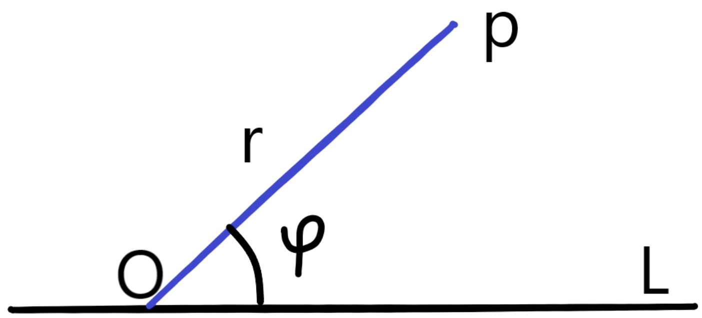

Figure 1.1: Polar coordinates

until it passes through a point p. The angle obtained by this rotation is the second polar coordinate of a point p, it is denoted by φ and is called a polar angle of a point p.

Obviously a polar angle in the case of a pole O is not defined. Thus a polar coordinate system defines a mapping which assigns to each point of a plane (with the exception of the pole O) the pair of real numbers (r, φ). Let $U = E^2 \setminus L$, $\mathbb{R}^2$ be a two-dimensional arithmetic space, whose coordinates are denoted by r, φ, $W = \{(r, \varphi) \in \mathbb{R}^2 : r > 0, 0 < \varphi < 2\pi\}$ and $\psi : U \to W$ be a mapping defined by polar coordinates. Then ψ is a bijection from $U \subset E^2$ onto open subset $W \subset \mathbb{R}^2$. If we endow a plane E^2 with the orthonormal frame $\{O, \vec{e}_1, \vec{e}_2\}$ whose origin coincides with the pole and positive direction of x-axis coincides with the polar axis L then the Cartesian coordinates x, y can be expressed in terms of polar coordinates as follows

$$x = r\,\cos\varphi, \quad y = r\,\sin\varphi. \tag{1.7}$$

Conversely the polar coordinates can be expressed in terms of Cartesian coordinates. Let $\mathbb{R}^2$ be a second copy of the 2-dimensional arithmetic space whose coordinates are

Cartesian coordinates x, y and V be an open subset of $\mathbb{R}^2$ defined by $V = \mathbb{R}^2 \setminus Ox$, where $Ox = \{(x, y) \in \mathbb{R}^2 : x \geq 0,\ y = 0\}$ is the x axis. Then the polar coordinates can be expressed in terms of Cartesian coordinates as follows

$$r = \sqrt{x^2 + y^2}, \quad \varphi = \begin{cases} \arccos \frac{x}{\sqrt{x^2+y^2}} & y \geq 0, \\ 2\pi - \arccos \frac{x}{\sqrt{x^2+y^2}} & y < 0. \end{cases}$$

Thus we see that the functions in these formulae determine mutually inverse bijections of two subsets V and W. We leave it as an exercise to a reader to show that these bijections are smooth.

The above description of polar coordinates shows that a polar coordinate system can be considered as a coordinate system in the sense of Definition (1.2). However in practical matters, for example, in the theory of plane curves it is convenient to use polar coordinates in a broader sense. In this approach we will assume that in order to find a polar angle of a point we can rotate a polar axis around the pole counterclockwise (positive angle) or clockwise (negative angle) any number of times. In this case a polar angle φ can be any real number and we will assume that polar coordinates $(r, \varphi \pm 2\pi k)$, where k is any integer, determine the same point of a plane.

Example 1.2. One of the most well-known coordinate systems in a 3-dimensional space is a cylindrical coordinate system. We fix a plane $\mathfrak{P}$ in three-dimensional affine Euclidean space E^3, a point O on this plane and two directed straight lines L, L' (i.e. endowed with fixed positive direction) that pass through point O. Thus the point O divides the directed straight lines L, L' into positive and negative parts. Line L' is perpendicular to the plane $\mathfrak{P}$, and line L lies in the plane $\mathfrak{P}$. Considering point O as the pole of a polar coordinate system, and line L as the polar axis we get the polar coordinate system on the plane $\mathfrak{P}$ described in detail in the previous example. Let us denote the coordinates of this polar coordinate system as r, φ. Consider some point p of an affine Euclidean space E^3 different from the point O. Let r, φ be the polar coordinates of the orthogonal projection p' of a point p onto the plane $\mathfrak{P}$, and z be the length of its orthogonal projection onto the directed straight line L' taken with plus if the projection of p lies on the positive part of L' and with minus otherwise. The set of numbers (r, φ, z) is called the cylindrical coordinates of a point p and the straight line L' is called a z-axis.

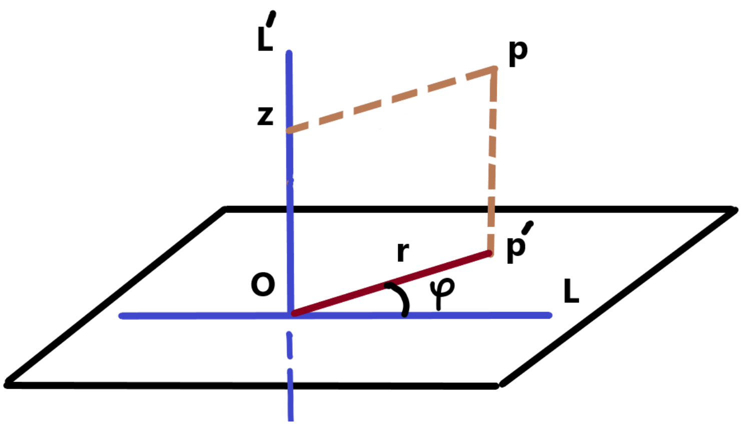

Figure 1.2: Cylindrical coordinates

If we now equip a 3-dimensional affine Euclidean space E^3 with a Cartesian coordinate system such that an origin of this coordinate system will be the point O, a positive part of

x-axis will be the polar axis L and a z-axis will be the line L' (with a positive direction defined on it) then the Cartesian coordinates x, y, z are expressed in terms of cylindrical coordinates r, φ, z as follows

$$x = r\cos\varphi, \;\; y = r\sin\varphi, \quad z = z. \tag{1.8}$$

This formula will be referred to as a transition from cylindrical coordinates to Cartesian ones. If we choose the open subset $U = E^3 \setminus L$ of an affine Euclidean space E^3 as a domain of a cylindrical coordinate system then a cylindrical coordinate system bijectively maps this subset to the open subset W of a 3-dimensional arithmetic space $\mathbb{R}^3$, where

$$W = \{(r, \varphi, z) \in \mathbb{R}^3 : r > 0, 0 < \varphi < 2\pi, -\infty < z < \infty\}. \tag{1.9}$$

Similar to the case of polar coordinates we can consider an open subset V of 3-dimensional arithmetic space $\mathbb{R}^3$ (whose coordinates are denoted by x, y, z) defined by $V = \mathbb{R}^3 \setminus Ox$, where

$$Ox = \{(x, y, z) \in \mathbb{R}^3 : x \geq 0, y = 0, z = 0\}.$$

Then the formula (1.8) defines the smooth bijection from W to V and the following formula (the transition from Cartesian coordinates to cylindrical ones)

$$r = \sqrt{x^2 + y^2 + z^2}, \;\; \varphi = \begin{cases} \arccos\frac{x}{\sqrt{x^2+y^2}} & y \geq 0, \\ 2\pi - \arccos\frac{x}{\sqrt{x^2+y^2}} & y < 0, \end{cases} \quad z = z,$$

defines its inverse which is also smooth. Hence a cylindrical coordinate system can be considered as a curvilinear coordinate system in a 3-dimensional affine Euclidean space E^3.

Example 1.3. A *spherical coordinate system* in a 3-dimensional affine Euclidean space E^3 is constructed similarly to a cylindrical coordinate system, that is, we fix a plane $\mathfrak{P}$, a point O on this plane and two directed straight lines L, L^prime passing through a point O where line L lies in a plane $\mathfrak{P}$ and line L' is perpendicular to $\mathfrak{P}$. Evidently the point O and the directed straight line L defines the polar coordinate system in the plane $\mathfrak{P}$. Let p be an arbitrary point of affine Euclidean space E^3 different from O. The first spherical coordinate of a point p is the distance between O and p. Let us denote this first spherical coordinate by r. Now let p' be the orthogonal projection of a point p to the plane $\mathfrak{P}$. Then the second spherical coordinate of a point p is the polar angle of the orthogonal projection p' in the polar coordinate system in the plane $\mathfrak{P}$ defined by O (pole) and directed line L (polar axis). Let us denote this second spherical coordinate by φ. The third spherical coordinate of a point A is the angle obtained by rotation of the positive part of line L' the shortest way around the point O until it passes through a point p. This third spherical coordinate will be denoted by ϑ. The set of three real numbers (r, φ, ϑ) is called spherical coordinates of a point p. A spherical coordinate system can be considered as a curvilinear coordinate system in E^3 in the sense of Definition (1.2) if its domain the open subset $U = E^3 \setminus (L \cup L')$ of E^3, that is, we remove from 3-dimensional affine Euclidean space E^3 the points of straight lines L, L'. Then a spherical coordinate system induces the bijection $p \in U \to (r, \varphi, \vartheta) \in W$ of U to the open subset $W \subset \mathbb{R}^3$, where

$$W = \{(r, \varphi, \vartheta) \in \mathbb{R}^3 : r > 0, 0 < \varphi < 2\pi, 0 < \vartheta < \pi\}.$$

If we endow a 3-dimensional space E^3 with a Cartesian coordinate system by placing its origin to the point O, choosing directed straight line L as x-axis, directed straight line L' as

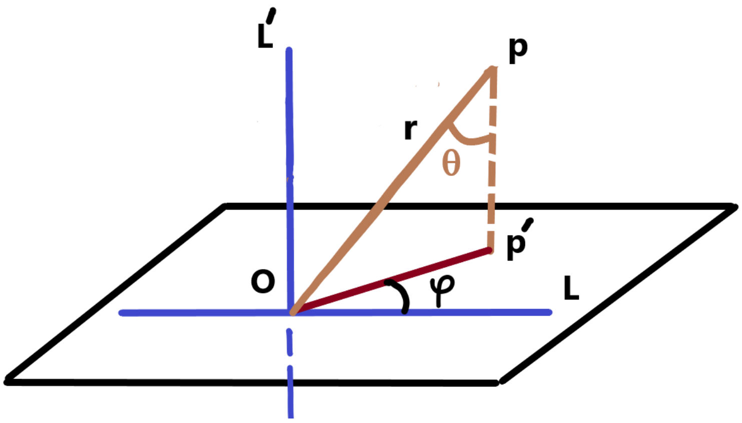

Figure 1.3: Spherical coordinates

z-axis and by constructing y-axis to be passing through O, perpendicular to x-axis, z-axis (and directed so that Cartesian coordinate system will be right-handed) then we can pass from spherical coordinates to the Cartesian coordinates by means of the formula

$$x = r\cos\varphi\sin\vartheta, \quad y = r\sin\varphi\sin\vartheta, \quad z = r\cos\vartheta. \tag{1.10}$$

The formula (1.10) defines the smooth bijection from the open set W to the open set $V \subset \mathbb{R}^3$, where $V = \mathbb{R}^3 \setminus V'$ and

$$V' = \{(x, y, z) \in \mathbb{R}^3 : x \geq 0, y = z = 0 \vee x = y = 0, -\infty < z < \infty\}.$$

Just as in the case of cylindrical coordinates one can find a formula for the transition from Cartesian to spherical coordinates and show that it induces the inverse smooth bijection from V to W. We leave all this as an exercise.

1.3 Algebra of smooth functions on Euclidean space

Let U be an open subset of Euclidean space E^n, where we do not exclude the possibility that this set may coincide with the whole space E^n. Let $f : U \to \mathbb{R}$ be a real-valued function, where U is a domain of this function. In what follows we will assume (unless otherwise stated) that the Euclidean space E^n is equipped with a Euclidean frame $\{O; \vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}$, which determines the Cartesian coordinates in E^n. A function f depends on a point x of an open set U, but we can identify a point x with its coordinates $x^1, x^2, \ldots, x^n$, and then a function f can be considered as a function of n real variables $f(x^1, x^2, \ldots, x^n)$. In order to simplify notations we will often write $f(x)$, implying that a point x is identified with its coordinates $x^1, x^2, \ldots, x^n$.

Smooth functions play an important role in differential geometry of Euclidean space E^n. A function $f : U \to \mathbb{R}$ is called a *smooth function* if it has continuous derivatives of all orders. The set of all smooth functions over an open subset U will be denoted by $C^\infty(U)$. What operations can we perform with smooth functions? Adding the values of two smooth functions $f, g \in C^\infty(U)$ at every point $x \in U$, we get the new function $f + g$, which is called the sum of functions f, g. Thus the value of the function $f + g$ at a point x is $f(x) + g(x)$ and this is called the pointwise addition of smooth functions. It is easy to show that the sum $f + g$ is also a smooth function, i.e. $f + g \in C^\infty(U)$. Hence the set of smooth functions is closed under the pointwise addition of smooth functions. Similarly, multiplying the value of a smooth function f at each point of U by a real number a, we get the function af and this operation is called the multiplication of functions by real numbers. Analogously, we can define the pointwise product fg of two smooth functions by the formula $(f\,g)(x) = f(x)\,g(x)$. It is also easy to show that af and fg are smooth functions, i.e. the set of smooth functions is closed under the pointwise multiplication of functions as well as the multiplication of smooth functions by real numbers.

What algebraic structure in the set of smooth functions is determined by the described above operations? If we consider only pointwise addition of smooth functions, then the set of all smooth functions is an additive Abelian group. It is easy to show that the pointwise addition of functions together with the multiplication of functions by real numbers yield the structure of vector space over the field of real numbers $\mathbb{R}$. Hence $C^\infty(U)$ is a vector space. If we add the pointwise multiplication of functions to these two algebraic operations, then the vector space $C^\infty(U)$ of smooth functions becomes an associative unital commutative algebra. *An associative unital algebra* is a vector space $\mathscr{A}$ (with an addition of elements u, v denoted by $u + v$ and a multiplication of elements by real numbers denoted by $a\,u$, where a is a real number) equipped with a mapping $(u, v) \in \mathscr{A} \times \mathscr{A} \to u \cdot v \in \mathscr{A}$, which is

1. bilinear, i.e. $(au + bv) \cdot w = au \cdot w + bv \cdot w,\ w \cdot (au + bv) = aw \cdot u + bw \cdot v$, where u, v, w are elements of a vector space $\mathscr{A}$ and a, b are real numbers,
2. associative, i.e. $u \cdot (v \cdot w) = (u \cdot v) \cdot w$,
3. there exists in $\mathscr{A}$ the element e such that $u \cdot e = e \cdot u = u$ for any element $u \in \mathscr{A}$.

A mapping $(u, v) \in \mathscr{A} \times \mathscr{A} \to u \cdot v \in \mathscr{A}$ is called a multiplication of algebra $\mathscr{A}$ and element e is called an *identity element* of $\mathscr{A}$. In what follows we will use the term algebra to mean an associative unital algebra. If the product of any two elements of algebra does not depend on the order in which we multiply these elements, i.e. $u \cdot v = v \cdot u$, then the algebra is called *commutative*.

It is easy to see that the pointwise multiplication of smooth functions is bilinear with respect to the pointwise addition of smooth functions and the pointwise multiplication of smooth functions by real numbers. It is also associative and commutative. The constant function $\mathbf{1}$, whose value at any point $x \in U$ is 1, i.e. $\mathbf{1}(x) = 1$, is the identity element for pointwise multiplication of smooth functions. Thus the set of all smooth functions $C^\infty(U)$ over an open subset U of Euclidean space E^n is a commutative algebra.

If we consider only two operations with functions, pointwise addition and multiplication, then the set of all smooth functions $C^\infty(U)$ becomes a commutative associative ring with the unit element and this structure is often used in literature.

An orthonormal frame $\mathfrak{E} = \{O; \vec{e}_1, \vec{e}_2, \dots, \vec{e}_n\}$ determines the Cartesian coordinate system in the affine Euclidean space E^n. This means that each point x has the coordinates $x^1, x^2, \dots, x^n$. These coordinates are real numbers and they are uniquely determined by a point x and two different points have different coordinates. For the purposes of differential geometry, it is useful to look at coordinates as functions. For these functions, we will use the same notations, which we use for coordinates, i.e. these functions will be denoted by $x^1, x^2, \dots, x^n$. This will not lead to confusion, because every time we specify, we consider x^i as a coordinate or as a function. If we consider x^i as a function, then its value at a point x is the ith coordinate of x, that is, $x^i(x) = x^i$. Obviously, every function x^i is a smooth function, because we have

$$\frac{\partial x^i}{\partial x^j} = \delta^i_j.$$

The functions $x^1, x^2, \dots, x^n$ are called *Cartesian coordinate functions*.

1.4 Parametrized curves in Euclidean space

In this section we assume that n-dimensional Euclidean space E^n is equipped with an orthonormal frame $\mathfrak{E} = \{O; \vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}$, which determines the Cartesian coordinates $x^1, x^2, \ldots, x^n$. A *parametrized curve* in E^n is a mapping

$$\xi : I \to E^n, \quad \xi : t \in I \mapsto (x^1(t), x^2(t), \ldots, x^n(t)) \in E^n, \tag{1.11}$$

where t is a parameter and $I \subset \mathbb{R}$ is an interval, which may be either finite or infinite. We will denote a parametrized curve by $(\xi = \xi(t), I)$ or simply by ξ. A parametrized curve can also be written in the form

$$\xi(t) = (x^1(t), x^2(t), \ldots, x^n(t)), \tag{1.12}$$

which will be called a *parametric equation of a parametrized curve* ξ. A parametrized curve ξ is called *smooth* if all functions $x^1(t), x^2(t), \ldots, x^n(t)$ in its parametric equation are smooth. Recall that a function with domain D is called smooth if it has derivatives of all orders and each of these derivatives is a continuous function on D. A smooth function is also referred to as an infinitely differentiable function. From now on, we will primarily consider smooth curves, so we will omit the word "smooth."

We can pass to radius vectors of the points of affine space E^n. Then the mapping (1.11) will assign to each value of the parameter $t \in I$ the radius vector of the point $\xi(t)$, which we denote by $\vec{\xi}(t)$. The parametric equation of a curve (1.12) can now be written in the form

$$\vec{\xi}(t) = (x^1(t), x^2(t), \ldots, x^n(t)), \tag{1.13}$$

where the right-hand side contains the coordinates of the vector $\vec{\xi}(t)$.

Equation (1.13) defines a mapping from the interval I into an n-dimensional vector space V^n. Such mappings are called *vector-valued functions.* In this case, $\vec{\xi}(t)$ is a vector-valued function of a single variable t.

It is worth noting that vector-valued functions can be differentiated and expanded into a Taylor series, with formulas analogous to those for ordinary real-valued functions. Operations from vector algebra, such as the dot product, cross product, and triple product, can be extended to vector-valued functions. Moreover, these operations satisfy the Leibniz rule for differentiation, for example, in the case of the dot product

$$\frac{d}{dt} < \vec{\xi}(t), \vec{\eta}(t) >=< \frac{d}{dt}(\vec{\xi}(t)), \vec{\eta}(t) > + < \vec{\xi}(t), \frac{d}{dt}(\vec{\eta}(t)) > .$$

For example, the vector-valued function $\vec{\xi}(t) = \vec{p} + t\,\vec{v}$ defines a line with direction vector $\vec{v}$ passing through the point with radius vector $\vec{p}$. The coordinate form of this vector equation is

$$\xi(t) = (p^1 + v^1\,t, p^2 + v^2\,t, \ldots, p^n + v^n\,t), \tag{1.14}$$

where $p^1, p^2, \ldots, p^n$ are coordinates of a point p and $\vec{v} = (v^1, v^2, \ldots, v^n)$ are coordinates of a vector $\vec{v}$.

The image $\operatorname{Im}\xi \subset E^n$ of a mapping $\xi : I \to E^n$ will be called the *image of a parametrized curve.* Thus, the image of a parametrized curve is the set of all points in the n-dimensional Euclidean space E^n as the parameter t varies within the interval I. This is precisely the image we visualize when referring to a curve. In what follows we will often refer to the image $\operatorname{Im}\xi$ of a parametrized curve (ξ, I) simply as a curve and a mapping $\xi : I \to E^n$ as its parametrization.

The basic notion of the differential geometry of curves is the notion of a tangent line to a curve. Let $p = \xi(t_0)$, $q = \xi(t_0 + \Delta t)$, where $t_0, t_0 + \Delta t \in I$, be two points of a parametrized curve ξ. The straight line that passes through these two points p, q is called a *secant* of a parametrized curve ξ. If Δt tends to zero, i.e. point q approaches to a point q along a parametrized curve ξ, then limiting position of a secant is called the *tangent line* to a parametrized curve at a point p. The tangent line at a point p is a linear approximation of a parametrized curve ξ at this point. In order to find this linear approximation, we can expand a vector-valued function $\vec{\xi} = \vec{\xi}(t)$ as power series in t at a point t_0. We get

$$\vec{\xi}(t) = \vec{\xi}(t_0) + \vec{\xi}\,'(t_0)\,\Delta t + \mathcal{O}(\Delta t), \tag{1.15}$$

where

$$\vec{\xi}\,'(t_0) = \frac{d\vec{\xi}(t)}{dt}\Big|_{t=t_0} = \Big(\frac{dx^1}{dt}\Big|_{t=t_0}, \frac{dx^2}{dt}\Big|_{t=t_0}, \dots, \frac{dx^n}{dt}\Big|_{t=t_0}\Big),$$

is the derivative of a vector-valued function and $\mathcal{O}(\Delta t)$ are the higher order terms. From (1.15) it follows that the parametric equation of a tangent line can be written in the form $\xi(t_0) + \tau\,\vec{\xi}\,'(t_0)$, where τ is the parameter of a tangent line. Thus the vector $\vec{\xi}\,'(t_0)$ is a directional vector of the tangent line. We will call this vector the *tangent vector* to a curve or the *velocity vector* of the parametrized curve ξ at a point p. It is often convenient to consider the velocity vector as a vector at a point p. In this case, we will write it in the form $\xi'(t_0) = (\xi(t_0); \vec{\xi}\,'(t_0))$. Thus $\vec{\xi}\,'(t_0) \in V^n$ and $\xi'(t_0) \in T_pE^n$.

Similarly the *acceleration vector* $\xi''(t_0)$ of a parametrized curve ξ at a point $p = \xi(t_0)$ is defined with the help of second order derivatives, that is,

$$\xi''(t_0) = \Big(p; \frac{d^2x^1}{dt^2}\Big|_{t=t_0}, \frac{d^2x^2}{dt^2}\Big|_{t=t_0}, \dots, \frac{d^2x^n}{dt^2}\Big|_{t=t_0}\Big). \tag{1.16}$$

A point $p = \xi(t_0)$ of a parametrized curve $(\xi = \xi(t), I)$ is called a *regular point* if the velocity vector $\xi'(t_0)$ at this point is a non-zero vector, that is, $\|\xi'(t_0)\| \neq 0$. A parametrized curve $(\xi = \xi(t), I)$ is called a *regular parametrized curve* if each point of this parametrized curve is a regular point. Since we will study only regular parametrized curves the condition of regularity will be tacitly assumed and the word "regular" will be usually omitted.

A point $p = \xi(t_0)$ of a parametrized curve ξ is called a *point of rectification* if the acceleration vector $\xi''(t_0)$ and the velocity vector $\xi'(t_0)$ at this point are collinear vectors, i.e. $\xi'(t_0)||\xi''(t_0)$. In what follows we will exclude from the consideration the points of rectification of a parametrized curve, that is, we will assume that in addition to the condition of regularity of a parametrized curve $\xi'(t) \neq 0$ we have also the condition of non-collinearity of the velocity vector and the acceleration vector $\xi'(t_0) \nparallel \xi''(t_0)$.

Naturally, if we consider the image of a parametrized curve $(\xi = \xi(t), I)$, then the set of points $\mathrm{Im}\,\xi$ can be parametrized in many different ways. By other words, we can have two different parametrized curves $(\xi = \xi(t), I)$ and $(\eta = \eta(\tau), J)$ with the same images, i.e. $\mathrm{Im}\,\xi = \mathrm{Im}\,\eta$. Indeed, consider a smooth function $t = t(\tau)$, whose domain is an interval $J = (c, d) \subset \mathbb{R}$ and a range is an interval $I = (a, b)$. Assume that the derivative $t'(\tau)$ with respect to τ is non-zero at each point $\tau \in J$. Clearly, $t = t(\tau)$ is either strictly increasing $(t'(\tau) > 0)$ or strictly decreasing function $(t'(\tau) < 0)$. The product $\xi \circ t : J \to E^n$, where t is considered as the mapping $t : \tau \in J \mapsto t(\tau) \in I$ determined by a function $t = t(\tau)$, is the parametrized curve $(\eta = \eta(\tau), J)$, where $\eta = \xi \circ t$. Then $\mathrm{Im}\,\xi = \mathrm{Im}\,\eta$. The parametrized curve $(\eta = \eta(\tau), J)$ is called a *reparametrization* of a parametrized curve $(\xi = \xi(t), I)$ by means of a function $t = t(\tau)$.

The coordinates of a velocity vector depend on a parametrization of a curve. Therefore it is natural to raise a question of how the coordinates of a velocity vector transform under

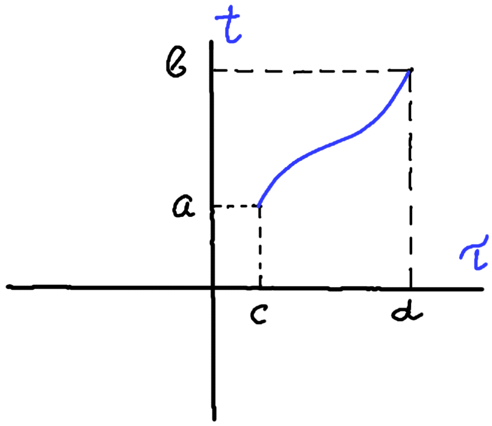

Figure 1.4: Function $t = t(\tau)$

a reparametrization of a curve. Let (ξ, I) and (η, J) be two different parametrizations of a curve, where $\eta = \xi \circ t$ and $t = t(\tau)$ is a function $t : J \to I$, whose derivative with respect τ is non-zero at each point $\tau \in J$. Let

$$\xi(t) = (x^1(t), x^2(t), \ldots, x^n(t)), \quad \eta(\tau) = (y^1(\tau), y^2(\tau), \ldots, y^n(\tau)),$$

where $t \in I, \tau \in J$, be the parametric equations of parametrized curves ξ and η. Then $y^i(\tau) = x^i(t(\tau))$. Making use of the rule of differentiation of a composite function we get

$$\frac{d}{d\tau}\big(y^i(\tau)\big) = \frac{d}{d\tau}\big(t(\tau)\big)\,\frac{d}{dt}\big(x^i(t)\big)\big|_{t=t(\tau)}.$$

Thus if we pass from one parametrization ξ to another η with the help of a function $t = t(\tau)$ then a velocity vector will transform according to a formula

$$\eta'(\tau) = t'(\tau)\,\xi'(t)\big|_{t=t(\tau)}, \; t'(\tau) = \frac{d}{d\tau}\big(t(\tau)\big), \tag{1.17}$$

which is called *a chain rule for a parametrized curve.* Differentiating both parts of the formula (1.17) one more time we obtain the formula for a transformation of acceleration vector when a curve $\xi(t)$ is reparametrized $\eta(\tau) = \xi(t(\tau))$

$$\eta''(\tau) = \big(t'(\tau)\big)^2\,\xi''(t)\big|_{t=t(\tau)} + t''(\tau)\,\xi'(t)\big|_{t=t(\tau)}. \tag{1.18}$$

Hence we proved the following statement.

Proposition 1.2. Let $(\xi = \xi(t), I)$ be a (regular) parametrized curve and $\eta = (\eta(\tau), J)$ be its reparametrization by means of a function $t = t(\tau), t : J \to I$. Then a velocity vector and an acceleration vector transform as follows

$$\begin{aligned}\eta'(\tau) &= t'(\tau)\,\xi'(t)\big|_{t=t(\tau)},\\ \eta''(\tau) &= \big(t'(\tau)\big)^2\,\xi''(t)\big|_{t=t(\tau)} + t''(\tau)\,\xi'(t)\big|_{t=t(\tau)}.\end{aligned}$$

From this proposition it follows that the condition of regularity of a parametrized curve is invariant under a reparametrization. Indeed the first formula in Proposition (1.2) shows that the reparametrized curve η is regular because of $\xi'(t) \neq 0$ and $t'(\tau) \neq 0$. It also follows from Proposition (1.2) that the condition $\xi'(t_0) \nparallel \xi''(t_0)$ (which excludes from consideration the points of rectification) does not depend on a parametrization of a curve. It follows from the first formula of Proposition (1.2) that the vector $\eta'(\tau)$ is collinear with the vector $\xi'(t)$. Next it is easy to see that the vector $\big(t'(\tau)\big)^2\,\xi''(t)$ is a non-zero vector. Indeed we have $t'(\tau) \neq 0$. The acceleration vector $\xi''(t)$ can not be zero vector because this would contradict with the assumption $\xi'(t_0) \nparallel \xi''(t_0)$. Now the second formula of Proposition (1.2) shows that acceleration vector $\eta''(\tau)$ is obtained by adding the non-zero vector $\big(t'(\tau)\big)^2\,\xi''(t)$ to vector, which is collinear to $\eta'(\tau)$. Thus the resulting vector $\eta''(\tau)$ cannot be collinear with vector $\eta'(\tau)$.

Let $(\xi = \xi(t), I)$ be a parametrized curve and $t_0, t_1 \in I$, where $t_0 < t_1$. Then the *length of an arc of a parametrized curve* ξ between two points t_0, t_1 is given by the integral

$$s = \int_{t_0}^{t_1} \|\xi'(t)\|\, dt. \tag{1.19}$$

The integrand is an infinitesimal element of an arc length and is often denoted by $ds = \|\xi'(t)\|\, dt$. We will also use this notation and call ds the arc length differential. By virtue of the fact, mentioned above, that the image of a parametrized curve does not change upon reparametrization, it can be proved that the length of an arc of a parametrized curve is invariant with respect to reparametrization (Exercise **??**).

A parametrized curve $(\xi = \xi(t), I)$ is called a *unit-speed parametrized curve* if for any point $t \in I$ we have $\|\vec{\xi}'(t)\| = 1$, i.e. the velocity vector at each point of a parametrized curve is a unit vector.

Proposition 1.3. If $(\xi = \xi(s), I)$ is a unit-speed parametrized curve, then

$$< \vec{\xi}''(s), \vec{\xi}'(s) >= 0.$$

Proof. We have $\|\vec{\xi}'(s)\| = 1$ or $< \vec{\xi}'(s), \vec{\xi}'(s) >= 1$. Differentiating the left-hand side of this relation with respect to s, we obtain

$$(\|\vec{\xi}'(s)\|^2)' =< \vec{\xi}''(s), \vec{\xi}'(s) > + < \vec{\xi}'(s), \vec{\xi}''(s) >= 2 < \vec{\xi}''(s), \vec{\xi}'(s) > .$$

The derivative of the right-hand side is equal to zero and we get

$$< \vec{\xi}''(s), \vec{\xi}'(s) >= 0.$$

□

Theorem 1.1. For any parametrized curve $(\xi = \xi(t), I)$ there exists a unit-speed parametrization.

Proof. Fix a point $t_0 \in I$ and define the function

$$s(t) = \int_{t_0}^{t} \|\xi'(u)\|\, du, \quad t \in I. \tag{1.20}$$

The domain of this function is an interval I. Let us denote the range of the function $s = s(t)$ by J. Then $s : t \in I \mapsto s(t) \in J$. It follows from the theory of functions that $s = s(t)$ is a smooth function. We can calculate the derivative of $s = s(t)$ by means of the general formula

$$F'(t) = \int_{g(t)}^{f(t)} F'_t(u,t)\,du + f'(t)\,F(f(t),t) - g'(t)\,F(g(t),t), \tag{1.21}$$

where

$$F(t) = \int_{g(t)}^{f(t)} F(u,t)\,du,$$

and $F(u,t), f(t), g(t)$ are smooth functions. In the case of the function $s = s(t)$ the first and last terms in (1.21) vanish because the integrand and the lower limit of integration in (1.20) do not depend on the parameter t. Thus

$$s'(t) = \frac{ds}{dt} = \frac{dt}{dt}\,\|\xi'(t)\| = \|\xi'(t)\|. \tag{1.22}$$

Since we consider only regular parametrized curves for any $t \in I$ it holds $s'(t) \neq 0$. Hence there exists uniquely determined inverse function for $s = s(t)$, which is also smooth. The inverse function will be denoted by $t = t(s)$ and $t : s \in J \mapsto t(s) \in I$. The derivative of the inverse function can be expressed in terms of the derivative of the function $s(t)$ as follows

$$t'(s) = \frac{1}{s'(t)}\Big|_{t=t(s)}. \tag{1.23}$$

Now we can construct a reparametrization of a parametrized curve $(\xi = \xi(t), I)$ with the help of the inverse function $t = t(s)$. Define the parametrized curve $(\eta = \eta(s), J)$, where $\eta(s) = \xi(t(s))$. Making use of the chain rule for parametrized curves and the formula (1.22) for the derivative of the function $s = s(t)$, we find

$$\|\eta'(s)\| = t'(s)\,(\|\xi'(t)\|)\Big|_{t=t(s)} = \frac{1}{s'(t)}\Big|_{t=t(s)}\,(\|\xi'(t)\|)\Big|_{t=t(s)} = 1.$$

Thus we see that the parametrized curve $(\eta = \eta(s), J)$ has the unit-speed parametrization and this ends the proof. □

From the proof of Theorem 1.1 it is clear that in order to construct a unit-speed parametrization of a parametrized curve one uses the length of arc as a parameter. This parameter is called *natural parameter* and unit-speed parametrization is called *natural parametrization*.

Example 1.4. Let us show, using a catenary as an example, how a unit-speed parametrization is constructed. We assume that a plane E^2 is equipped with an orthonormal frame that defines a Cartesian coordinate system, whose coordinates we will denote by x, y. The curve, that is called *catenary*, is a plane curve and usually given by the equation

$$y = a\,\cosh\,(\frac{x}{a}), \tag{1.24}$$

where $a > 0$ is a real number. This curve is of mechanical origin. If we have a sufficiently heavy, flexible, inextensible cable that sags under the action of gravity, being fixed at the ends at the same level, then the shape of this cable is the catenary. Equation (1.24) is easy to write in a parametric form. To do this, we can choose the variable x as a parameter t. Then a catenary can be written in the form of a parametrized curve as follows

$$\xi(t) = (t, a\,\cosh\,(\frac{t}{a})),\ I = \mathbb{R}.$$

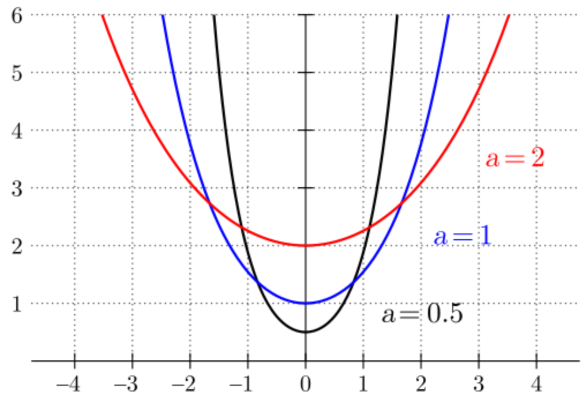

Figure 1.5: Catenary

Differentiating with respect to t, we find the velocity vector and its length

$$\xi'(t) = (\xi(t);\, 1, \sinh(\frac{t}{a})), \quad ||\xi'(t)|| = \sqrt{1+\sinh^2(\frac{t}{a})} = \cosh(\frac{t}{a})$$

From this we obtain the arc length function of the catenary

$$s(t) = \int_0^t \cosh(\frac{u}{a})\, du = a \int_0^t \cosh(\frac{u}{a})\, d(\frac{u}{a}) = a\, \sinh(\frac{u}{a})\Big|_0^t = a\, \sinh(\frac{t}{a}).$$

Note that the lower limit of integration in this formula shows that we have chosen the vertex of the catenary as the reference point for the arc length, that is, the point of its intersection with axis y. Now we have to find the inverse function for $s = a\,\sinh(\frac{t}{a})$. We have

$$t = a\,\sinh^{-1}(\frac{s}{a}) = a\,\ln\frac{s+\sqrt{s^2+a^2}}{a}.$$

Here $\sinh^{-1}$ stands for the inverse function of sinh and we used the formula $\sinh^{-1} x = \ln(x+\sqrt{x^2+1})$. Thus, we obtain a natural parametrization of the catenary

$$\beta(s) = (a\,\sinh^{-1}(\frac{s}{a}), a\,\cosh(\sinh^{-1}(\frac{s}{a}))). \tag{1.25}$$

The second function of this parametrization can be simplified by means of $\cosh(\sinh^{-1} x) = \sqrt{1+x^2}$, the application of which gives the final result

$$\beta(s) = (a\,\sinh^{-1}(\frac{s}{a}), \sqrt{s^2+a^2}).$$

Example 1.5. In this example, consider the natural parametrization of a helix. A helix is a spatial curve that is defined by the parametric equation $(\xi = \xi(t), I)$, where

$$\xi(t) = (a\,\cos t, a\,\sin t, bt), \quad I = \mathbb{R}$$

and $a > 0, b > 0$ are constant real numbers. First of all, let us calculate the arc length function of a helix. We have

$$\begin{aligned}
\xi'(t) &= (\xi(t); -a\,\sin t, a\,\cos t, b), \\
||\xi'(t)|| &= \sqrt{a^2+b^2}, \\
s(t) &= \int_0^t \sqrt{a^2+b^2}\, du = \sqrt{a^2+b^2}\, t.
\end{aligned}$$

In this case, the inverse function is very easy to find, in the last formula we should simply express t through s. We have

$$t = \frac{s}{\sqrt{a^2+b^2}}.$$

Hence the unit-speed parametrization of a helix is

$$\eta(s) = (a\,\cos\frac{s}{\sqrt{a^2+b^2}}, a\,\sin\frac{s}{\sqrt{a^2+b^2}}, \frac{b\,s}{\sqrt{a^2+b^2}}). \tag{1.26}$$

1.5 Vector fields along a parametrized curve

A concept of a vector field plays an important role in modern differential geometry. Vector fields play an important role in applications of differential geometry in various fields of physics and mechanics. For example, the distribution of velocities of flow of a fluid at some fixed moment of time can be described by means of a geometric notion of a vector field.

Let U be an open subset of n-dimensional Euclidean space E^n. Then a *vector field* $\mathtt{X}$ in U is a mapping $\mathtt{X} : U \to T(U)$ such that $\pi \circ \mathtt{X} = \mathrm{id}_U$, where $\pi : T(U) \to U$ is the projection and id_U is the identity mapping of U. Thus a vector field is a mapping which assigns to each point p of U a vector at a point p. This vector at a point p will be denoted by $\mathtt{X}_p$. A subset U will be called a domain of a vector field $\mathtt{X}$. It is evident that a vector field $\mathtt{X}$ is uniquely determined by n real-valued functions $X^1, X^2, \ldots, X^n$ defined on U, that is, $X^i : U \to \mathbb{R}$, where $\mathtt{X}_p = (p; X^1(p), X^2(p), \ldots, X^n(p))$, and $X^i(p)$ is a value of an ith function at a point p. A function X^i will be referred to as an ith component of a vector field $\mathtt{X}$. If there is a coordinate system in an open subset U with coordinates $x^1, x^2, \ldots, x^n$ the the components of a vector field $\mathtt{X}$ are functions of coordinates $X^i(x^1, x^2, \ldots, x^n)$, that is, functions of n real variables. In what follows we will assume that all components $X^1, X^2, \ldots, X^n$ are smooth functions, that is, $X^1, X^2, \ldots, X^n \in C^\infty(U)$. By other words, we will consider only smooth vector fields omitting the word "smooth". Obviously a vector field $\mathtt{X}$ is completely determined by the following vector-valued function

$$\overrightarrow{\mathtt{X}} = (X^1, X^2, \ldots, X^n) : U \to \mathbb{R}^n, \quad \overrightarrow{\mathtt{X}}_p = (X^1(p), X^2(p), \ldots, X^n(p)).$$

Example 1.6. Let $f \in C^\infty(U)$ be a smooth function. Define a vector field $\mathrm{grad}\, f$ by the following formula

$$(\mathrm{grad}\, f)_p = \Big(p; \frac{\partial f}{\partial x^1}\Big|_p, \frac{\partial f}{\partial x^2}\Big|_p, \ldots, \frac{\partial f}{\partial x^n}\Big|_p\Big),$$

where $p \in U$. The vector field ∇f is called the gradient of a function f. Obviously the gradient of a smooth function is a smooth vector field. The corresponding vector-valued function of the gradient of a function will be denoted $\overrightarrow{\nabla} f$, i.e.

$$\overrightarrow{\mathrm{grad}}\, f = \Big(\frac{\partial f}{\partial x^1}, \frac{\partial f}{\partial x^2}, \ldots, \frac{\partial f}{\partial x^n}\Big).$$

We can perform the following operations with vectors: addition, multiplication by real numbers, inner product and in three-dimensional space also a cross product. We can extend in obvious way (pointwise) all these operations to vector fields, i.e. for example we can define the sum of two vector fields $\mathtt{X} + \mathtt{Y}$ as the vector field, whose value at a point $p \in U$ is the vector $\mathtt{X}_p + \mathtt{Y}_p$. Analogously we can define a vector field $\mathtt{X}$ multiplied by a real number a and the inner product of two vector fields $< \mathtt{X}, \mathtt{Y} >$, where $< \mathtt{X}, \mathtt{Y} >$ is the function. It is easy to show that if $\mathtt{X}, \mathtt{Y}$ are smooth vector fields, then $\mathtt{X} + \mathtt{Y}, a\,\mathtt{X}$ are also smooth vector fields and $< \mathtt{X}, \mathtt{Y} >$ is the smooth function. Let us denote the set of all smooth vector fields on an open set $U \subset E^n$ by $\mathcal{D}(U)$. It is clear that the operation of addition of two vector fields determines the structure of additive Abelian group on $\mathcal{D}(U)$. The addition of vector fields and their multiplication by real numbers turn the set $\mathcal{D}(U)$ into the vector space over $\mathbb{R}$ and it should be mentioned that this vector space is infinite dimensional.

The operation of multiplying vector fields by real numbers can be slightly changed, assuming that a number depends on a point of open set U, i.e. it is a function defined on U. In this case we get a new operation of multiplying vector fields by smooth functions. Given a smooth function $f \in C^\infty(U)$ and a vector field $\mathtt{X}$, we define the vector field $f\,\mathtt{X}$, whose value at a point p is the vector $f(p)\,\mathtt{X}_p$. It is easy to show that this operation has the following algebraic properties:

1. $f\,(\mathtt{X} + \mathtt{X}) = f\,\mathtt{X} + f\,\mathtt{Y}$,
2. $(f + g)\,\mathtt{X} = f\,\mathtt{X} + g\,\mathtt{X}$,
3. $(f\,g)\,\mathtt{X} = f\,(g\,\mathtt{X})$,
4. $\mathbf{1}\,\mathtt{X} = \mathtt{X}$.

These properties show that multiplication of vector fields by smooth functions determines on the Abelian group of vector fields $\mathcal{D}(U)$ the structure of *module over the ring of smooth functions* $C^\infty(U)$.

The concept of a vector field is very useful in the theory of curves. A vector field along a curve is a very natural concept and is analogous to a vector field in Euclidean space. A vector field along a parametrized curve $(\xi = \xi(t), I)$ is a mapping, which assigns to each point of a parametrized curve a vector at this point. A vector field along a parametrized curve $(\xi = \xi(t), I)$ will be denoted by $\mathtt{X}(t)$. Hence,

$$\mathtt{X}(t) = (\xi(t); X^1(t), X^2(t), \ldots, X^n(t)),$$

where $X^i(t)$ is a real-valued function of one variable t, whose domain is $I \subset \mathbb{R}$. In what follows we will assume that for any $i = 1, 2, \ldots, n$ a function $X^i(t)$ is smooth.. The set of

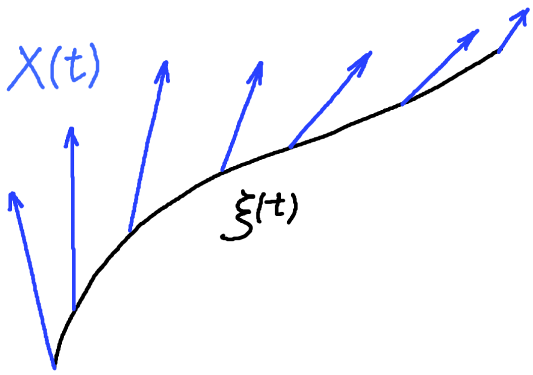

Figure 1.6: Vector field along a curve

all smooth vector fields along a parametrized curve $(\xi = \xi(t), I)$ will be denoted by D_ξ.

Thus, if at each point of a parametrized curve we have a vector that smoothly depends on a point of parametrized curve, then the set of all these vectors is the vector field along a parametrized curve. For example, at each point of a parametrized curve, we defined the velocity vector and the set of all velocity vectors determines the vector field, which will be called a *tangent vector field* or *vector field of velocity* of a parametrized curve. The velocity vector field will be denoted by $\xi'(t)$. Analogously, we can define an *acceleration vector field* of a parametrized curve with the help of acceleration vector. The acceleration vector field of a parametrized curve will be denoted by $\xi''(t)$.

One more important operation with functions, which can be extended to vector fields, is a differentiation. This operation is very important and will lead us to a concept of covariant differentiation of a vector field. We extend differentiation from functions to vector fields as follows: If $\mathtt{X}(t)$ is a vector field along a parametrized curve $\xi = \xi(t)$, then its derivative with respect to t is a new vector field $\mathtt{X}'(t)$, whose component functions are the derivatives of component functions of $\mathtt{X}(t)$ with respect to t. Thus,

$$\mathtt{X}'(t) = (\xi(t); (X^1(t))', (X^2(t))', \dots, (X^n(t))'),$$

where $\mathtt{X}(t) = (\xi(t); X^1(t), X^2(t), \dots, X^n(t))$. From this definition it can be seen that differentiation of a vector field $\mathtt{X}(t)$ essentially reduces to differentiation of its vector-valued function $\vec{\mathtt{X}}(t)$, i.e. $\mathtt{X}'(t) = (\xi(t); \vec{\mathtt{X}}'(t))$. It follows from this definition that differentiation is a linear operation on vector fields, i.e. $(a\,\mathtt{X} + \mathtt{Y})' = a\,\mathtt{X}'(t) + \mathtt{Y}'(t)$, where a is a real number. It is a simple exercise to prove the following properties of differentiation of vector fields

$$(f(t)\,\mathtt{X}(t))' = f'(t)\,\mathtt{X}(t) + f(t)\,\mathtt{X}'(t), \quad (<\mathtt{X}(t), \mathtt{Y}(t)>)' = <\mathtt{X}'(t), \mathtt{Y}(t)> + <\mathtt{X}(t), \mathtt{Y}'(t)> .$$

Proposition 1.4. Let $\mathtt{X}(t)$ be a vector field along a parametrized curve $(\xi = \xi(t), I)$ of constant length, i.e. for any $t \in I$ it holds $\|\mathtt{X}(t)\| = c$, where $c \geq 0$ is a real number. Then

$$<\mathtt{X}(t), \mathtt{X}'(t)> = 0,$$

i.e. the derivative of a constant length vector field $\mathtt{X}(t)$ is orthogonal to $\mathtt{X}(t)$.

Proof. For any $t \in I$ we have $<\mathtt{X}(t), \mathtt{X}(t)> = c^2$. Differentiating the both sides of this relation and making use of the property of differentiation of inner product of two vector fields, we obtain

$$(<\mathtt{X}(t), \mathtt{X}(t)>)' = <\mathtt{X}'(t), \mathtt{X}(t)> + <X(t), \mathtt{X}'(t)> = 2 <\mathtt{X}(t), \mathtt{X}'(t)> = 0,$$

and it follows immediately from this that the inner product of $\mathtt{X}(t)$ and $\mathtt{X}'(t)$ is zero. We note that this proposition is a generalization of Proposition 1.3. □

1.6 Curvature of a parametrized curve

A curvature of a parametrized curve at some point of this curve can be measured by means of its deviation from a tangent line at this point. Let us consider a unit-speed parametrized curve $(\eta = \eta(s), I)$. Assuming that η is parametrized by a natural parameter s we do not lose a generality of a structure considered below since in Section (1.4) it is proved that any (regular) parametrized curve can be parametrized with the help of a natural parameter (Theorem 1.1). Hence a velocity vector $\eta'(s)$ is a unit vector at any point of η, i.e. $\|\eta'(s)\| = 1$. Let p be some point of a parametrized curve η corresponding to a value $s_0 \in I$ of natural parameter s, i.e. $p = \eta(s_0)$. Let us give the value s_0 of natural parameter s some increment of Δs so that $s_0, s_0 + \Delta s \in I$ and denote the point of a parametrized curve η corresponding to $s_0 + \Delta s$ by q, i.e. $q = \eta(s_0 + \Delta s)$. Then point q can be considered as the displacement of the point p along a parametrized curve η. The velocity vectors are vectors at a point and in order to have their vector part we can represent them as follows

$$\eta'(s_0) = (p; \vec{\eta}'(s_0)), \eta'(s_0 + \Delta s) = (q; \vec{\eta}'(s_0 + \Delta s)).$$

Let us denote the angle between vectors $\vec{\eta}'(s_0)$ and $\vec{\eta}'(s_0 + \Delta s)$ by $\Delta\phi$. Note that velocity vector η' changes when the point p moves along the curve η to the point q and it happens because η is curved. However, due to the fact that we use the natural parametrization, the velocity vector remains unit vector, that is, there is no change in its length. This means that all the change in the velocity vector η' occurs due to the change in its direction. The figure (1.7) clearly shows that the direction of a velocity vector changes the faster, the more curved a line. Thus, we can measure how curved a line η is using the rate of change of the angle $\Delta\phi$ when the point moves along a line.

Definition 1.3. The curvature $\kappa(s)$ of a parametrized curve $\eta = \eta(s)$ at a point $s_0 \in I$ is the limit

$$\kappa(s_0) = \lim_{\Delta s \to 0} \frac{\Delta\phi}{\Delta s}.$$

If vectors $\vec{\eta}'(s_0), \vec{\eta}'(s_0 + \Delta s)$ are applied from a point p, then the end points of these vectors lie on the unit circle with the center at a point p. Let us denote $\Delta\eta'(s_0) = (p; \vec{\eta}'(s_0 + \Delta s) - \vec{\eta}'(s_0))$. Then it holds

$$\lim_{\Delta s \to 0} \frac{\Delta\phi}{\|\Delta\eta'(s_0)\|} = 1.$$

Indeed, we have $\|\Delta\eta'(s_0)\| = 2\,\sin\frac{\Delta\phi}{2}$. Thus

$$\lim_{\Delta s \to 0} \frac{\Delta\phi}{\|\Delta\eta'(s_0)\|} = \lim_{\Delta\phi \to 0} \frac{\Delta\phi}{2\,\sin\frac{\Delta\phi}{2}} = \lim_{\Delta\phi \to 0} \frac{\Delta\phi/2}{\sin(\Delta\phi/2}) = 1$$

Now, we can express curvature as follows

$$\begin{aligned}
\kappa(s_0) &= \lim_{\Delta s \to 0} \frac{\Delta\phi}{\Delta s} = \lim_{\Delta s \to 0} \frac{\Delta\phi}{\|\Delta\eta'(s_0)\|} \frac{\|\Delta\eta'(s_0)\|}{\Delta s} \\
&= \lim_{\Delta s \to 0} \frac{\Delta\phi}{\|\Delta\eta'(s_0)\|} \lim_{\Delta s \to 0} \frac{\|\Delta\eta'(s_0)\|}{\Delta s} = \lim_{\Delta s \to 0} \frac{\|\Delta\eta'(s_0)\|}{\Delta s} = \|\eta''(s)\|\Big|_{s=s_0}
\end{aligned}$$

Thus we proved that the curvature $\kappa(s)$ of a unit-speed parametrized curve $\eta = \eta(s)$ at a point s is equal to the length of acceleration vector at this point.

1.7 Plane curves and their curvatures

In this section we consider the curvature of a parametrized curve in the case of an affine Euclidean plane E^2. Throughout this section we assume that the Euclidean plane E^2 is endowed with an orthonormal right-handed oriented frame $\{O; \vec{e}_1, \vec{e}_2\}$ and the Cartesian coordinates induced by this frame will be denoted by x, y. Our aim in this section is to define a signed curvature of a plane curve and to derive formula for its calculation in Cartesian coordinates.

Before we move on to considering the curvature of a plane curve we will consider ways to define a curve on a plane. The first and basic way to define a curve is parametric, that

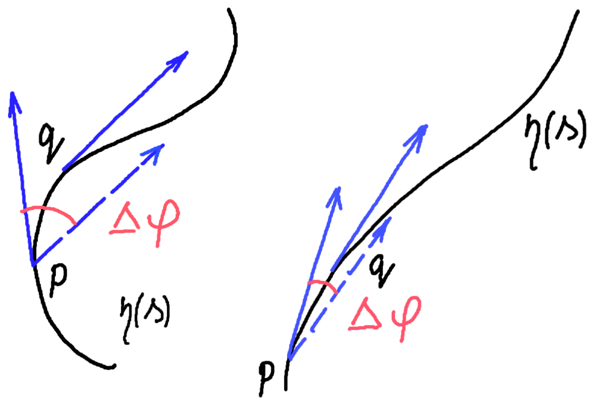

Figure 1.7: Curvature of a line

is, a curve is given by a parametric equation $\xi(t) = \big(x(t), y(t)\big)$, where $t \in I$. However there are other ways to define a curve on a plane E^2. Let $y = f(x)$ be a smooth function and $I \subset \mathbb{R}$ be its domain. Then the graph of a function f, i.e. the set of points

$$\operatorname{graph} f = \{(x, f(x) \in E^2 : x \in I\}, \tag{1.27}$$

can be considered as a curve on a plane E^2. This way to define a plane curve can be considered as a particular case of a parametrized curve. Indeed taking the first coordinate x as a parameter t we can define a curve (1.27) by means of the parametric equation $\xi(t) = (t, f(t)), t \in I$. On the other hand if we have a parametrized curve $\xi(t) = (x(t), y(t))$ then we can present this curve at least locally as a graph of a function. Let $p = \xi(t_0)$ be a point of a curve ξ. Since ξ is a regular curve at least one of the derivatives $x'(t_0), y'(t_0)$ at a point p has to be non-zero. Assume $x'(t_0) \neq 0$. Then there exists a neighbourhood of t_0 where t can be expressed by means of the $x(t)$ function's inverse as $t = t(x)$. Then the graph of the function $y = y(t(x))$ will locally (in a neighbourhood of t_0) coincide with a curve ξ. Thus we see that the way to define a plane curve by means of a parametric equation is locally equivalent to the way to define a curve as a graph of a smooth function.

Let a plane curve is defined as a graph of a smooth function $f : I \to \mathbb{R}$. Let $x_0 \in I$. In order to find the equation of a tangent line to a curve $y - f(x)$ we expand a function $f(x)$ in a Taylor series in a neighbourhood of a point $x = x_0$

$$y = y_0 + f'(x_0)(x - x_0) + \mathcal{O}(\Delta x),$$

where $y_0 = f(x_0), \Delta x = x - x_0$ and $\mathcal{O}(\Delta x)$ are terms of second and higher order. Omitting all terms of second and higher order we get the equation of a tangent line to a curve $y = f(x)$ at a point $x = x_0$

$$y - y_0 = f'(x_0)(x - x_0). \tag{1.28}$$

One can write (1.28) in the form of a parametric equation $x = x_0 + t, y = y_0 + f'(x_0)t$ which shows that $\mathtt{v} = (p; 1, f'(x_0))$ can be taken as a velocity vector of a curve $y = f(x)$ at a point $p = (x_0, y_0)$.

A plane curve can be also defined with the help of an equation $F(x, y) = c$, where $F(x, y)$ is a smooth function of two variables and c is a real number. In this case a curve is the set of all points of a plane E^2 whose coordinates satisfy an equation $F(x, y) = c$. Let us denote this set of solutions by $\mathfrak{C}$. Additionally we will assume that the set of solutions of an equation $F(x,y)=c$ is non-empty, i.e. $\mathfrak{C}\neq\emptyset$, and at each point of a curve we have $(F'_x)^2+(F'_y)^2>0$. Let $p=(x_0,y_0)$ be a point of a curve $F(x,y)=c$ such that $F'_y|_p\neq 0$. Then according to the implicit function theorem there exists a neighbourhood of a point p, where y can be expressed as a smooth function of x, i.e. $y=f(x)$, and

$$\frac{df}{dx}=-\frac{F'_x}{F'_y}.$$

Thus the set of solutions of equation $F(x,y)=0$ is a curve on a plane, which is locally, that is, in a neighbourhood of each point, can be represented as a graph of a function.

Assume that a plane curve is defined by an equation $F(x,y)=c$ and according to the implicit function theorem in a neighbourhood of a point $p=(x_0,y_0)$ of this curve we have $y=f(f)$, where $y_0=f(x_0)$ and $F(x,f(x))\equiv c$. Then

$$dF(x,y)=F'_x(p)\,dx+F'_y(p)\,dy=0\ \Rightarrow\ F'_x(p)+F'_y(p)\,f'(x_0)=0.$$

The left-hand side of this equation can be considered as a scalar product of two plane vectors $(p;F'_x(p),F'_y(p))$ and $(p;1,f'(x_0))$. But the latter vector is a velocity vector of a curve at a point p. Hence the vector $(p;F'_x(p),F'_y(p))$ is perpendicular to the velocity vector of a curve and can be taken as a normal vector of a curve $F(x,y)=0$ at a point p. A straight line which passes a point of a curve and is perpendicular to a tangent line to curve at this point is called a normal line of a curve. From previous considerations it follows that equation of the normal line of a curve $F(x,y)=c$ at a point $p=(x_0,y_0)$ of this curve is

$$F'_x(p)(x-x_0)+F'_y(p)(y-y_0)=0. \tag{1.29}$$

It is worth to mention that a function $F(x,y)$ standing at the left-hand side of an equation of a curve induces the gradient vector field ∇F in a domain of a function $F(x,y)$. The restriction of this vector field to a curve $\mathfrak{C}$ gives the vector field along this curve and this vector field is perpendicular to the velocity vector field of a curve. Hence $\nabla F\big|_{\mathfrak{C}}$ is a normal vector field along a curve $F(x,y)=c$.

The curvature of a plane curve can be defined with the help of Definition 1.3, when $n=2$. However, in the case of a plane curve, we can use the right-handed orientation of a basis to endow the curvature of a parametrized curve with a sign. This can be made by means of a complex structure of a plane E^2. Let p be a point of a plane. We rotate this point around the origin O counterclock-wise by right angle, get the point q and this determines the transformation of a plane E^2, which will be denoted by J. Hence, $J:E^2\to E^2$ and $J(p)=q$. This transformation is called the *complex structure of a plane* E^2. The transformation J can be applied to vectors of the associated vector space V^2. Indeed, given a vector $\vec{v}\in V^2$, we apply it from the origin, then rotate around the origin counterclock-wise by right angle and denote the obtained vector by $J(\vec{v})$. It is easy to see that $J:V^2\to V^2$ is a linear transformation, i.e. we have $J(a\,\vec{v}+b\,\vec{w})=a\,J(\vec{v})+b\,J(\vec{w})$. This linear mapping has the following properties:

1. $J^2=-\mathrm{id}_{E^2}$;
2. $<J(\vec{v}),J(\vec{w})>=<\vec{v},\vec{w}>$;
3. $<J(\vec{v}),\vec{v}>=0$.

Evidently, if we apply the transformation J to the vectors of basis, then $J(\vec{e}_1)=\vec{e}_2, J(\vec{e}_2)=-\vec{e}_1$. Hence, if $\vec{v}=v^1\,\vec{e}_1+v^2\,\vec{e}_2$, then

$$J(\vec{v})=J(v^1\,\vec{e}_1+v^2\,\vec{e}_2)=v^1\,J(\vec{e}_1)+v^2\,J(\vec{e}_2)=-v^2\,\vec{e}_1+v^1\,\vec{e}_2.$$

Thus, the transformation J sends a vector with coordinates v^1, v^2 to the vector with coordinates $-v^2, v^1$. This implies that J maps a point of E^2 with coordinates x, y to the point with coordinates $-y, x$, i.e. $J(x, y) = (-y, x)$.

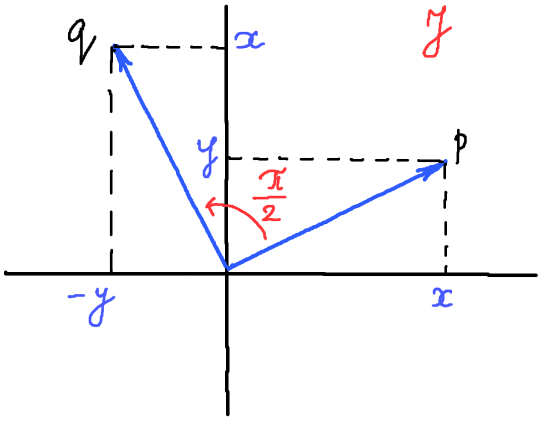

Figure 1.8: Complex structure of a plane

Consider a unit-speed parametrized plane curve $\eta = \eta(s)$. According to Proposition 1.3, the acceleration vector $\eta''(s)$ is perpendicular to the velocity vector $\eta'(s)$. Hence, the acceleration vector and the vector $J(\eta'(s))$ are collinear.

Definition 1.4. Let $\eta = \eta(s)$ be a unit-speed parametrized plane curve. Then the signed curvature $\kappa(s)$ of a unit-speed parametrized plane curve ξ is defined by

$$\eta''(s) = \kappa(s)\, J(\eta'(s)).$$

From this definition we easily get the formula for the curvature of a unit-speed parametrized plane curve

$$\kappa(s) =< \eta''(s), J(\eta'(s)) > . \tag{1.30}$$

Taking into account that the acceleration vector $\eta''(s)$ and $J(\eta'(s))$ are collinear vectors and $||J(\eta'(s))|| = 1$, we conclude $|\kappa(s)| = \|\eta''(s)\|$ and this shows that Definition 1.4 is consistent with the previously given general Definition 1.3. Hence, in the case of a parametrized plane curve the absolute value of the curvature is equal to the length of acceleration vector. This is the reason why the curvature of a parametrized plane curve is called *signed curvature.*

The calculation of curvature in a unit-speed parametrization is often a difficult task, because first we need to find this unit-speed parametrization, which requires calculation of inverse function for arc length function. Thus, it is desirable to have a formula for curvature in an arbitrary parametrization. In order to find this formula, we begin with a parametrized curve $(\xi = \xi(t), I)$, which is not necessarily a unit-speed parametrized curve. But, according to Theorem 1.1, there exists a unit-speed parametrization of this curve, which will be denoted by $\eta = \eta(s)$. We know that this unit-speed parametrization is constructed by means of the arc length function $s = s(t)$ of ξ, i.e. $\xi(t) = \eta(s(t))$. Thus, it is natural to define the curvature $\kappa(t)$ of a parametrized curve $\xi = \xi(t)$ by putting $\kappa(t) = \kappa(s)|_{s=s(t)}$, where $\kappa(s)$ is the curvature of the unit-speed parametrized curve $\eta(s)$. From this definition and the formula (1.30) we obtain

$$\kappa(t) =< \eta''(s), J(\eta'(s)) > |_{s=s(t)}. \tag{1.31}$$

Differentiating the formula $\xi(t) = \eta(s(t))$ and using the chain rule for parametrized curves, we get

$$\xi'(t) = s'(t)\,\eta'(s)\big|_{s=s(t)} \quad \Longrightarrow \quad \eta'(s)\big|_{s=s(t)} = \frac{\xi'(t)}{\|\xi'(t)\|}. \tag{1.32}$$

Differentiating the above formula one more time, we get

$$s'(t)\,\eta\,''(s)\big|_{s=s(t)} = \frac{\|\xi\,'(t)\|\,\xi\,''(t) - (\|\xi\,'(t)\|)'\,\xi\,'(t))}{\|\xi\,'(t)\|^2}.$$

Substituting $s'(t) = \|\xi'(t)\|$ we can express the accelerator vector $\eta\,''(s)$ as follows

$$\eta\,''(s)\big|_{s=s(t)} = \frac{1}{\|\xi\,'(t)\|^2}\,\xi\,''(t) - \frac{(\|\xi\,'(t)\|)'}{\|\xi\,'(t)\|^3}\,\xi\,'(t). \tag{1.33}$$

Making use of (1.32) and linearity of complex structure J we get

$$J(\eta\,'(s))\big|_{s=s(t)} = J\Big(\frac{\xi\,'(t)}{\|\xi\,'(t)\|}\Big) = \frac{J(\xi\,'(t))}{\|\xi\,'(t)\|}.$$

Substituting this formula and (1.33) into (1.31) and taking into account that $\xi\,'(t)$ is perpendicular to $J(\xi\,'(t))$, i.e. $<\xi\,'(t), J(\xi\,'(t)) >= 0$, we finally get

$$\kappa(t) = \frac{<\xi\,''(t), J(\xi\,'(t))>}{\|\xi\,'(t)\|^3}. \tag{1.34}$$

We can write the above formula for the curvature of a parametrized curve $\xi(t)$ in Cartesian coordinates of the plane E^2. Let $\xi(t) = (x(t), y(t))$, then

$$\begin{aligned} \xi\,'(t) &= (\xi(t); x'(t), y'(t)), \quad \xi\,''(t) = (\xi(t); x''(t), y''(t)), \\ J(\xi\,'(t)) &= (\xi(t); -y'(t), x'(t)), \quad \|\xi\,'(t)\| = \sqrt{(x'(t))^2 + (y'(t))^2}. \end{aligned}$$

Substituting these formulae into (1.34) we obtain

$$\kappa(t) = \frac{x'(t)\,y''(t) - x''(t)y'(t)}{\big[(x'(t))^2 + (y'(t))^2\big]^{3/2}}, \tag{1.35}$$

or

$$\kappa(t) = \frac{1}{\big[(x'(t))^2 + (y'(t))^2\big]^{3/2}} \begin{vmatrix} x'(t) & y'(t) \\ x''(t) & x''(t) \end{vmatrix}.$$

The resulting formula (1.35) is applicable in the case of an arbitrary parametrization of a curve and we do not need previously to find a natural parametrization of a curve.

Let us consider some particular cases of formula (1.35). If a plane curve is defined as a graph of a function $y = f(x)$ then we can parametrize this curve as follows $\xi(x) = (x, f(x))$. Hence $\xi'(x) = (\xi(x); 1, f'(x)), \xi''(x) = (\xi(x); 0, f''(x))$ and

$$\kappa(x) = \frac{f''(x)}{[1 + (f'(x))^2]^{3/2}}. \tag{1.36}$$

If a plane curve is defined in an implicit form by an equation $F(x, y) = c$ we can use previous formula (1.36) assuming that locally an equation $F(x, y) = c$ is resolved and we have the explicit function $y = f(x)$ satisfying $F(x, f(x)) \equiv c$. Then

$$f'(x) = -\frac{F'_x}{F'_y}. \tag{1.37}$$

In order to express the second order derivative $f''(x)$ in terms of partial derivatives of a function $F(x,y)$ we differentiate the relation $F'_x + F'_y f'(x) = 0$, which immediately follows from $dF(x,y) = F'_x\,dx + F'_y\,dy = 0$ when $y = f(x)$. Analogously we find

$$\begin{aligned} d(F'_x + F'_y f'(x)) &= dF'_x + (dF'_y) f'(x) + F'_y\,df'(x) \\ &= F''_{xx}\,dx + F''_{xy}\,dy + (F''_{xy}\,dx + F''_{yy}\,dy)\,f'(x) + F'_y\,f''(x)\,dx. \end{aligned}$$

Taking into account that y is a function of x, i.e. $y - f(x)$, we obtain

$$F'_y\,f''(x) + F''_{xx} + 2\,F''_{xy}\,f'(x) + F''_{yy}\,dy)\,(f'(x))^2 = 0.$$

Substituting from (1.37) and expressing $f''(x)$ in terms of the partial derivatives we get

$$f'' = -\frac{F''_{xx}(F'_y)^2 - 2\,F''_{xy}\,F'_x F'_y + F''_{yy}\,(F'_x)^2}{(F'_y)^3}. \tag{1.38}$$

Substituting (1.37) and (1.38) into (1.36) we finally find the formula for a curvature of a curve defined by $F(x,y) = c$

$$\kappa = -\frac{F''_{xx}(F'_y)^2 - 2\,F''_{xy}\,F'_x F'_y + F''_{yy}\,(F'_x)^2}{[(F'_x)^2 + (F'_y)^2]^{3/2}}. \tag{1.39}$$

1.8 Osculating circle of a plane curve

In this section, we shall restrict our consideration to curves in the plane. At each point of a regular curve, its tangent line is well-defined. The tangent can be viewed as a line that approximates the curve with a certain degree of accuracy in a sufficiently small neighborhood of the point of tangency. We can make this notion more precise.

To do so, assume that two curves $\xi(s)$ and $\eta(s)$ pass through a point p, i.e., $\xi(s_0) = \eta(s_0) = p$, where s is the natural parameter. Suppose the curves are parameterized in such a way that for $s > s_0$, the points $p_1 = \xi(s)$ and $p_2 = \eta(s)$ lie on the same side of p. The deviation of the curve η from the curve ξ is defined as the length of the segment $\delta = \|p_1 - p_2\|$. Consequently, we have

$$\overrightarrow{p_1 p_2} = \vec{\eta}(s) - \vec{\xi}(s)$$

Expanding in a Taylor series at the point s_0, we obtain

$$\begin{aligned} \vec{\xi}(s) &= \vec{\xi}(s_0) + \vec{\xi}'(s_0)\,(s - s_0) + \frac{1}{2}\vec{\xi}''(s_0)\,(s - s_0)^2 + \ldots + \frac{1}{n!}\vec{\xi}^{(n)}(s_0)\,(s - s_0)^n + \ldots, \\ \eta(s) &= \vec{\eta}(s_0) + \vec{\eta}'(s_0)\,(s - s_0) + \frac{1}{2}\vec{\eta}''(s_0)\,(s - s_0)^2 + \ldots + \frac{1}{n!}\vec{\eta}^{(n)}(s_0)\,(s - s_0)^n + \ldots, \end{aligned}$$

Hence

$$\begin{aligned} \delta = \|\vec{\xi}'(s_0) - \eta'(s_0)\|\,(s - s_0) &+ \frac{1}{2}\|\vec{\xi}''(s_0) - \vec{\eta}''(s_0)\|(s - s_0)^2 \\ &+ \ldots + \frac{1}{n!}\|\vec{\xi}^{(n)}(s_0) - \vec{\eta}^{(n)}(s_0)\|\,(s - s_0)^n + \ldots \end{aligned}$$

We say that two curves ξ, η have contact of order n at the point p if condition

$$\vec{\xi}'(s_0) = \vec{\eta}'(s_0),\ \vec{\xi}''(s_0) = \vec{\eta}''(s_0),\ \ldots, \vec{\xi}^{(n)}(s_0) = \vec{\eta}^{(n)}(s_0). \tag{1.40}$$

holds. It is evident that this is equivalent to the deviation δ of one curve from the other at the point p being an infinitesimal of order at least $n+1$ with respect to $s - s_0$.

Suppose we are given a parametrized curve $\xi(t) = (x(t), y(t)), t \in I$ in a plane E^2. We now pose the problem of finding a curve that has second-order contact with a given curve ξ at a certain point $p_0 = \xi(t_0)$. Note that in this case, the deviation of the given curve ξ from the sought curve must be an infinitesimal of order three. In order to find a curve such that it has second-order contact with ξ we will use a circle.

We assume that the curvature $\kappa(t_0)$ of a curve ξ is non-zero at a point p. A circle that passes through a point p and has a second-order contact with a curve ξ at a point p is called an *osculating circle.* Our aim now is to find a radius and a center of osculating circle at a point p. Let us denote a center of an osculating circle by p_0, its position vector by $\vec{r}$ and a radius of osculating circle by R. Then we have $\|\vec{\xi}(t_0) - \vec{r}\| = R^2$, where $\vec{\xi}(t_0)$ is a position vector of a point p. Let $\vec{\xi}(t)$ be a position vector of a point q of a curve ξ close enough to a

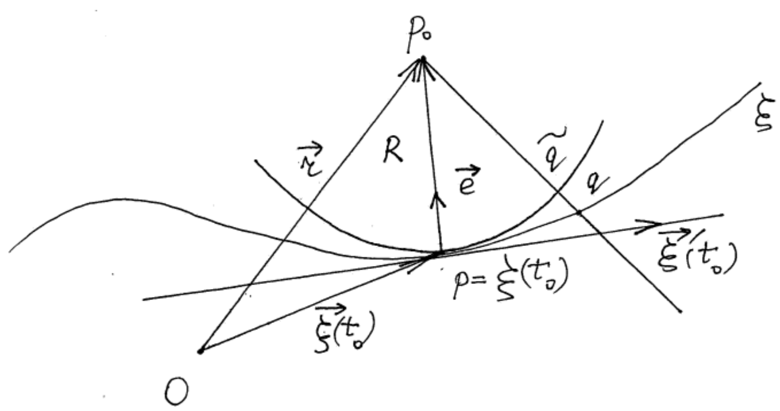

Figure 1.9: Osculating circle

point p. Let us connect point q with a center of an osculating circle p_0 by a straight line segment and denote the point of intersection of this line segment with an osculating circle by $\tilde{q}$. The deviation of a curve ξ from osculating circle is equal to the length of the segment $q\tilde{q}$ and can be written as $\|q - \tilde{q}\| = \|q - p_0\| - \|\tilde{q} - p_0\|$. This deviation must be infinitesimal of the third order with respect to $t - t_0$. It is easy to see that $\|q - p_0\| - \|\tilde{q} - p_0\|$ and $\|q - p_0\|^2 - \|\tilde{q} - p_0\|^2$ are infinitesimals of the same order. Indeed we have

$$\lim_{t \to t_0} \frac{\|q - p_0\|^2 - \|\tilde{q} - p_0\|^2}{\|q - p_0\| - \|\tilde{q} - p_0\|} = \lim_{t \to t_0} (\|q - p_0\| + \|\tilde{q} - p_0\|) = 2R,$$

because $\|\tilde{q}-p_0\|$ is a radius of an osculating circle and $\lim_{t\to t_0}\|q-p_0\| = \|p-p_0\| = R$. Now the deviation can be expressed as follows

$$\delta(t) = \|\vec{\xi}(t) - \vec{r}\|^2 - R^2.$$

Expanding the deviation function $\delta(t)$ into a Taylor series, we must obtain a series that begins with terms of the third order, that is,

$$\delta(t) = \frac{1}{3!}\,\delta'''(t_0)(t-t_0)^3 + \dots$$

Hence we must have

$$\delta(t_0) = \delta'(t_0) = \delta''(t_0) = 0.$$

These three conditions are necessary and sufficient for a circle centered at point p with radius R to have a second-order contact with a curve ξ, that is, to be an osculating circle.

The figure shows that vector $\vec{pp_0}$ is perpendicular to the tangent vector $\xi'(t_0)$. It follows that by turning the unit vector

$$\vec{e} = \frac{\xi'(t_0)}{\|\xi'(t_0)\|}$$

at a right angle counterclockwise with the help of the complex structure J, we can write the position vector of a center of osculating circle $\vec{r}$ in the form

$$\vec{r} = \vec{\xi}(t_0) + R \cdot J(\vec{e}). \tag{1.41}$$

Let us show that this vector satisfies the conditions $\delta(t_0) = \delta'(t_0) = 0$. We have

$$\delta(t_0) = \|\vec{\xi}(t_0) - \vec{\xi}(t_0) - R \cdot J(\vec{e})\|^2 - R^2 = R^2 - R^2 = 0.$$

In the case of the first order derivative of the deviation function $\delta(t)$ we have

$$\delta'(t_0) = 2 < \vec{\xi}(t_0) - \vec{r}, \vec{\xi}'(t_0) >= -2\frac{R}{\|\vec{\xi}'(t_0)\|} < J(\vec{\xi}'(t_0)), \vec{\xi}'(t_0) >= 0,$$

because of $J(\vec{\xi}'(t_0)) \perp \vec{\xi}'(t_0)$. The second order derivative of the deviation function $\delta(t)$ at a point $t = t_0$ can be written as follows

$$\delta''(t_0) = 2\left(\|\vec{\xi}'(t_0)\|^2 + < \vec{\xi}(t_0) - \vec{r}, \vec{\xi}''(t_0) > \right).$$

Substituting (1.41) into the resulting expression and equating it to zero, we obtain

$$\|\vec{\xi}'(t_0)\|^2 - \frac{R}{\|\vec{\xi}'(t_0)\|} < \vec{\xi}''(t_0), J(\vec{\xi}'(t_0)) >= 0.$$

Making use of the formula for curvature (1.35) we find the radius of an osculating circle

$$R = \frac{1}{\kappa(t_0)}. \tag{1.42}$$

Due to the relation (1.42) the radius of an osculating curve is also called a *radius of curvature* of a curve ξ at a point $\xi(t_0)$. The center of an osculating circle is called a *center of curvature* of a curve ξ at a point $\xi(t_0)$. Substituting (1.42) into (1.41) we find the position vector of a center of curvature

$$\vec{r} = \vec{\xi}(t_0) + \frac{1}{\kappa(t_0)} \cdot J\left(\frac{\vec{\xi}'(t_0)}{\|\vec{\xi}'(t_0)\|}\right). \tag{1.43}$$

Using the affine structure of plane E^2 we can pass from the position vectors $\vec{r}, \vec{\xi}(t_0)$ to the corresponding points p_0, p and write the center of curvature as a point of a plane

$$p_0 = p + \frac{1}{\kappa(t_0)} \cdot J\Big(\frac{\vec{\xi}'(t_0)}{\|\vec{\xi}'(t_0)\|}\Big). \tag{1.44}$$

Now note that an osculating circle is defined at each point of a curve $\xi = \xi(t)$, that is, for any value of the parameter $t \in I$. Thus we can consider the set of points in a plane E^2 that are the centers of the osculating circles of this curve. Obviously this set of points is a plane curve which is called *evolute* and the parametric equation of the evolute of a curve $\xi = \xi(t)$ follows from (1.43) or (1.44) and has the form

$$\mathfrak{e}(t) = \vec{\xi}(t) + \frac{1}{\kappa(t)} \cdot J\Big(\frac{\vec{\xi}'(t)}{\|\vec{\xi}'(t)\|}\Big) \tag{1.45}$$

1.9 Plane curves in polar coordinates

So far we have used Cartesian coordinates in Euclidean affine space E^n, which are constructed by means of an orthonormal frame $\mathfrak{C} = \{O; \vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}$. However, in Euclidean space E^n, besides Cartesian coordinates, there are another coordinate systems, which are called curvilinear coordinates. We will consider a general approach to this type of coordinates in subsequent chapters of this book, but now it is useful to consider the special case of such coordinates on a plane E^2.

A parameterized curve ξ in polar coordinates of a plane can be defined in a similar way to how it is done in Cartesian coordinates. This means that one can define a parameterized curve as a mapping $\xi : I \to E^2$, which in polar coordinates r, ϕ is determined by a parametric equation $\xi(t) = (r(t), \phi(t)), t \in I$, where $r(t), \phi(t)$ are smooth functions. However, it is usually convenient to choose the polar angle ϕ as a parameter. In this case the parametric equation of a parametrized curve has a simple form $r = r(\phi)$ and we will determine a parametrized curve with the help of this equation. If a parametrized curve is given in polar coordinates of a plane by equation $r = r(\phi)$, we can easily derive its parametric equation in Cartesian coordinates, if we construct an orthonormal frame $\mathfrak{C} = \{O; \vec{e}_1, \vec{e}_2\}$ as follows: Origin O coincides with the pole, $\vec{e}_1$ is the unit directional vector of the polar axis (pointing in the positive direction of the polar axis) and $\vec{e}_2$ is obtained by counterclockwise rotation of $\vec{e}_1$ around the origin O by the right angle. Then so constructed Cartesian coordinates can be expressed in terms of polar coordinates as follows

$$x = r\,\cos\phi, \quad y = r\,\sin\phi. \tag{1.46}$$

Thus, the parametric equation of a curve $r = r(\phi)$ in Cartesian coordinates is

$$\xi(\phi) = (r(\phi)\cos\phi, r(\phi)\sin\phi), \tag{1.47}$$

and the parameter is a polar angle ϕ. Hence, we can apply previously developed formulae to calculate the coordinates of velocity vector, acceleration vector, the arc length and the curvature of a curve $r = r(\phi)$.

In the case of polar coordinates, we can construct a basis for the tangent space T_pE^2 of a plane at a point p similar to the basis $\{\mathsf{e}_{p,1}, \mathsf{e}_{p,2}\}$ in Cartesian coordinates. But vectors of this basis will depend on a point of plane. In order to construct a basis for T_pE^2 in polar coordinates, we use the coordinate lines. Let p be a point of plane E^2, whose polar coordinates are r, ϕ. If we fix the polar angle ϕ of a point p and let polar radius r to vary within some interval $(r-\delta, r+\delta)$, then we get the first coordinate line passing through a point p and the parametric equation of this straight line is (t, ϕ), where $r-\delta < t < r+\delta$. Analogously, the second coordinate line is the circle, whose radius is r and center is the pole O. The parametric equation of the second coordinate line is (r, τ), where $\phi-\delta < \tau < \phi+\delta$. We will use the velocity vectors of these coordinates lines to construct the orthonormal basis for the tangent space T_pE^2. In order to find the coordinates of velocity vectors, we pass from polar coordinates to Cartesian ones, because we know how to calculate the coordinates of velocity vectors only in Cartesian coordinates. The parametric equation of the first coordinate line in Cartesian coordinates is $(t\cos\phi, t\sin\phi)$ and of the second line is $(r\cos\tau, r\sin\tau)$. Thus, the first velocity vector is $(p; \cos\phi, \sin\phi)$ and the second one is $(p; -r\sin\phi, r\cos\phi)$. It is easy to see that the first vector is a unit vector, the length of the second vector is r and vectors are orthogonal to each other. Hence, the vectors

$$\begin{aligned} \mathsf{e}_{p,r} &= (p; \cos\phi, \sin\phi) = \cos\phi\,\mathsf{e}_{p,1} + \sin\phi\,\mathsf{e}_{p,2}, \\ \mathsf{e}_{p,\phi} &= (p; -\sin\phi, \cos\phi) = -\sin\phi\,\mathsf{e}_{p,1} + \cos\phi\,\mathsf{e}_{p,2}, \end{aligned}$$

form the orthonormal basis for tangent space T_pE^2 in polar coordinates. The above formulae can be written in the matrix form

$$\begin{pmatrix} \mathsf{e}_{p,r} & \mathsf{e}_{p,\phi} \end{pmatrix} = \begin{pmatrix} \mathsf{e}_{p,1} & \mathsf{e}_{p,2} \end{pmatrix} \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix},$$

and we see that the orthonormal basis in polar coordinates $\mathsf{e}_{p,r}, \mathsf{e}_{p,\phi}$ can be obtained by counterclockwise rotation of the canonical orthonormal basis $\mathsf{e}_{p,1}, \mathsf{e}_{p,2}$ around a point p by an angle ϕ.

To pass from vectors to the vector fields induced by them, we will omit the point p in the corresponding notation. Hence $\mathsf{e}_1, \mathsf{e}_2$ will stand for the vector fields induced by the canonical basis $\vec{e}_1, \vec{e}_2$ and $\mathsf{e}_r, \mathsf{e}_\phi$ for the vector fields induced by $\mathsf{e}_{p,r}, \mathsf{e}_{p,\phi}$. Thus we have

$$\mathsf{e}_r = \cos\phi\,\mathsf{e}_1 + \sin\phi\,\mathsf{e}_2, \tag{1.48}$$

$$\mathsf{e}_\phi = -\sin\phi\,\mathsf{e}_1 + \cos\phi\,\mathsf{e}_2,, \tag{1.49}$$

and

$$\begin{pmatrix} \mathsf{e}_r & \mathsf{e}_\phi \end{pmatrix} = \begin{pmatrix} \mathsf{e}_1 & \mathsf{e}_2 \end{pmatrix} \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}.$$

Now we can calculate the velocity vector field v along a curve given in polar coordinates by equation $r = r(\phi)$, and we calculate the coordinates of this velocity vector field in the orthonormal frame field $\mathsf{e}_r, \mathsf{e}_\phi$ induced by the polar coordinates. In Cartesian coordinates defined by a canonical frame field $\mathsf{e}_1, \mathsf{e}_2$ this velocity vector field has the coordinates

$$r'\cos\phi - r\,\sin\phi, r'\sin\phi + r\,\cos\phi. \tag{1.50}$$

We can write them in a matrix form as follows

$$\begin{pmatrix} r'\cos\phi - r\,\sin\phi \\ r'\sin\phi + r\,\cos\phi \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} r' \\ r \end{pmatrix}.$$

Since the matrix

$$\begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$$

transforms the coordinates of the velocity vector field v in frame field $\mathsf{e}_r, \mathsf{e}_\phi$ into coordinates in the frame field $\mathsf{e}_1, \mathsf{e}_2$, we conclude that r', r are the coordinates of the velocity vector field in frame field $\mathsf{e}_r, \mathsf{e}_\phi$. Hence

$$\mathsf{v} = r'\;\mathsf{e}_r + r\;\mathsf{e}_\phi. \tag{1.51}$$

Hence, $\|\mathtt{v}\| = \sqrt{(r')^2+r^2}$ and the length of the arc between two points $(r(\phi_1),\phi_1)$, $(r(\phi_2),\phi_2)$ of a curve $r=r(\phi)$, where $\phi_1<\phi_2$, can be calculated in polar coordinates with the help of the formula

$$s=\int_{\phi_1}^{\phi_2}\sqrt{(r')^2+r^2}\,d\phi. \tag{1.52}$$

In order to calculate the curvature of a plane curve given in polar coordinates by equation $r=r(\phi)$ we need to calculate the coordinates of acceleration vector field $\mathtt{a}$ along a curve $r=r(\phi)$. We do this by differentiating the coordinates of the velocity vector field (1.50) with respect ϕ

$$(r''-r)\,\cos\phi-2r'\sin\phi,(r''-r)\,\sin\phi+2r'\cos\phi,$$

and writing it in the matrix form

$$\begin{pmatrix}(r''-r)\,\cos\phi-2r'\sin\phi\\(r''-r)\,\sin\phi+2r'\cos\phi\end{pmatrix}=\begin{pmatrix}\cos\phi & -\sin\phi\\ \sin\phi & \cos\phi\end{pmatrix}\begin{pmatrix}r''-r\\2\,r'\end{pmatrix}.$$

Thus the acceleration vector field in polar coordinates can be written as

$$\mathtt{a}=(r''-r)\,\mathtt{e}_r+2\,r'\,\mathtt{e}_\phi. \tag{1.53}$$

The coordinates of the acceleration vector $\mathtt{a}$ can be calculated in a different way. To do this, we differentiate both sides of equation (1.51) with respect to polar angle ϕ. We obtain

$$\mathtt{a}=\mathtt{v}'=r''\,\mathtt{e}_r+r'\,\mathtt{e}_r'+r'\,\mathtt{e}_\phi+r\,\mathtt{e}_\phi'. \tag{1.54}$$

Now the question is what the derivatives of the vector fields $\mathtt{e}_r$ and $\mathtt{e}_\phi$ are equal to. We consider these vector fields as vector fields along a curve $r=r(\phi)$ and differentiate them by the parameter of this curve, which is the polar angle of a point, that is, we differentiate along this curve. However, it is easy to see that these vector fields do not depend on the polar radius (see formulas (1.48,1.49). Thus, their change is completely determined by the movement of the point along the coordinate line ϕ, that is, the circle. From this it follows that the derivatives of these vector fields can be calculated by differentiating the formulae (1.48,1.49) with respect to variable ϕ. This differentiation gives

$$\mathtt{e}_r'=\mathtt{e}_\phi,\mathtt{e}_\phi'=-\mathtt{e}_r.$$

Substituting this result onto (1.49) we get (1.53).

Now applying complex structure transformation to the velocity vector field (at each point of a curve $r=r(\phi)$) we find

$$J(\mathtt{v})=-r\,\mathtt{e}_r+r'\mathtt{e}_\phi.$$

Making use of the formula (1.34) for a curvature we finally get the curvature of a curve in polar coordinates

$$\kappa=\frac{<\mathtt{a},J(\mathtt{v})>}{||\mathtt{v}||^{3/2}}=\frac{r^2+2(r')^2-r\,r''}{[r^2+(r')^2]^{3/2}}. \tag{1.55}$$

Example 1.7. One of the most famous examples of a curve defined by a parametric equation in polar coordinates is the logarithmic spiral. It is given by the equation $r=ae^{b\phi}$, where a,b are real numbers such that $a>0,b\neq 0$. This curve has many wonderful properties. We will calculate the arc length of a logarithmic spiral between two polar angle values. We find

$$r'=ab\,e^{b\phi},\;\;(r')^2+r^2=a^2(b^2+1)\,e^{2b\phi}.$$

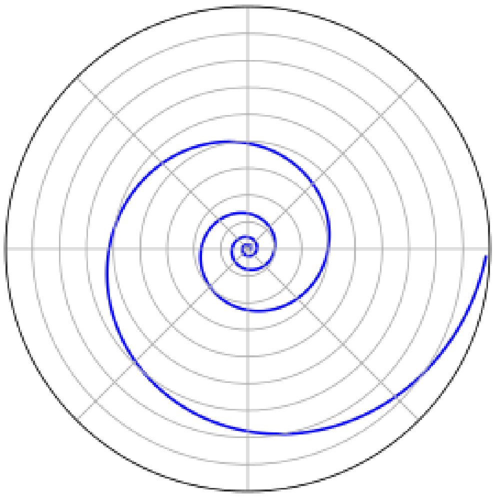

Figure 1.10: Logarithmic spiral

Then the length of arc of logarithmic spiral between two its points $r(\phi_1), r(\phi_2)$, where $\phi_1 < \phi_2$ can be calculated by means of the formula (1.52) as follows

$$\begin{aligned} s &= \int_{\phi_1}^{\phi_2} \sqrt{a^2(b^2+1)\, e^{2b\phi}}\, d\phi = a\sqrt{b^2+1}\int_{\phi_1}^{\phi_2} e^{b\phi} d\phi \\ &= a\frac{\sqrt{b^2+1}}{b}\int_{\phi_1}^{\phi_2} e^{b\phi} d(b\phi) = a\frac{\sqrt{b^2+1}}{b} e^{b\phi}\Big|_{\phi_1}^{\phi_2} = \frac{\sqrt{b^2+1}}{b}\big(r(\phi_2)-r(\phi_1)\big). \end{aligned}$$

We calculate the curvature of the logarithmic spiral using formula (1.55). We have

$$r'' = ab^2\, e^{b\phi},\quad r^2+2(r')^2-r\,r'' = a^2(b^2+1)\, e^{2b\phi},$$

and

$$\kappa = \frac{r^2+2(r')^2-r\,r''}{[r^2+(r')^2]^{3/2}} = \frac{a^2(b^2+1)\, e^{2b\phi}}{a^3(b^2+1)^{3/2}\, e^{3b\phi}} = \frac{1}{r\sqrt{b^2+1}}.$$

1.10 Bartels-Frenet-Serret equations

In this subsection we consider parametrized curves in three dimensional Euclidean space E^3, construct a moving frame along a parametrized curve and derive Bartels-Frenet-Serret formulas for a moving frame, which express the derivatives of vector fields constituting moving frame, in terms of these vector fields themselves. In the first part of this section, we will use the natural parametrization of a curve.

Let $(\eta = \eta(s), I)$ be a unit-speed parametrized curve in three dimensional Euclidean space E^3 and $\mathtt{T}$ be the vector field of velocity along a unit-speed parametrized curve η. Hence, $\mathtt{T}(s) = \eta'(s)$ and $\|\mathtt{T}(s)\| = 1$. In this section we assume that the curvature $\kappa(s)$ of a unit-speed parametrized curve η is non-zero at any point of this curve, i.e. for any $s \in I$ we have $\kappa(s) \neq 0$. The derivative of $\mathtt{T}$ is the vector field of acceleration along η and $\mathtt{T}' \perp \mathtt{T}$, $\|\mathtt{T}'(s)\| = ||\eta''(s)|| = \kappa(s)$. Thus, we can make $\mathtt{T}'$ to be a unit vector field by normalizing it, i.e. we introduce the vector field

$$\mathtt{N}(s) = \frac{1}{\kappa(s)}\mathtt{T}'(s). \tag{1.56}$$

This vector field is orthogonal to the vector field of velocity $\mathtt{T}$ and $\|\mathtt{N}(s)\| = 1$ for any $s \in I$. The vector field $\mathtt{N}$ will be referred to as a *normal vector field* along a parametrized curve η. In order to have a three-dimensional frame at each point of a curve η, we introduce one more vector field $\mathtt{B}$ along a curve η, which is called a *binormal vector field* and is defined as cross-product of vector field of velocity $\mathtt{T}$ and the normal vector field $\mathtt{N}$. Hence $\mathtt{B} = \mathtt{T} \times \mathtt{N}$. Three vector fields $\{\mathtt{T}, \mathtt{N}, \mathtt{B}\}$ form an orthonormal frame for the tangent space $T_{\eta(s)}E^3$ at each point of a unit-speed parametrized curve $\eta(s)$ and this frame will be referred to as a *moving frame of a parametrized curve.* Now our aim is to find formulae, which show how the derivatives of the vector fields $\mathtt{T}', \mathtt{N}', \mathtt{B}'$, constituting the moving frame, can be expanded in terms of vector fields themselves. The geometric essence of these formulae is that they show the speed of rotation of the three-dimensional frame $\{\mathtt{T}, \mathtt{N}, \mathtt{B}\}$ when a point moves along a curve. Indeed, all three vectors of this frame are mutually orthogonal unit vectors, and when a point moves along a curve, the lengths of these vectors and their relative position do not change. Geometrically, this means that when a point moves along a curve, these three vectors rotate as a single rigid body. This means that the derivatives of these vector fields will show the speed of this rotation.

It follows from the definition (1.56) of the normal vector field N that

$$\mathtt{T}' = \kappa\,\mathtt{N}. \tag{1.57}$$

Thus, we have expressed the derivative of the vector field of velocity in terms of vector fields of moving frame $\{\mathtt{T}, \mathtt{N}, \mathtt{B}\}$. This equation is called the first Bartels-Frenet-Serret equation.

Now consider the derivative of binormal vector field $\mathtt{B}$. Differentiating the cross-product $\mathtt{T} \times \mathtt{N}$ we get

$$\mathtt{B}' = \mathtt{T}' \times \mathtt{N} + \mathtt{T} \times \mathtt{N}'. \tag{1.58}$$

The first term on the right-hand side of (1.58) is equal to the zero vector and this follows from the first Bartels-Frenet-Serret equation (1.57). The second term in (1.58) (and hence the vector field $\mathtt{B}'$) is perpendicular to vector fields $\mathtt{T}$ and $\mathtt{N}'$. But we have one more vector field, which is also perpendicular to vector fields $\mathtt{T}$ and $\mathtt{N}'$, this is the normal vector field $\mathtt{N}$. Therefore, the derivative of binormal vector field $\mathtt{B}'$ is collinear to the normal vector field $\mathtt{N}$ and there exists a uniquely defined scalar function $\tau(s)$ such that

$$\mathtt{B}' = -\tau\,\mathtt{N}. \tag{1.59}$$

This function τ is called a *torsion of the space curve* and it is an important geometric characteristic of a curve, the geometric meaning of which we will explore later in this section. The equation (1.59) is called the third Bartels-Frenet-Serret equation.

Now we consider the derivative of the normal vector field N. As $\{\mathtt{T}, \mathtt{N}, \mathtt{B}\}$ is an orthonormal frame, we can expand the derivative $\mathtt{N}'$ as follows

$$\mathtt{N}' = <\mathtt{N}', \mathtt{T}> \mathtt{T} + <\mathtt{N}', \mathtt{N}> \mathtt{N} + <\mathtt{N}', \mathtt{B}> \mathtt{B}.$$

As $\mathtt{N}$ is a constant length vector field, it follows from Proposition 1.4 that $<\mathtt{N}, \mathtt{N}'> = 0$ and the middle term in the above expansion vanishes. In order to find the coefficient function $<\mathtt{N}', \mathtt{T}>$ in the first term of expansion, we differentiate the both sides of relation $<\mathtt{N}, \mathtt{T}> = 0$ and get $<\mathtt{N}', \mathtt{T}> = -<\mathtt{N}, \mathtt{T}'>$. Making use of the first Bartels-Frenet-Serret equation $\mathtt{T}' = \kappa\,\mathtt{N}$ we find $<\mathtt{N}',\mathtt{T}>= -\kappa$. Analogously, differentiating the both sides of the relation $<\mathtt{N},\mathtt{B}>=0$ we get $<\mathtt{N}',\mathtt{B}>+<\mathtt{N},\mathtt{B}'>=0$. Hence $<\mathtt{N}',\mathtt{B}>=-<\mathtt{N},\mathtt{B}'>$ and substituting $\mathtt{B}' = -\tau\,\mathtt{N}$ we get $<\mathtt{N}',\mathtt{B}>=\tau\,\mathtt{B}$. Thus, we have

$$\mathtt{N}' = -\kappa\,\mathtt{T} + \tau\,\mathtt{B}, \tag{1.60}$$

and this equation is called the second Bartels-Frenet-Serret equation.

Summarizing the reasoning presented above, we obtain the following result. Let $\eta = (\eta(s), I)$ be a unit-speed parametrized curve in space, where the curvature $\kappa(s)$ at each point of the curve is nonzero. Then the moving frame $\{\mathtt{T},\mathtt{N},\mathtt{B}\}$ of this curve satisfies the Bartels-Frenet-Serret equation

$$\mathtt{T}' = \kappa\,\mathtt{N}, \tag{1.61}$$
$$\mathtt{N}' = -\kappa\,\mathtt{T} + \tau\,\mathtt{B}, \tag{1.62}$$
$$\mathtt{B}' = -\tau\,\mathtt{N}, \tag{1.63}$$

where $\tau =<\mathtt{N}',\mathtt{B}>$ is called a torsion of a unit-speed parametrized curve $(\eta = \eta(s), I)$. The structure of Bartels-Frenet-Serret formulae becomes very clear if we write them in matrix form

$$\begin{pmatrix} \mathtt{T}' \\ \mathtt{N}' \\ \mathtt{B}' \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} \mathtt{T} \\ \mathtt{N} \\ \mathtt{B} \end{pmatrix}.$$

We see that the third-order square matrix on the right-hand side of this matrix equation is skew-symmetric. This fact is very important and will be explained later.

Let us determine the geometric meaning of the torsion of a space curve. From the third equation of the Bartels-Frenet-Serret formulas, it follows that the rate of rotation of the binormal vector $\mathtt{B}(s)$ is equal to the absolute value of the torsion $|\tau(s)|$. However, the binormal vector is perpendicular to the osculating plane of the curve at each of its points. Therefore, the absolute value of the torsion represents the rate of rotation of the osculating plane. It is evident that if the curve is planar, meaning it lies entirely in some fixed plane P, then its osculating plane at every point coincides with P. In this case, the binormal vector remains constant, implying that its derivative is zero. Since the principal normal vector $\mathtt{N}$ is nonzero, we conclude that the torsion of such a curve is zero. Conversely, it can be proven that if the torsion of a curve is zero, then the curve lies entirely in some plane. Indeed let $\tau(s) = 0$. Then $\mathtt{B}'(s) = 0$. Hence binormal vector is constant and we denote it by $\mathtt{B}_0$. Then $<\mathtt{T}(s),\mathtt{B}_0>=0$ or we can write $<\eta'(s),\mathtt{B}_0>=0$. Hence

$$\frac{d}{ds}(<\vec{\eta}(s),\mathtt{B}_0>) = 0.$$

We see that the function $<\vec{\eta}(s),\mathtt{B}_0>$ is constant. Let $<\vec{\eta}(s),\mathtt{B}_0>=d$, where d is a real number, $\mathtt{B}_0 = (a,b,c), \vec{\eta}(s) = (x(s),y(s),z(s))$. Then for any $s \in I$ we have $a\,x(s) + b\,y(s) + c\,z(s) = d$. Therefore, every point $(x(s),y(s),z(s))$ of the curve $\eta(s)$ lies in a plane, and the curve is planar.

The curvature and torsion of a space curve are fundamental characteristics that describe its geometry. Computing the curvature requires evaluating second-order derivatives of the position vector of the curve, while computing the torsion involves third-order derivatives. One might expect that there exist additional geometric characteristics of the curve that depend on higher-order derivatives of the position vector, but this turns out not to be the case. All other geometric properties can be expressed in terms of the curvature, the torsion, and their derivatives. Consequently, a curve is completely determined, up to rigid motions, by its curvature and torsion. The equations $\kappa = \kappa(s)$, $\tau = \tau(s)$ are called the *natural equations* of the curve. The following theorem holds:

Theorem 1.2. Let $\kappa(s)$ and $\tau(s)$ be two smooth functions defined on an interval $I = [0, l]$ with $\kappa(s) > 0$. Suppose that a point p_0 is given in space, along with three mutually orthogonal vectors T_0, N_0, B_0 satisfying $B_0 = T_0 \times N_0$. Then there exists a unique curve $(\eta(s), I)$ in space such that:

- $\eta(0) = p_0$,
- s is the arc length of the curve measured from p_0,
- $\{T_0, N_0, B_0\}$ is the Bartels-Frenet-Serret frame at p_0,
- $\kappa(s)$ and $\tau(s)$ are the curvature and torsion of the curve, respectively.

It should be noted that the formulas (1.61), (1.62), (1.63) are a powerful tool in the study of space curves. Besides, these formulas have several important applications in theoretical physics and mechanics. It is worth to mention a historical aspect of the discovery of these formulas [**Abramov:2004finest**]. These formulas were published by F. Frenet in his dissertation in [**Frenet:1847**] in 1847 and, independently of him, these formulas were obtained by another French mathematician, J.A. Serret, and published in [**Serret:1851**] in 1851. In this regard, these formulas are usually called the Frenet-Serret formulas. However, it emerged later that the formulas (1.61), (1.62), (1.63) were discovered by M. Bartels before the French mathematicians F. Frenet and J.A. Serret ever published them. M. Bartels was a professor of mathematics at the University of Dorpat (now Tartu) in Estonia. He used to present his results of studies in differential geometry of curves in lectures for students. One of his students, C.E. Senff, took notes of M. Bartels lectures and later included these notes into his book "Principal theorems from the theory of curves and surfaces", which was published in 1831. Therefore, from a historical point of view, it would be correct to call the formulas (1.61), (1.62), (1.63) Bartels-Frenet-Serret formulas, as it was proposed in [**Lumiste:1997mbr**].

Example 1.8. As an example, let us find Bartels-Frenet-Serret equations for a helix. Recall that in Example (1.5), a natural (unit-speed) parametrization of a helix was found and we will use it in this example. The unit-speed parametrization of a helix is given by

$$\eta(s) = (a\,\cos\frac{s}{\sqrt{a^2+b^2}}, a\,\sin\frac{s}{\sqrt{a^2+b^2}}, \frac{b\,s}{\sqrt{a^2+b^2}}).$$

To simplify the form of formulae and make them more compact, we introduce the notation $\alpha = \sqrt{a^2+b^2}$. Hence differentiating with respect to natural parameter s we get the velocity vector field

$$\mathtt{T}(s) = \eta'(s) = (\eta(s); -\frac{a}{\alpha}\sin\frac{s}{\alpha}, \frac{a}{\alpha}\cos\frac{s}{\alpha}, \frac{b}{\alpha}).$$

Differentiating the velocity vector field $\mathtt{T}$ we find the acceleration vector field

$$\mathtt{T}'(s) = \eta''(s) = (\eta(s); -\frac{a}{\alpha^2}\cos\frac{s}{\alpha}, -\frac{a}{\alpha^2}\sin\frac{s}{\alpha}, 0).$$

Since the curvature of a spatial curve in unit-speed parametrization is equal to the length of the acceleration vector, we find the curvature of a helix

$$\kappa(s) = ||\eta''(s)|| = \frac{a}{\alpha^2} = \frac{a}{a^2+b^2}.$$

Note that the curvature of a helix is the constant non-zero number. Next we find the normal vector field of a helix

$$\mathtt{N}(s) = \frac{1}{\kappa(s)}\mathtt{T}'(s) = (\eta(s); -\cos\frac{s}{\alpha}, -\sin\frac{s}{\alpha}, 0).$$

The cross product $\mathtt{T} \times \mathtt{N}$ gives us the binormal vector field $\mathtt{B}$. Hence

$$\mathtt{B}(s) = (\eta(s); \frac{b}{\alpha}\sin\frac{s}{\alpha}, -\frac{b}{\alpha}\cos\frac{s}{\alpha}, \frac{a}{\alpha}).$$

Differentiating the binormal vector field we obtain

$$\mathtt{B}'(s) = (\eta(s); \frac{b}{\alpha^2}\cos\frac{s}{\alpha}, \frac{b}{\alpha^2}\sin\frac{s}{\alpha}, 0).$$

The last formula shows that the derivative of the binormal vector field $\mathtt{B}'$ is collinear to the normal vector field $\mathtt{N}$ and the corresponding coefficient function, when we express $\mathtt{B}'$ in terms of $\mathtt{N}$, taken with a minus, is equal to the torsion $\tau(s)$ of a helix. Hence

$$\tau(s) = \frac{b}{\alpha^2} = \frac{b}{a^2 + b^2}.$$

We leave the verification of the second Bartels-Frenet-Serret equation to reader as a small exercise.

1.11 Computational Formulas for Curvature and Torsion

We have derived Bartels-Frenet-Serret formulae based on the assumption that a parametrized curve is regular and it is parameterized by the natural parameter s. Since such a parametrization is always possible, the obtained result is general for the entire class of regular curves. However, we can find a form of Bartels-Frenet-Serret equations in the case of an arbitrary parametrization of a regular curve. To extend these equations to an arbitrary parametrization of a regular curve, we assume that we are given a regular parametrized curve $\xi = (\xi(t), I)$. Let $\eta = (\eta(s), J)$ be the unit-speed parametrization of ξ. Then we know that $\xi(t) = \eta(s(t))$, where $s(t)$ is the arc length function of ξ, i.e.

$$s(t) = \int_{t_0}^{t} ||\xi'(u)||\, du,$$

where $t_0 \in I$. Thus, using the arc length function $s(t)$, one can switch from a unit-speed parametrization $\eta(s)$ to a given parametrization $\xi(t)$. Hence it follows that we can extend any quantity (function, vector field) defined for a given curve in the natural parametrization and depending on the natural parameter s to an arbitrary parametrization by replacing the natural parameter s with the arc length function $s = s(t)$. According to this approach, we define the moving frame, curvature and torsion of a curve $(\xi = \xi(t), I)$ (i.e. in parametrization $\xi(t)$) as follows

$$\begin{aligned} \mathtt{T}(t) &= \mathtt{T}(s)\Big|_{s=s(t)}, \ \mathtt{N}(t) = \mathtt{N}(s)\Big|_{s=s(t)}, \ \mathtt{B}(t) = \mathtt{B}(s)\Big|_{s=s(t)}, \\ \kappa(t) &= \kappa(s)\big|_{s=s(t)}, \ \tau(t) = \tau(s)\big|_{s=s(t)}. \end{aligned}$$

It is important to note here that neither the values of the vector fields of the moving frame, nor the values of curvature and torsion at the points of the curve, change, only the value of the parameter that corresponds to them changes.

Now our goal is to find the form of Bartels-Frenet-Serret equations in the case of an arbitrary parametrization $\xi(t)$. We have

$$\mathtt{T}'(t) = s'(t)\,\mathtt{T}'(s)\big|_{s=s(t)} = ||\xi'(t)||\,\Big(\kappa(s)\,\mathtt{N}(s)\Big)\Big|_{s=s(t)} = ||\xi'(t)||\,\kappa(t)\,\mathtt{N}(t).$$

When deriving this equation, we used the chain rule for parametrized curves and the first Bartels-Frenet-Serret equation in the case of unit-speed parametrization. Passing to vector fields and functions (i.e, omitting parameter t in the above equation) and denoting the function $s'(t) = ||\xi'||$ by υ, we get

$$\mathtt{T}' = \upsilon\,\kappa\,\mathtt{N}. \tag{1.64}$$

Similarly one can derive the second and third Bartels-Frenet-Serret equations for a parametrized curve $(\xi = \xi(t), I)$. Thus, we got the following result: If $(\xi = \xi(t), I)$ is a regular parametrized curve and the curvature of this curve is non-zero at each point of a curve, then the moving frame $\{\mathtt{T}, \mathtt{N}, \mathtt{B}\}$ satisfies the equations

$$\begin{aligned} \mathtt{T}' &= \upsilon\kappa\,\mathtt{N}, &(1.65)\\ \mathtt{N}' &= -\upsilon\kappa\,\mathtt{T} + \upsilon\tau\,\mathtt{B}, &(1.66)\\ \mathtt{B}' &= -\upsilon\tau\,\mathtt{N}, &(1.67)\end{aligned}$$

which will be referred to as *generalized Bartels-Frenet-Serret equations.*

Our next goal is to derive formulae for the moving frame, curvature and torsion of a parametrized curve $\xi = (\xi(t), I)$. To do this, we need the derivatives of $\xi(t)$ up to the third order inclusive. Differentiating the both sides of $\xi(t) = \eta(s(t))$ with respect to t we get

$$\xi'(t) = s'\,\eta'(s)\big|_{s=s(t)}.$$

Taking into account that $\eta'(s) = \mathtt{T}(s)$ and the definition of $\mathtt{T}(t)$, we can put the previous formula into the form $\xi' = \upsilon\,\mathtt{T}$. Differentiating one more time and applying the first generalized Bartels-Frenet-Serret equation (1.65) we obtain

$$\xi'' = \upsilon'\,\mathtt{T} + \upsilon\,T' = \upsilon'\,\mathtt{T} + \upsilon^2\,\kappa\,\mathtt{N}.$$

Analogously, i.e. differentiating the both sides of the above formula and making use of Bartels-Frenet-Serret equations, we calculate the third order derivative of $\xi(t)$. Hence we have

$$\begin{aligned} \xi' &= \upsilon\,\mathtt{T}, &(1.68)\\ \xi'' &= \upsilon'\,\mathtt{T} + \upsilon^2\,\kappa\,\mathtt{N}, &(1.69)\\ \xi''' &= (\upsilon'' - \upsilon^3\kappa^2)\,\mathtt{T} + (3\upsilon\,\upsilon'\,\kappa + \upsilon^2\,\kappa')\,\mathtt{N} + \upsilon^3\kappa\,\tau\,\mathtt{B}. &(1.70)\end{aligned}$$

Using these equations, we can calculate the curvature and torsion of a curve. First of all, we consider the vector product of the vector fields ξ' and ξ''. We find

$$\xi' \times \xi'' = \upsilon^3\kappa\,\mathtt{B} \;\Rightarrow\; ||\xi' \times \xi''|| = \upsilon^3\kappa.$$

Hence

$$\kappa = \frac{||\xi' \times \xi''||}{||\xi'||^3}. \tag{1.71}$$

In order to find the torsion of a curve ξ we consider the triple product of the vector fields ξ', ξ'', ξ'''. From (1.68),(1.69), (1.70) it follows that

$$(\xi', \xi'', \xi''') = \upsilon^6\kappa^2\tau.$$

Hence

$$\tau = \frac{(\xi', \xi'', \xi''')}{||\xi' \times \xi''||^2}. \tag{1.72}$$

2 Vector fields and Forms

ONE of the most important concepts of modern differential geometry is the notion of a vector field. The importance of this concept lies in the fact that it can be used not only in differential geometry, but also in mechanics and theoretical physics. Imagine that we were able to take a snapshot of the flow of a current fluid and at each point we constructed a velocity vector of infinitesimal particles of this fluid. We obtain the distribution of vectors in a certain region of space, which would naturally be called a vector field. Maxwell's great insight was the idea of describing electromagnetic phenomena with the help of a concept of an electromagnetic field. A vector field, considered as the distribution of vectors in a certain region of space, is very clear and easy to understand. However, it turns out that an approach to vector field based on directional derivative is more effective and useful. The concept of a differential form is not so visual, it is more abstract and in order to explain a notion of a differential form we will use the dual space of a vector space. Despite this, the calculus of differential forms plays an exceptionally important role not only in differential geometry, but also in field theories.

2.1 Vector fields in Euclidean space

In this section we continue the study of vector fields in an affine Euclidean space E^n started in the previous chapter (Section 1.5). Recall that a vector field in an open subset U of an affine space E^n is defined as a smooth mapping $\mathtt{X} : p \in U \to \mathtt{X}_p \in T_pE^n$, which attributes to each point p of U a tangent vector of an affine space E^n at this point p. We can give this definition by means of the disjoint union of tangent spaces to E^n over an open subset U, i.e. $T(U) = \cup_{p\in U} T_pE^n$, as a smooth mapping $\mathtt{X} : U \to T(U)$, which satisfies $\pi \circ \mathtt{X} = \mathrm{id}_U$, where $\pi : T(U) \to U$ is the projection defined by the formula $\pi(\mathtt{v}_p) = p$. The set of all smooth vector fields in an open subset U is denoted by $\mathcal{D}(U)$ and this set becomes the vector space if we define pointwise the addition of vector fields and multiplication by real numbers. One can also define a multiplication of vector fields by smooth functions $f\mathtt{X}$, where f is a smooth function, $\mathtt{X}$ is a vector field and $f\mathtt{X}$ is a vector field, whose value at a point $p \in U$ is the vector $f(p)\mathbb{X}_p$. Multiplication by smooth functions together with previously defined addition of vector fields and their multiplication by real numbers makes the set of all vector fields $\mathcal{D}(U)$ a module over the algebra of smooth functions $C^\infty(U)$. In the case of an affine Euclidean space E^n we have an inner product and the tangent space T_pE^n at a point p is an Euclidean space and this Euclidean structure induces the inner product of vector fields $< \mathtt{X}, \mathtt{Y} >$ as follows: The inner product $< \mathtt{X}, \mathtt{Y} >$ is the function on U, whose value at a point $p \in U$ is the inner product of tangent vectors $< \mathtt{X}_p, \mathtt{Y}_p >$.

So far we have considered the notion of a vector field in a coordinate free form. This approach has certain advantages, for instance, it makes possible to find invariant structures, that is, structures which are independent of a choice of coordinate system. Such a structure depends directly on the geometry of a space and is not affected by an accidental nature of a choice of coordinate system. However an introduction of a coordinate system allows one to perform calculations and this is the reason why we break this symmetry and choose some specific coordinate system. The most convenient coordinate system is a system of affine coordinates. We will choose it as the basic coordinate system and show how a vector field can be described by means of affine coordinates. First we fix a reference frame $\{O; \vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}$ in an affine space E^n and denote the corresponding affine coordinates by $x^1, x^2, \ldots, x^n$. Then at any point $p \in E^n$ we have the frame $\{\mathtt{e}_{p,1}, \mathtt{e}_{p,2}, \ldots, \mathtt{e}_{p,n}\}$ of the tangent space T_pE^n. Hence the set of all vectors $\mathtt{e}_{p,i}$, where i is fixed and p runs over affine space E^n, determines the constant vector field, which will be denoted by $\mathtt{D}_i$. Thus the value of a vector field $\mathtt{D}_i$ at a point p (we denote it by $(\mathtt{D}_i)_p$) is $(\mathtt{D}_i)_p = \mathtt{e}_{p,i}$. In the case when E^n is an affine Euclidean space, i.e. there is an inner product, we will use Cartesian coordinates, i.e. we will assume that a reference frame $\{O; \vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}$ is orthonormal. Making use of the multiplication of a vector field by a function, one can expand any vector field $\mathtt{X} \in \mathcal{D}(U)$ into the sum of the basic vector fields $\mathtt{D}_1, \mathtt{D}_2, \ldots, \mathtt{D}_n$ as follows

$$\mathtt{X} = \sum_{i=1}^{n} X^i\, \mathtt{D}_i,$$

where $X^1, X^2, \ldots, X^n$ are smooth functions defined on U. Hence in affine coordinates a vector field $\mathtt{X}$ is uniquely determined by the functions X^i and we will call these functions *components of a vector field* $\mathtt{X}$ in affine coordinates.

It is easy to see that previously defined operations with vector fields can be expressed in terms of components of these vector fields as follows. If $(X^1, X^2, \ldots, X^n)$ and $(Y^1, Y^2, \ldots, Y^n)$ are the components of vector fields $\mathtt{X}, \mathtt{Y}$ respectively, then the components of the sum $\mathtt{X} + \mathtt{Y}$ are the functions $(X^1 + Y^1, X^2 + Y^2, \ldots, X^n + Y^n)$. If E^n is an affine Euclidean space we have the inner product of two vector fields $< \mathtt{X}, \mathtt{Y} >$ and this inner product can be written in the Cartesian coordinates as follows

$$< \mathtt{X}, \mathtt{Y} >= \sum_{i,j=1}^{n} X^i Y^i.$$

If we multiply a vector field $\mathtt{X}$ by a smooth function f then the components of the vector field $f\,\mathtt{X}$ are the functions $(f\,X^1, f\,X^2, \ldots, f\,X^n)$.

In Cartesian coordinates a vector field $\mathtt{X}$ can be expressed in terms of the basic fields $\mathtt{D}_1, \mathtt{D}_2, \ldots, \mathtt{D}_n$ by means of inner product as follows

$$\mathtt{X} = \sum_{i=1}^{n} < \mathtt{X}, \mathtt{D}_i > \mathtt{D}_i.$$

Hence in order to calculate the components of a vector field $\mathtt{X}$ in Cartesian coordinates one can use the formula $X^i =< \mathtt{X}, \mathtt{D}_i >$.

2.2 Directional derivative

The concept of a vector field is similar to the concept of a function. Indeed a function assigns to each point of a domain a real number and a vector field assigns a vector. Methods for studying the behavior of a function are based on the study of the derivative of a function. Similarly the study of the behavior of a vector field, for instance its rate of change, is based on a differentiation of a vector field. This section is devoted to differentiation of vector fields. But we will begin this section with a concept of directional derivative of a function.

Let $f \in C^\infty(U)$ be a smooth function and $\mathtt{v}_p = (p; \vec{v}) \in T_pE^n$ be a tangent vector at a point $p \in U$. A tangent vector $\mathtt{v}_p$ determines the straight line passing through a point p, whose parametric equation can be written in the form $p+t\vec{v}$. The restriction of a function f to the straight line $p+t\vec{v}$ will be denoted by $f(p+t\vec{v})$. It is easy to see that this restriction of a function f is the function of one variable t.

Definition 2.1. A directional derivative of a function f at a point p is defined by the formula

$$\mathtt{v}_p \triangleright f = \frac{d}{dt}\big(f(p+t\vec{v})\big)\big|_{t=0}.$$

Thus the directional derivative is a mapping $\mathtt{v}_p : C^\infty \to \mathbb{R}$. This mapping is linear and satisfies the Leibniz rule, i.e. for any smooth functions f, g and any real numbers $a, b \in \mathbb{R}$ one has the following properties

1. $\mathtt{v}_p \triangleright (a\, f + b\, g) = a\, \mathtt{v}_p \triangleright f + b\, \mathtt{v}_p \triangleright g$,
2. $\mathtt{v}_p \triangleright (f\, g) = (\mathtt{v}_p \triangleright f)\, g + f\, (\mathtt{v}_p \triangleright g)$.

Definition (2.1) is given in a coordinate free form. If we now assume that an affine space E^n is endowed with a system of affine coordinates $x^1, x^2, \ldots, x^n$ then a function f is the function of n variables $f(x^1, x^2, \ldots, x^n)$ and a vector $\vec{v}$ can be written in coordinates $\vec{v} = (v^1, v^2, \ldots, v^n)$. Now making use of Definition (2.1) we find

$$\mathtt{v}_p \triangleright f = \frac{d}{dt}\big(f(p+t\vec{v})\big)\big|_{t=0} = \sum_{i=1}^{n} \frac{d}{dt}(p^i + t\, v^i)\Big|_{t=0} \frac{\partial f}{\partial x^i}\Big|_p = \sum_{i=1}^{n} v^i\, \frac{\partial f}{\partial x^i}\Big|_p. \tag{2.1}$$

One can use this formula to compute a directional derivative of a function in affine coordinates. For instance applying (2.1) we find

$$\mathtt{e}_{p,i} \triangleright f = \frac{\partial f}{\partial x^i}\Big|_p. \tag{2.2}$$

In the case of Cartesian coordinates we can write this formula by means of an inner product and the gradient as follows

$$\mathtt{v}_p \triangleright f =< \mathtt{v}_p, (\operatorname{grad} f)_p > . \tag{2.3}$$

We can extend the notion of directional derivative to vector fields. Let $\mathtt{X} \in \mathcal{D}(U)$ be a vector field. Then according to the definition a vector field $\mathtt{X}$ assigns to each point p of U a tangent vector $\mathtt{X}_p$. Given a smooth function $f \in C^\infty(U)$ one can find a directional derivative $\mathtt{X}_p \triangleright f$. This directional derivative is a real number. Hence one can assign to each point p of U a real number, that is, one has the real-valued function defined on U. Let us denote this function by $\mathtt{X} \triangleright f$. Then according to the previous considerations the value of the function $\mathtt{X} \triangleright f$ at a point p is the real number $\mathtt{X}_p \triangleright f$. For example making use of (2.2) we find $\mathtt{D}_i \triangleright f = \partial f / \partial x^i$. Hence in affine coordinates for any vector field $\mathtt{X} = \sum_{i=1}^n X^i\, \mathtt{D}_i$ we have

$$\mathtt{X} \triangleright f = \sum_{i=1}^{n} X^i\, (\mathtt{D}_i \triangleright f) = \sum_{i=1}^{n} X^i\, \frac{\partial f}{\partial x^i}. \tag{2.4}$$

It follows from (2.3) that in the case of Cartesian coordinates we can write the last formula in the form

$$\mathtt{X} \triangleright f = \sum_{i=1}^{n} < \mathtt{X}, \operatorname{grad} f > . \tag{2.5}$$

Formula (2.4) shows that in affine coordinates a vector field $\mathtt{X}$ induces the first order differential operator

$$\mathtt{X} \to \sum_{i=1}^{n} X^i \frac{\partial}{\partial x^i}. \tag{2.6}$$

It also shows that if f is a smooth function then $\mathtt{X}[f]$ is also a smooth function. Thus a vector field $\mathtt{X}$ induces the mapping $f \in C^\infty \to \mathtt{X}[f] \in C^\infty$ which is linear and satisfies the Leibniz rule, that is, for any two smooth functions f, g and any two real numbers a, b we have

1. $\mathtt{X} \triangleright (a\, f + b\, g) = a\, (\mathtt{X} \triangleright f) + b\, (\mathtt{X} \triangleright g)$,
2. $\mathtt{X} \triangleright (f\, g) = (\mathtt{X} \triangleright f)\, g + f\, (\mathtt{X} \triangleright g)$.

Definition 2.2. Let $\mathcal{A}$ be a unital associative algebra. Then a linear mapping $\delta : \mathcal{A} \to \mathcal{A}$ is said to be an *algebra derivation* if it satisfies the Leibniz rule

$$\delta(u\, v) = \delta(u)\, v + u\, \delta(v),$$

where $u, v \in \mathcal{A}$.

From this definition it follows that a vector field can be considered as a derivation of the algebra of smooth functions on U. It can be proved that any derivation of the algebra of smooth functions $C^\infty(U)$ can be uniquely written in affine coordinates in the form of first order differential operator (2.6). Thus in what follows we will consider a vector field as a derivation of an algebra of smooth functions and identify it with a first order differential operator (2.6), i.e.

$$\mathtt{X} = \sum_{i=1}^{n} X^i \frac{\partial}{\partial x^i}. \tag{2.7}$$

Particularly we have

$$\mathtt{D}_i = \frac{\partial}{\partial x^i}, \quad (\mathtt{D}_i)_p = \mathtt{e}_{p,i} = \frac{\partial}{\partial x^i}\Big|_p.$$

Now the affine basis $\{\mathtt{e}_{p,1}, \mathtt{e}_{p,2}, \ldots, \mathtt{e}_{p,n}\}$ for a tangent space T_pE^n at a point p can be written in the form

$$\{\frac{\partial}{\partial x^1}\Big|_p, \frac{\partial}{\partial x^2}\Big|_p, \ldots, \frac{\partial}{\partial x^n}\Big|_p\}. \tag{2.8}$$

In the case of Cartesian coordinates this basis is orthonormal and we have

$$< \frac{\partial}{\partial x^i}\Big|_p, \frac{\partial}{\partial x^j}\Big|_p >= \delta_{ij}. \tag{2.9}$$

Let us now consider the question of vector fields in curvilinear coordinates. To do this, we formulate in a suitable way a scheme for working with vector fields in affine coordinates, in particular, in Cartesian coordinates. In this case we start with an affine frame $\{O; \vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}$ which induces a system of affine coordinates $x^1, x^2, \ldots, x^n$. Then we move in a parallel way the vectors of basis $\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n$ to each point p of a space E^n with the help of notion of a vector at a point. Thus we get n constant vector fields $\mathtt{D}_1, \mathtt{D}_2, \ldots, \mathtt{D}_n$ and at each point of a space E^n these vector fields form the basis for a tangent space T_pE^n. In order to extend this approach to curvilinear coordinates we can consider the vectors of basis for a tangent space from a slightly different point of view. Let p be a point in space E^n. A straight line $p + t\,\vec{e}_i$ is called the ith *coordinate line* of an affine coordinate system passing through a point p. Hence n coordinate lines pass through each point in a space. It is easy to see that the tangent vector to a coordinate line $p + t\,\vec{e}_i$ at a point p is the ith vector of basis $\mathtt{e}_{p,i}$ of a tangent space T_pE^n. On the other hand if we compute a directional derivative of a function f along this coordinate line then

$$\frac{d}{dt}\big(f(p + t\,\vec{e}_i)\big)\Big|_{t=0} = \frac{\partial f}{\partial x^i}\Big|_p,$$

and we can identify an ith vector of basis with partial derivative with respect to ith coordinate at a point p.

Let us now apply this approach to curvilinear coordinates. Hence we assume that we have a system of curvilinear coordinates $\psi : U \subset E^n \to W \subset \mathbb{R}^n$ which is C^∞-compatible with affine coordinates $x^1, x^2, \dots, x^n$. The curvilinear coordinates will be denoted by $\tilde{x}^1, \tilde{x}^2, \dots, \tilde{x}^n$ and

$$x^1 = f^1(\tilde{x}^1, \tilde{x}^2, \dots, \tilde{x}^n),\ x^2 = f^2(\tilde{x}^1, \tilde{x}^2, \dots, \tilde{x}^n), \dots, x^n = f^n(\tilde{x}^1, \tilde{x}^2, \dots, \tilde{x}^n), \tag{2.10}$$

will be the functions of transition from an affine coordinates to curvilinear coordinates. Let $p = (\tilde{x}_0^1, \tilde{x}_0^2, \dots, \tilde{x}_0^n)$ be a point of U. Then

$$\xi_i(t) = (\tilde{x}_0^1, \tilde{x}_0^2, \dots, \tilde{x}_0^{i-1}, \tilde{x}_0^i + t, \tilde{x}_0^{i+1}, \dots, x_0^n)$$

is an ith coordinate line of a system of curvilinear coordinates passing through a point p. Calculating a directional derivative of a function f along a coordinate line $\xi_i(t)$ we find

$$\frac{d}{dt}\big(\xi_i(t)\big)\Big|_{t=0} = \sum_{k=0}^{n} \frac{\partial f}{\partial \tilde{x}^k}\Big|_p \frac{d\tilde{x}^k}{dt}\Big|_{t=0} = \frac{\partial f}{\partial \tilde{x}^i}\Big|_p$$

This formula shows that for any point $p \in U$ the tangent vectors

$$\frac{\partial}{\partial \tilde{x}^1}\Big|_p, \frac{\partial}{\partial \tilde{x}^2}\Big|_p, \dots, \frac{\partial}{\partial \tilde{x}^n}\Big|_p \tag{2.11}$$

form the basis for a tangent space T_pE^n in curvilinear coordinates $\tilde{x}^1, \tilde{x}^2, \dots, \tilde{x}^n$. Hence in curvilinear coordinates the vector fields

$$\tilde{\mathtt{D}}_1 = \frac{\partial}{\partial \tilde{x}^1},\ \tilde{\mathtt{D}}_2 = \frac{\partial}{\partial \tilde{x}^2},\ \dots\ ,\tilde{\mathtt{D}}_n = \frac{\partial}{\partial \tilde{x}^n} \tag{2.12}$$

form the basis of a tangent space T_pE^n at each point of an affine space E^n.

It is useful to find how the basic vector fields (2.12) in curvilinear coordinates can be expressed in terms of the basic vector fields in affine coordinates. Consider a smooth function f defined in affine coordinates $x^1, x^2, \dots, x^n$. We can pass from affine coordinates to curvilinear coordinates by means of the transition functions (2.10) and we obtain the composite function whose partial derivative with respect to $\tilde{x}^i$ can be written as follows

$$\frac{\partial f}{\partial \tilde{x}^i} = \frac{\partial x^j}{\partial \tilde{x}^i}\frac{\partial f}{\partial x^j}.$$

Since f is an arbitrary smooth function we find the transformation rule for basic vector fields

$$\tilde{\mathtt{D}}_i = \sum_{j=1}^{n} \frac{\partial x^j}{\partial \tilde{x}^i}\,\mathtt{D}_j. \tag{2.13}$$

It is worth to mention that this formula describes a transition from one basis for a tangent space T_pE^n to another (at each point p of U) because the Jacobian of this transition (the determinant of the matrix $A = (A_i^j) = (\partial x^j/\partial \tilde{x}^i)$) is non-zero (see Proposition 1.1).

Formula (2.13) is very important because it leads to several important conclusions. First we see that in curvilinear coordinates the basic vector fields $\tilde{\mathtt{D}}_1, \tilde{\mathtt{D}}_2, \ldots, \tilde{\mathtt{D}}_n$ are not constant vector fields. Hence the basis (2.11) for a tangent space T_pE^n depends on a point of a space and it changes when one moves from one point of a space to another. Second formula (2.11) is important when E^n is an affine Euclidean space. Indeed in curvilinear coordinates we can not use formulae of Euclidean geometry for inner product and a length, but formula (2.11) shows a way how to calculate inner product of tangent vectors if we use curvilinear coordinates. Let us consider the case when E^n is an affine Euclidean space. Then

$$
\begin{aligned}
< \tilde{\mathtt{D}}_i, \tilde{\mathtt{D}}_j > \ &= \ < \sum_{k=1}^{n} \frac{\partial x^k}{\partial \tilde{x}^i}\, \mathtt{D}_k, \sum_{m=1}^{n} \frac{\partial x^m}{\partial \tilde{x}^j}\, \mathtt{D}_m >= \sum_{k,m=1}^{n} \frac{\partial x^k}{\partial \tilde{x}^i}\, \frac{\partial x^m}{\partial \tilde{x}^j} < \mathtt{D}_k, \mathtt{D}_m > \\
&= \sum_{k,m=1}^{n} \frac{\partial x^k}{\partial \tilde{x}^i}\, \frac{\partial x^m}{\partial \tilde{x}^j}\, \delta_{km} = \sum_{k=1}^{n} \frac{\partial x^k}{\partial \tilde{x}^i}\, \frac{\partial x^k}{\partial \tilde{x}^j}.
\end{aligned}
$$

Let us define the square matrix of nth order $\tilde{g} = (\tilde{g}_{ij})$, where

$$\tilde{g}_{ij} = \sum_{k=1}^{n} \frac{\partial x^k}{\partial \tilde{x}^i}\, \frac{\partial x^k}{\partial \tilde{x}^j}.$$

The matrix $\tilde{g}$ is referred to as a *matrix of Euclidean metric in curvilinear coordinates*. Obviously the matrix of Euclidean metric in Cartesian coordinates g is the unit matrix $g = (\delta_{ij})$ because $< \mathtt{D}_i, \mathtt{D}_j >= \delta_{ij}$. The square of the arc differential expressed in terms of coordinate differentials is usually referred to as the *Euclidean metric*. In Cartesian coordinate the Euclidean metric can be written as

$$ds^2 = (dx^1)^2 + (dx^2)^2 + \ldots + (dx^n)^2 = \sum_{i=1}^{n} \delta_{ij} dx^i\, dx^j.$$

Hence in curvilinear coordinates the Euclidean metric has the form

$$ds^2 = \sum_{i,j=1}^{n} \tilde{g}_{ij} dx^i\, dx^j = \sum_{i,j,k=1}^{n} \frac{\partial x^k}{\partial \tilde{x}^i}\, \frac{\partial x^k}{\partial \tilde{x}^j} dx^i\, dx^j.$$

2.3 Exterior algebra of a vector space

In this section we will use Einstein's convention to write sums containing quantities with indices. This convention is that if the same index occurs twice in an expression, once as a superscript and the second time as a subscript, then one takes the sum over this index, but the symbol of the sum is omitted.

Let V be an n-dimensional vector space. A *linear form* on a vector space V is a function $\varrho : V \to \mathbb{R}$ which satisfies the linearity condition $\varrho(a\,\vec{v} + b\,\vec{w}) = a\,\varrho(\vec{v}) + b\,\varrho(\vec{w})$, where $a, b \in \mathbb{R}$ and $\vec{v}, \vec{w} \in V$. We will also call a linear form a *1-form*. The set of all 1-forms defined on a vector space V is the vector space if one defines the addition of 1-forms and their multiplication by real numbers as follows

$$(\varrho + \varsigma)(\vec{v}) = \varrho(\vec{v}) + \varsigma(\vec{v}),\ (a\,\varrho)(\vec{v}) = a\,\varrho(\vec{v}). \tag{2.14}$$

The vector space of 1-forms is called a *dual space* to a vector space V and it will be denoted by V^*. It can be proved that the dual space V^* is n-dimensional vector space. If $\{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}$ is a basis for a vector space V then this basis induces the dual basis

$\{e^1, e^2, \ldots, e^n\}$ for the dual vector space and a 1-form e^i of the dual basis is defined by $e^i(\vec{e}_j) = \delta^i_j$. Then any 1-form ϱ can be expressed in terms of the dual basis as follows

$$\varrho = \varrho_i \, e^i, \quad \varrho_i \in \mathbb{R}.$$

Note that in this formula we use the Einstein convention. A *bilinear form* on a vector space V is a function of two vector variables $\vartheta : V \times V \to \mathbb{R}$ which is linear in both variables, i.e.

$$\vartheta(a\,\vec{v}_1 + b\,\vec{v}_2, \vec{w}) = a\,\vartheta(\vec{v}_1, \vec{w}) + b\,\vartheta(\vec{v}_2, \vec{w}),$$

and the analogous property holds for the second variable $\vec{w}$. We will shortly call a bilinear form a 2-form. The set of 2-forms is the vector space if one defines the sum of two 2-forms and the multiplication by real numbers similarly to (2.14). A 2-form ϑ is said to be a skew-symmetric 2-form if for any two vectors $\vec{v}, \vec{w}$ it is satisfies

$$\vartheta(\vec{v}, \vec{w}) = -\vartheta(\vec{w}, \vec{v}). \tag{2.15}$$

Evidently the set of skew-symmetric 2-forms is the subspace of the vector space of all 2-forms and this subspace will be denoted by $\wedge^2 V^*$. Generally a n-form on a vector space V is a function of n vector variables $\varsigma : V \times V \times \ldots \times V \to \mathbb{R}$ which is linear in every variable. The set of n-forms on a vector space V is a vector space if one defines the sum of two n-forms and the multiplication of n-forms by real numbers similarly to (2.14). A l-form is said to be a skew-symmetric if its satisfies

$$\varsigma(\vec{v}_{i_1}, \vec{v}_{i_2}, \ldots, \vec{v}_{i_l}) = (-1)^{\mathtt{p}} \varsigma(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_l), \tag{2.16}$$

where $\mathtt{p}$ is a parity of a permutation $(i_1, i_2, \ldots, i_l)$. Hence a skew symmetric l-form does not change if we perform an even permutation of its vector variables and it changes a sign in the case of odd permutation. The set of skew-symmetric l-forms is the subspace in the vector space of all skew-symmetric l-forms and this subspace will be denoted by $\wedge^l V^*$.

The wedge product of two 1-forms $\varrho_1, \varrho_2 \in V^*$ is a 2-form $\varrho_1 \wedge \varrho_2$ defined by

$$(\varrho_1 \wedge \varrho_2)(\vec{v}, \vec{w}) = \begin{vmatrix} \varrho_1(\vec{v}) & \varrho_1(\vec{w}) \\ \varrho_2(\vec{v}) & \varrho_2(\vec{w}) \end{vmatrix} = \varrho_1(\vec{v})\,\varrho_2(\vec{w}) - \varrho_1(\vec{w})\,\varrho_2(\vec{v}). \tag{2.17}$$

It is easy to see that for any 1-forms $\varrho_1, \varrho_2, \varrho_3$ and real numbers a, b we have the following algebraic properties

1. $V^* \wedge V^* \to \wedge^2 V^*$, i.e. the wedge product of two 1-forms is a skew-symmetric 2-form,
2. $\varrho_1 \wedge \varrho_2 = -\varrho_2 \wedge \varrho_1$, i.e. wedge product is anti-symmetric operation,
3. $(a\,\varrho_1 + b\,\varrho_2) \wedge \varrho_3 = a\,\varrho_1 \wedge \varrho_3 + b\,\varrho_2 \wedge \varrho_3$, i.e. wedge product is bilinear operation.

We can extend the wedge product (2.17) to any k-form ϑ and m-form ς by saying that $\vartheta \wedge \varsigma$ is a l-form, where $l = k + m$, defined by

$$(\vartheta \wedge \varsigma)(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_l) = \sum_{\substack{1 \leq i_1 < \ldots < i_k \leq n \\ 1 \leq j_1 < \ldots < j_m \leq n}} (-1)^{\mathtt{p}}\,\vartheta(\vec{v}_{i_1}, \vec{v}_{i_2}, \ldots, \vec{v}_{i_k}) \varsigma(\vec{v}_{j_1}, \vec{v}_{j_2}, \ldots, \vec{v}_{j_m}), \tag{2.18}$$

where $(i_1, i_2, \ldots, i_k, j_1, j_2, \ldots, j_m)$ is a permutation of the integers $(1, 2, \ldots, l)$ and $\mathtt{p}$ is the parity of this permutation. Particularly if $k = m = 1$ then the general formula (2.18) takes on the form of wedge product of two 1-forms (2.17). It can be verified that the wedge product $\vartheta \wedge \varsigma$ defined in (2.18) is the l-ary skew-symmetric form. The wedge product has the following algebraic properties

1. if $\vartheta \in \wedge^k V^*$, $\varsigma \in \wedge^m V^*$ then $\vartheta \wedge \varsigma \in \wedge^{k+m} V^*$,
2. $(\varrho \wedge \vartheta) \wedge \varsigma = \varrho \wedge (\vartheta \wedge \varsigma)$, (associativity)
3. if $\vartheta \in \wedge^k V^*$, $\varsigma \in \wedge^m V^*$ then $\vartheta \wedge \varsigma = (-1)^{km} \varsigma \wedge \vartheta$,
4. $(a\,\varrho_1 + b\,\varrho_2) \wedge \varrho_3 = a\,\varrho_1 \wedge \varrho_3 + b\,\varrho_2 \wedge \varrho_3$, where a, b are real numbers.

In order to make the notation of vector space of forms uniform, we will denote the dual space V^* as follows $\wedge^1 V^*$. Hence the sum of vector spaces $V^* + \wedge^2 V^* + \ldots + \wedge^l V^* + \ldots$, i.e. the vector space of all formal finite linear combinations of forms with real coefficients, endowed with the wedge product, is the associative algebra. It is useful to make it a unital algebra by adding the vector space $\wedge^0 V^* = \mathbb{R}$. The real numbers $\mathbb{R}$ can be considered as a one-dimensional vector space spanned by the unity 1. Hence $\wedge^0 V^* = \{a\,1 : a \in \mathbb{R}\}$. The elements of the one-dimensional vector space $\wedge^0 V^*$ will be referred to as 0-forms. Hence a 0-form here is simply a real number. Naturally we extend the wedge product to the 0-forms by saying that a wedge product of a 0-form and a k-form is multiplication of a k-form by a real number defined in (2.14). It is worth to mention here that due to (2.14) it is natural to define $\varrho \wedge a = \varrho\, a = a\,\varrho$, that is, forms commute with real numbers. Now the unity 1 is the identity element of the exterior algebra because $1 \wedge \varrho = \varrho \wedge 1 = \varrho$.

Definition 2.3. The unital associative algebra $(\wedge V^*, \wedge)$, where $\wedge V^*$ is the sum of vector spaces $\wedge^0 V^* + V^* + \ldots + \wedge^k V^* + \ldots$ and $\wedge$ is the wedge product, is called an *exterior algebra* of a vector space V.

The vector space of the exterior algebra is the sum of vector spaces of k-forms $\wedge^k V^*$. But this sum is not infinite as it may seem at the first glance. Actually this sum ends at the vector space of n-forms $\wedge^n V^*$ (n is the dimension of a vector space V), that is, all vector spaces $\wedge^l V^*, l > n$ are trivial, i.e. consist of the zero vector. The reason for this can be understood if we consider the structure of vector spaces $\wedge^k V^*$ with the help of a basis.

Let $\{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}$ be a basis for V. Then we have the dual basis of 1-forms $\{e^1, e^2, \ldots, e^n\}$ for the dual vector space V^*. Now if we take into account that wedge product is anti-symmetric, i.e. for any integers i, j it holds $e^i \wedge e^j = -e^j \wedge e^i$ then $e^i \wedge e^i = 0$. Thus if a sequence of integers $i_1, i_2, \ldots, i_k$, where each integer of this sequence can take values from 1 to n, contains two identical integers then, because of anti-symmetry of the wedge product, the product $e^{i_1} \wedge e^{i_2} \wedge \ldots \wedge e^{i_k}$ will be 0. Therefore we will get a non-zero product of basic 1-forms e^i only in the case when all integers in a sequence $i_1, i_2, \ldots, i_k$ are distinct. On the other hand, if we rearrange the integers $i_{\sigma(1)}, i_{\sigma(2)}, \ldots, i_{\sigma(k)}$, where $(\sigma(1), \sigma(2), \ldots, \sigma(k))$ is a permutation of integers $1, 2, \ldots, k$, we get the product, which is expressed as follows

$$e^{i_{\sigma(1)}} \wedge e^{i_{\sigma(2)}} \wedge \ldots \wedge e^{i_{\sigma(k)}} = (-1)^{\mathtt{p}(\sigma)} e^{i_1} \wedge e^{i_2} \wedge \ldots \wedge e^{i_k},$$

where $\mathtt{p}(\sigma)$ is the parity of a permutation σ. These considerations lead to the conclusion that all products $e^{i_1} \wedge e^{i_2} \wedge \ldots \wedge e^{i_k}$, where $1 \leq i_1 < i_2 < \ldots < i_k \leq n$, are independent and can be shown to form a basis for a vector space of k-forms. Thus the dimension of a vector space $\wedge^k V^*$ is C^n_k, i.e. dim $\wedge^k V^* = C^n_k$. Now it becomes clear that the longest non-zero product of basic 1-forms e^i that we can form is the n-form $e^1 \wedge e^2 \wedge \ldots \wedge e^n$ which span the 1-dimensional vector space $\wedge^n V^*$ and every vector space of k-forms, where $k > n$, is trivial, i.e. $\wedge^k V^* = \{0\}$ for $k > n$. Hence

$$\wedge V^* = \wedge^0 V^* + \wedge^1 V^* + \wedge^2 V^* + \ldots + \wedge^n V^*.$$

Any k-form ϱ can be uniquely expressed in terms of wedge products of basic 1-forms e^i as follows

$$\varrho = \sum_{1 \le i_1 < i_2 < \ldots < i_k \le n} \varrho_{i_1 i_2 \ldots i_k}\, e^{i_1} \wedge e^{i_2} \wedge \ldots \wedge e^{i_k}, \tag{2.19}$$

where $\varrho_{i_1 i_2 \ldots i_k}$ are real numbers. This formula can be written in a more compact form. Let $\mathcal{I} = \{i_1, i_2, \ldots, i_k\}$ $(k \le n)$ be a subset of the set of integers from 1 to n, i.e. if we denote $\mathcal{N} = \{1, 2, \ldots, n\}$ then $\mathcal{I} \subset \mathcal{N}$. Recall that the concept of a set does not imply ordering of elements. Hence we can choose natural ordering $1 \le i_1 < i_2 < \ldots < i_k \le n$. Now we associate to every subset $\mathcal{I} \subset \mathcal{N}$ the basic k-form as follows

$$\mathcal{I} \subset \mathcal{N} \;\; \rightarrow \;\; e^{\mathcal{I}} = e^{i_1} \wedge e^{i_2} \wedge \ldots \wedge e^{i_k}. \tag{2.20}$$

We assume that a subset $\mathcal{I}$ may be empty. In this case we associate to the empty set the identity element of the exterior algebra $\wedge V^*$, that is, $e^{\emptyset} = 1$. Now formula (2.19) can be written in a compact form as follows

$$\varrho = \sum_{\mathcal{I} \subset \mathcal{N}} \varrho_{\mathcal{I}}\, e^{\mathcal{I}}. \tag{2.21}$$

There is one more way to express a k-form in terms of the basic k-forms and in this case we do not need to use ordered sequences of indexes, which makes calculations with forms in some cases easier. If we assume that the coefficients $\varrho_{i_1 i_2 \ldots i_k}$ in the sum (2.19) are skew-symmetric in subscripts, that is,

$$\varrho_{i_{\sigma(1)} i_{\sigma(2)} \ldots i_{\sigma(k)}} = (-1)^{\mathtt{p}(\sigma)}\, \varrho_{i_1, i_2, \ldots, i_k},$$

where σ is a permutation and $\mathtt{p}(\sigma)$ is its parity, and we take the sum over all sequences $i_1, i_2, \ldots, i_k$ then due to the anti-symmetry of the wedge product we will get every term $k!$ times. Thus we can express any k-form as follows

$$\varrho = \frac{1}{k!}\, \varrho_{i_1 i_2 \ldots i_k}\, e^{i_1} \wedge e^{i_2} \wedge \ldots \wedge e^{i_k}. \tag{2.22}$$

In the above formula we can use the Einstein convention because now every subscript independently of others takes values from 1 to n.

Let us consider the question of transformation of the coefficients $\varrho_{i_1 i_2 \ldots i_k}$ of a k-form when one passes from one basis of a vector space V to another. Let $\{\vec{e}_1', \vec{e}_2', \ldots, \vec{e}_n'\}$ be another basis for a vector space V and $\vec{e}_i' = A_i^j\, \vec{e}_j$, where $A = (A_i^j)$ is a transition matrix. Let $\{\tilde{e}^1, \tilde{e}^2, \ldots, \tilde{e}^n\}$ be the dual basis for a basis $\{\vec{e}_1', \vec{e}_1', \ldots, \vec{e}_1'\}$. Then $e^i = A_j^i\, \tilde{e}^j$.

In order to find a transformation law for the coefficients of a k-form ρ it is useful to consider the determinant of a submatrix of a transition matrix A. Let $1 \le i_1 < i_2 < \ldots < i_k \le n$ be numbers of rows and $1 \le j_1 < j_2 < \ldots < j_k \le n$ be numbers of columns of a matrix A. Then the elements of a transition matrix A located at the intersections of these rows and columns form the submatrix of matrix A. Let us denote this submatrix by $A^{(i_1 i_2 \ldots i_k)}_{(j_1 j_2 \ldots j_k)}$. It should be noted that here we mean the following. The elements of the first row of the submatrix are the elements of a matrix A located at the intersection of the row i_1 with columns $j_1, j_2, \ldots, j_k$, the elements of the second row of the submatrix are the elements of a matrix A located at the intersection of the row i_2 with columns $j_1, j_2, \ldots, j_k$ and so on. From the properties of a determinant it follows that Det $\big(A^{(i_1 i_2 \ldots i_k)}_{(j_1 j_2 \ldots j_k)}\big)$ is skew-symmetric in superscripts as well as in subscripts. Indeed if we do a permutation of the superscripts $i_1, i_2, \ldots, i_k$ then this will be equivalent to a permutation of the rows in the determinant

$\operatorname{Det}\big(A^{(i_1 i_2 \ldots i_k)}_{(j_1 j_2 \ldots j_k)}\big)$. If we do a permutation of the subscripts $j_1, j_2, \ldots, j_k$ then this will be equivalent to a permutation of the columns in determinant $\operatorname{Det}\big(A^{(i_1 i_2 \ldots i_k)}_{(j_1 j_2 \ldots j_k)}\big)$. Hence

$$\begin{aligned}
\operatorname{Det}\big(A^{(i_{\sigma(1)} i_{\sigma(2)} \ldots i_{\sigma(k)})}_{(j_1 j_2 \ldots j_k)}\big) &= (-1)^{\mathtt{p}} \operatorname{Det}\big(A^{(i_1 i_2 \ldots i_k)}_{(j_1 j_2 \ldots j_k)}\big),\\
\operatorname{Det}\big(A^{((i_1 i_2 \ldots i_k)}_{(j_{\sigma(1)} j_{\sigma(2)} \ldots j_{\sigma(k)})}\big) &= (-1)^{\mathtt{p}} \operatorname{Det}\big(A^{(i_1 i_2 \ldots i_k)}_{(j_1 j_2 \ldots j_k)}\big),
\end{aligned}$$

where $\mathtt{p}$ is the parity of a permutation σ.

Let ϱ be a k-form. We can write it either in the coordinates of the first basis $\{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}$ or in the coordinates of the second basis $\{\vec{e}'_1, \vec{e}'_2, \ldots, \vec{e}'_n\}$, that is,

$$\varrho = \frac{1}{k!}\, \varrho_{i_1 i_2 \ldots i_k}\, e^{i_1} \wedge e^{i_2} \wedge \ldots \wedge e^{i_k}, \quad \varrho = \frac{1}{k!}\, \tilde{\varrho}_{i_1 i_2 \ldots i_k}\, \tilde{e}^{i_1} \wedge \tilde{e}^{i_2} \wedge \ldots \wedge \tilde{e}^{i_k}. \tag{2.23}$$

Making use of $e^i = A^i_j\, \tilde{e}^j$ we find for $1 \le j_1 < j_2 < \ldots < j_k \le n$

$$\tilde{\varrho}_{j_1 j_2 \ldots j_k} = \sum_{1 \le i_1 < i_2 < \ldots < i_k \le n} \operatorname{Det}\big(A^{(i_1 i_2 \ldots i_k)}_{(j_1 j_2 \ldots j_k)}\big)\, \varrho_{i_1 i_2 \ldots i_k}. \tag{2.24}$$

Taking into account the skew-symmetry of a determinant $\operatorname{Det}(A^{(i_1 i_2 \ldots i_k)}_{(j_1 j_2 \ldots j_k)})$ both with respect to the superscripts and subscripts we can drop the condition for the ordering of indices and then

$$\tilde{\varrho}_{j_1 j_2 \ldots j_k} = \frac{1}{k!} \operatorname{Det}\big(A^{(i_1 i_2 \ldots i_k)}_{(j_1 j_2 \ldots j_k)}\big)\, \varrho_{i_1 i_2 \ldots i_k}. \tag{2.25}$$

Definition 2.4. Assume that in each coordinate system $\{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}$ of a vector space V we are given a set of real numbers $\{\varrho_{i_1 i_2 \ldots i_k}\}$ indexed by the set of subscripts $i_1 i_2 \ldots i_k$, where each subscript runs from 1 to n, and $\varrho_{i_1 i_2 \ldots i_k}$ are skew-symmetric under a permutation of subscripts. If $\{\varrho_{i_1 i_2 \ldots i_k}\}$ transform under transition from one coordinate system to another according to the law (2.25) then in a vector space V we have the *covariant skew-symmetric tensor of kth order* ϱ and $\{\varrho_{i_1 i_2 \ldots i_k}\}$ are components of this tensor in a coordinate system $\{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}$.

2.4 Differential forms

In this section the exterior algebra of a vector space described in the previous section will be used to construct differential forms in an affine Euclidean space E^n.

At each point p of an affine Euclidean space E^n we have the tangent space T_pE^n. Let us denote by $T^*_pE^n$ the dual space of a tangent space T_pE^n. This dual space is a vector space of dimension n and will be called a *cotangent space* at a point p. Hence we can consider the exterior algebra $\wedge T^*_pE^n$. According to the considerations of the previous section the exterior algebra $\wedge T^*_pE^n$ is the sum of vector spaces

$$\wedge T^*_pE^n = \wedge^0 T^*_pE^n + \wedge^1 T^*_pE^n + \ldots + \wedge^n T^*_pE^n,$$

where $\wedge^k T^*_pE^n$ is a vector space of k-forms defined on a tangent space T_pE^n and $\wedge^1 T^*_pE^n = T^*_pE^n$. Let $U \subset E^n$ be an open subset in E^n. In analogy with a tangent space we define $\wedge^k T^*(U) = \cup_{p \in U} \wedge^k T^*_pE^n, \wedge T^*(U) = \cup_{p \in U} \wedge T^*_pE^n$, where $\cup$ is a disjoint union of vector spaces. The projection $\pi : \wedge^k T^*(U) \to U$ is defined in a natural way, i.e. if ϑ is a k-form defined on a tangent space T_pE^n then $\pi(\vartheta) = p$.

Definition 2.5. A smooth mapping $\omega : U \to \wedge^k T^*(U)$, which satisfies $\pi \circ \omega = \mathrm{id}_U$, is called a *differential k-form* defined on an open subset U.

Hence a differential k-form attributes to each point of an open subset of an affine Euclidean space a k-form and this k-form depends smoothly on a point of U. A vector space of differential k-forms defined on an open subset $U \subset E^n$ will be denoted by $\Omega^k(U)$. Here the vector space structure of $\Omega^k(U)$ is defined by pointwise addition of differential k-forms and by pointwise multiplication by real numbers, that is, $(\omega + \theta)_p = \omega_p + \theta_p$ and $(a\,\omega)_p = a\,\omega_p$, where $\omega, \theta \in \Omega^k(U), a \in \mathbb{R}$. As in the case of vector fields, we can define one more operation with differential k-forms and this is multiplication of a differential k-form ω by a smooth function $f \in C^\infty(U)$ defined pointwise $(f\,\omega)_p = f(p)\,\omega_p$. Usually we will multiply a differential k-form by a function f from the left and write $f\,\omega$, but it is natural to define $\omega\, f = f\,\omega$, that is, differential forms commute with smooth functions.

The multiplication of differentials k-forms by smooth functions has the following properties

- $(f + g)\,\omega = f\,\omega + g\,\omega$,
- $f\,(\omega + \theta) = f\,\omega + g\,\theta$,
- $(f\,g)\,\omega = f\,(g\,\omega)$,
- $\mathbf{1}\,\omega = \omega$, where $\mathbf{1}$ is the constant function whose value at each point p is 1.

Hence the multiplication of differential k-forms by smooth functions defines the structure of *module over the ring of smooth functions* on $\Omega^k(U)$.

At each point p of an affine space E^n the tangent space T_pE^n is a vector space of dimension n. Therefore, according to the previous section, at every point p one can construct the exterior algebra $\wedge T_p^*E^n$. This algebra can be extended to the vector space of differential forms if we define a wedge product of two differential forms pointwise. In order to do this we consider the sum of vector spaces $\Omega^k(U)$ whose elements are all possible linear combinations of differential forms with real coefficients, i.e. $\sum_{i=1}^n a_i\,\omega^{(i)}$, where $\omega^{(i)} \in \Omega^i(U), a_i \in \mathbb{R}$. We denote this sum of vector spaces by $\Omega(U)$, that is,

$$\Omega(U) = \Omega^0(U) + \Omega^1(U) + \ldots + \Omega^n(U),$$

and $\Omega^0(U)$ is the vector space of smooth functions $C^\infty(U)$. It is clear that the multiplication of differential k-forms by smooth functions induces the structure of module on $\Omega(U)$ if we put $f\,\sum_{i=1}^n \omega^{(i)} = \sum_{i=1}^n f\omega^{(i)}$, where $\omega^{(i)} \in \Omega^i(U)$. Now we define the wedge product of any two differential forms pointwise $(\omega \wedge \theta)_p = \omega_p \wedge \theta_p$ and extend it in an obvious way to $\Omega(U)$. Note that the wedge product of a zero-form (function) and a differential k-form ($k > 0$) is the multiplication of a differential k-form by a function. Evidently the constant function $\mathbf{1}$, whose value at any point of an affine space E^n is the unity 1, plays a role of identity element for the wedge product.

Hence $(\Omega(U), \wedge)$ is a unital associative algebra which has the following properties

- if $\omega \in \Omega^k(U), \theta \in \Omega^m(U)$ then $\omega \wedge \theta \in \Omega^{k+m}(U)$,
- $(\omega \wedge \theta) \wedge \rho = \omega \wedge (\theta \wedge \rho)$ (associativity),
- $\omega \wedge \theta = (-1)^{km}\theta \wedge \omega$, where $\omega \in \Omega^k(U), \theta \in \Omega^m(U)$,
- $(f\,\omega + g\,\theta) \wedge \rho = f\,\omega \wedge \rho + g\,\omega \wedge \rho$, where f, g are smooth functions.

The algebra $(\Omega(U), \wedge)$ will be referred to as an *algebra of differential forms defined on an open subset* U.

Definition 2.6. A unital associative algebra $(\mathcal{A}, \cdot)$ is said to be a $\mathbb{Z}$-graded algebra if it is a direct sum of subspaces $\mathcal{A}^i \subset \mathcal{A}$ labelled by integers $i \in \mathbb{Z}$, that is, $\mathcal{A} = \oplus_{i\in\mathbb{Z}} \mathcal{A}^i$ and for any $k, m \in \mathbb{Z}, u \in \mathcal{A}^k, v \in \mathcal{A}^m$ it holds $u \cdot v \in \mathcal{A}^{k+m}$. If for any $k, m \in \mathbb{Z}, u \in \mathcal{A}^k, v \in \mathcal{A}^m$ a multiplication of a graded algebra $\mathcal{A}$ satisfies $u \cdot v = (-1)^{km} v \cdot u$ then graded algebra $\mathcal{A}$ is referred to as a graded commutative algebra.

It follows from this definition that the algebra of differential forms $(\Omega(U), \wedge)$ defined on an open subset U of an affine space E^n is a $\mathbb{Z}$-graded commutative algebra whose subspaces $\Omega^k(U)$ are trivial for $k < 0$ and $k > n$.

Let $\psi : U \subset E^n \to W \subset \mathbb{R}^n$ be a system of coordinates $(x^1, x^2, \ldots, x^n)$ (not necessarily affine coordinates) C^∞-compatible with affine coordinates. Then the vectors at a point $p \in U$

$$\frac{\partial}{\partial x^1}\Big|_p, \frac{\partial}{\partial x^2}\Big|_p, \ldots, \frac{\partial}{\partial x^n}\Big|_p \tag{2.26}$$

form the basis for the tangent space T_pE^n. The dual basis for the vector space $T_p^*E^n$ (or $\wedge^1 T_p^*E^n$) will be denoted by

$$(dx^1)_p, (dx^2)_p, \ldots, (dx^n)_p. \tag{2.27}$$

Hence we have

$$(dx^i)_p \Big(\frac{\partial}{\partial x^j}\Big|_p\Big) = \delta^i_j.$$

Let ω be a differential k-form and $\omega_p, p \in U$ be its value at a point p. According to Definition (2.5) $\omega_p \in \wedge^k T_p^*E^n$. Formula (2.22) shows that we can write a differential k-form ω at a point p as follows

$$\omega_p = \frac{1}{k!}\, \omega_{i_1 i_2 \ldots i_k}(p)\, (dx^{i_1})_p \wedge (dx^{i_2})_p \wedge \ldots \wedge (dx^{i_k})_p, \tag{2.28}$$

Since p is an arbitrary point of U we can omit point p in formula (2.28) and then we will obtain a formula for a differential k-form ω in coordinates $x^1, x^2, \ldots, x^n$. In this case the numbers $\omega_{i_1 i_2 \ldots i_k}(p)$ in (2.28) become functions $\omega_{i_1 i_2 \ldots i_k}$ of the coordinates $x^1, x^2, \ldots, x^n$ of a point. The requirement of smoothness for a differential form considered as a mapping $U \to \wedge T^*(U)$ now means that all functions $\omega_{i_1 i_2 \ldots i_k}$ are smooth, i.e. $\omega_{i_1 i_2 \ldots i_k} \in C^\infty(U)$. Hence a differential k-form can be written in coordinates $x^1, x^2, \ldots, x^n$ as the following expression

$$\omega = \frac{1}{k!}\, \omega_{i_1 i_2 \ldots i_k}(x)\, dx^{i_1} \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_k}, \tag{2.29}$$

where x stands for the set of coordinates $x^1, x^2, \ldots, x^n$ of a variable point of U.

At each point p of an open subset $U \subset E^n$ a differential k-form ω is a k-form ω_p, that is, a real-valued multilinear skew-symmetric function of k vectors. On the other hand if we are given k vector fields $\mathtt{X}_1, \mathtt{X}_2, \ldots, \mathtt{X}_k \in \mathcal{D}(U)$ then at each point $p \in U$ we have k tangent vectors $(\mathtt{X}_1)_p, (\mathtt{X}_2)_p, \ldots, (\mathtt{X}_k)_p$. Thus at each point p we can calculate the value of a k-form ω_p on k tangent vectors and this is a real number. Hence we have the function

$$p \in U \to \omega_p\big((\mathtt{X}_1)_p, (\mathtt{X}_2)_p, \ldots, (\mathtt{X}_k)_p\big).$$

This function is called the value of a differential k-form ω on vector fields $\mathtt{X}_1, \mathtt{X}_2, \ldots, \mathtt{X}_k$ and we will denote this function by $\omega(\mathtt{X}_1, \mathtt{X}_2, \ldots, \mathtt{X}_k)$.

Obviously the value of a differential k-form on k vector fields is skew-symmetric under a permutation of vector fields, that is,

$$\omega(\mathtt{X}_{\sigma(1)}, \mathtt{X}_{\sigma(2)}, \ldots, \mathtt{X}_{\sigma(k)}) = (-1)^{\mathtt{p}}\, \omega(\mathtt{X}_1, \mathtt{X}_2, \ldots, \mathtt{X}_k),$$

where σ is a permutation of integers $\{1, 2, \ldots, k\}$ and $\mathtt{p}$ is the parity of σ. Due to the multilinearity of a k-form we have one more property

$$\begin{aligned}\omega(\mathtt{X}_1, \mathtt{X}_2, \ldots, \mathtt{X}_i + f\,\mathtt{Y}, \ldots, \mathtt{X}_k) &= \omega(\mathtt{X}_1, \mathtt{X}_2, \ldots, \mathtt{X}_{i-1}, \mathtt{X}_i, \mathtt{X}_{i+1}, \ldots, \mathtt{X}_k) \\ &\quad + f\,\omega(\mathtt{X}_1, \mathtt{X}_2, \ldots, \mathtt{X}_{i-1}, \mathtt{Y}, \mathtt{X}_{i+1}, \ldots, \mathtt{X}_k), \end{aligned} \tag{2.30}$$

where f is a smooth function defined on U and $1 \leq i \leq k$. If a differential k-form and vector fields are written in coordinates as follows

$$\omega = \frac{1}{k!}\,\omega_{i_1 i_2 \ldots i_k}(x)\, dx^{i_1} \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_k},\ \mathtt{X}_1 = X^i_1 \frac{\partial}{\partial x^i},\ \mathtt{X}_2 = X^i_2 \frac{\partial}{\partial x^i}, \ldots, \mathtt{X}_k = X^i_k \frac{\partial}{\partial x^i}$$

then the value of differential k-form ω on vector fields $\mathtt{X}_1, \mathtt{X}_2, \ldots, \mathtt{X}_k$ can be written as follows

$$\omega(\mathtt{X}_1, \mathtt{X}_2, \ldots, \mathtt{X}_k) = \frac{1}{k!}\,\omega_{i_1 i_2 \ldots i_k}(x) \begin{vmatrix} X^{i_1}_1 & X^{i_1}_2 & \ldots & X^{i_1}_k \\ X^{i_2}_1 & X^{i_2}_2 & \ldots & X^{i_2}_k \\ \ldots & \ldots & \ldots & \ldots \\ X^{i_k}_1 & X^{i_k}_2 & \ldots & X^{i_k}_k \end{vmatrix}. \tag{2.31}$$

Given a differential k-form ω one can find the expression of this form in coordinates $x^1, x^2, \ldots, x^n$ by means of the basic vector fields $\mathtt{D}_i = \partial/\partial x^i$, where $i = 1, 2, \ldots, n$, as follows

$$\omega_{i_1 i_2 \ldots i_k} = \omega(\mathtt{D}_{i_1}, \mathtt{D}_{i_2}, \ldots, \mathtt{D}_{i_k}) = \omega\left(\frac{\partial}{\partial x^{i_1}}, \frac{\partial}{\partial x^{i_2}}, \ldots, \frac{\partial}{\partial x^{i_k}}\right). \tag{2.32}$$

An important structure in the case of differential forms is an exterior differential. It is an exterior differential that underlies the theory of de Rham's cohomologies, which is used to study the global structure of a manifold. We will define an exterior differential step by step and start with 0-forms, i.e. functions. It is worth to mention here that any differential form is uniquely defined if we show how to calculate its value on arbitrary vector fields (the number of vector fields depend on the degree of a differential form). Let f be a smooth function defined on an open subset $U \subset E^n$. The exterior differential of a function f is the 1-form df defined by

$$df(\mathtt{X}) = \mathtt{X} \triangleright f. \tag{2.33}$$

It follows from this definition that in the case of a 0-form the exterior differential d raises the degree of a form by one, i.e. $d : \Omega^0(U) \to \Omega^1(U)$. It also follows from (2.33) that d is a linear mapping. In order to find the expression of the 1-form df in coordinates we calculate the value of this form on basic vector fields $\mathtt{D}_i$. We obtain

$$df(\mathtt{D}_i) = \mathtt{D}_i \triangleright f = \frac{\partial f}{\partial x^i}.$$

Hence

$$df = \frac{\partial f}{\partial x^1}\, dx^1 + \frac{\partial f}{\partial x^2}\, dx^2 + \ldots + \frac{\partial f}{\partial x^n}\, dx^n = \frac{\partial f}{\partial x^i}\, dx^i. \tag{2.34}$$

The expression on the right-hand side of the obtained formula is called in the differential calculus of functions the total differential of a function. Thus at the level of functions we can identify the exterior differential with the total differential and this leads to an important conclusion that d satisfies the Leibniz rule

$$d(f\,h) = (df)\,h + f\,(dh). \tag{2.35}$$

It should be noted that a difference between the exterior differential and the total differential appears when we pass from first order differential to higher-order differentials. Formula (2.34) also explains why the dual basis of a cotangent space $T^*_pE^n$ was denoted as differentials of coordinates $(dx^1)_p, (dx^2)_p, \ldots, (dx^n)_p$. Indeed the coordinates can be considered as functions on space (coordinate functions). Then according to exterior calculus the exterior differential of a coordinate function at some point in space is an element of a cotangent space. However it is worth remembering that in the calculus of differential forms, in contrast to the differential calculus of functions, the differentials at point p are generators of the exterior algebra $\wedge T^*_pE^n$, that is, we can multiply them using exterior multiplication.

Now let ω be a 1-form defined on U. The exterior differential of 1-form ω is the 2-form $d\omega$ defined by

$$d\omega(\mathtt{X},\mathtt{Y}) = \mathtt{X} \rhd \big(\omega(\mathtt{Y})\big) - \mathtt{Y} \rhd \big(\omega(\mathtt{X})\big) - \omega([\mathtt{X},\mathtt{Y}]). \tag{2.36}$$

It follows from this definition that, just as in the case of functions, the exterior differential raises the degree of a 1-form by one, that is, it assigns a 2-form to a 1-form. The linearity of the exterior differential in the case of 1-forms also easily follows from Formula (**??**). Hence in this case d is a linear mapping $d : \Omega^1(U) \to \Omega^2(U)$. Thus we have the sequence of vector spaces

$$\Omega^0(U) \xrightarrow{d} \Omega^1(U) \xrightarrow{d} \Omega^2(U). \tag{2.37}$$

The exterior differential has a very important property $d^2 = d \circ d = 0$. Indeed let f be a smooth function and $\mathtt{X},\mathtt{Y}$ be two vector fields defined on U. Then

$$\begin{aligned} d^2 f(\mathtt{X},\mathtt{Y}) &= \big(d\,(df)\big)(\mathtt{X},\mathtt{Y}) = \mathtt{X} \rhd \big(df(\mathtt{Y})\big) - \mathtt{Y} \rhd \big(df(\mathtt{X})\big) - df([\mathtt{X},\mathtt{Y}]) \\ &= \mathtt{X} \rhd (\mathtt{Y} \rhd f) - \mathtt{Y} \rhd (\mathtt{X} \rhd f) - [\mathtt{X},\mathtt{Y}] \rhd f = [\mathtt{X},\mathtt{Y}] \rhd f - [\mathtt{X},\mathtt{Y}] \rhd f = 0. \end{aligned}$$

Any 1-form ω can be expressed in coordinates $x^1, x^2, \ldots, x^n$ as follows $\omega = \omega_i\, dx^i$. Now our aim is to find the expression for the 2-form $d\omega$ in coordinates $x^1, x^2, \ldots, x^n$. First of all we know that any 2-form can be written as follows $d\omega = \frac{1}{2}\omega_{ij} dx^i \wedge dx^j$. In order to find the coefficient functions ω_{ij} we use $\omega_{ij} = \omega(\mathtt{D}_i, \mathtt{D}_j)$. Hence

$$\omega_{ij} = d\omega(\mathtt{D}_i,\mathtt{D}_j) = \mathtt{D}_i \rhd \big(\omega(\mathtt{D}_j)\big) - \mathtt{D}_j \rhd \big(\omega(\mathtt{D}_i)\big) - \omega([\mathtt{D}_i,\mathtt{D}_j]).$$

Next we have $\omega(\mathtt{D}_i) = \omega_i, \omega(\mathtt{D}_j) = \omega_j$ and $[\mathtt{D}_i,\mathtt{D}_j] = [\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}] = 0$ because partial derivatives commute. We obtain

$$\omega_{ij} = \mathtt{D}_i \rhd \omega_j - \mathtt{D}_j \rhd \omega_i = \frac{\partial \omega_j}{\partial x^i} - \frac{\partial \omega_i}{\partial x^j},$$

and

$$\omega = \omega_i\, dx^i \ \to d\omega = \frac{1}{2!}\Big(\frac{\partial \omega_j}{\partial x^i} - \frac{\partial \omega_i}{\partial x^j}\Big)\, dx^i \wedge dx^j.$$

Now we extend the exterior differential to any k-form ω by the following definition.

Definition 2.7. Let ω be a k-form and $\mathtt{X}_1, \mathtt{X}_2, \ldots, \mathtt{X}_{k+1}$ be vector fields. Then the exterior differential $d\omega$ is the $(k+1)$-form defined by

$$\begin{aligned} d\omega(\mathtt{X}_1,\mathtt{X}_1,\ldots,\mathtt{X}_{k+1}) = {} & \sum_i (-1)^{i-1}\mathtt{X}_i \rhd \big(\omega\,(\mathtt{X}_1,\ldots,\hat{\mathtt{X}}_i,\ldots,X_{k+1})\big) \\ & + \sum_{i<j} (-1)^{i+j}\,\omega([\mathtt{X}_i,\mathtt{X}_j],\mathtt{X}_1,\ldots,\hat{\mathtt{X}}_i,\ldots,\hat{\mathtt{X}}_j,\ldots,\mathtt{X}_{k+1}), \end{aligned}$$

where a hat $\hat{}$ over a vector field $\mathtt{X}_i$ means that this vector field is omitted.

Theorem 2.1. The exterior differential has the following properties:

1. the exterior differential is a linear mapping $d : \Omega^i(U) \to \Omega^{i+1}(U)$,

2. for any differential k-form ω and any differential form θ it holds
$$d(\omega \wedge \theta) = (d\omega) \wedge \theta + (-1)^k \,\omega \wedge (d\theta),$$

3. $d^2 = d \circ d = 0$.

If $\omega = \frac{1}{k!}\omega_{i_1 i_2 \ldots i_k}\, dx^{i_1} \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_k}$ then

$$d\omega = \sum_{j_1 < \ldots < j_{k+1}} (-1)^{m-1} \frac{\partial \omega_{j_1 \ldots \hat{j}_m \ldots j_{k+1}}}{\partial x^{j_m}}\, dx^{j_1} \wedge \ldots \wedge dx^{j_{k+1}}, \tag{2.38}$$

where $\hat{j}_m$ means that j_m is omitted.

Proof. The linearity of the exterior differential follows from the definition of the sum of two differential k-forms and the multiplication of a differential form by a real number. Indeed if $\omega, \theta \in \Omega^k(U)$ and $a \in \mathbb{R}$ then, by definition,

$$\begin{aligned} (\omega + \theta)(\mathtt{X}_1, \mathtt{X}_2, \ldots, \mathtt{X}_k) &= \omega(\mathtt{X}_1, \mathtt{X}_2, \ldots, \mathtt{X}_k) + \theta(\mathtt{X}_1, \mathtt{X}_2, \ldots, \mathtt{X}_k), \\ (a\,\omega)(\mathtt{X}_1, \mathtt{X}_2, \ldots, \mathtt{X}_k) &= a\,\big(\omega(\mathtt{X}_1, \mathtt{X}_2, \ldots, \mathtt{X}_k)\big). \end{aligned} \tag{2.39}$$

Applying Definition (2.7) and using the linearity of directional derivative we obtain $d(\omega+\theta) = d\omega + d\theta$ and $d(a\,\omega) = a\,d\omega$.

In order to prove the second property we will use coordinates $x^1, x^2, \ldots, x^n$. It can be easily checked that the proposed proof can be applied in any coordinate system. Hence the second property takes places in any coordinate system. First of all let us note that due to the linearity of the exterior differential we need to prove the second property only in the case of differential forms

$$\omega = f\, dx^{i_1} \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_k}, \quad \theta = g\, dx^{j_1} \wedge dx^{j_2} \wedge \ldots \wedge dx^{j_l}, \quad f, g \in C^\infty(U) \tag{2.40}$$

because any differential form can be written as the sum of this kind of differential forms. Second it is sufficient to prove the second property only in the case of basic vector fields $\mathtt{D}_i$ because any vector field can be written as the sum of basic vector fields multiplied by functions and a differential form has the property (2.30). First we prove that the exterior differential of a k-form ω, where ω is the first form in (2.40), can be written in coordinates as follows
$$d\omega = df \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_k}.$$

Let $\mathtt{D}_1, \mathtt{D}_2, \ldots, \mathtt{D}_{k+1}$ be basic vector fields. Making use of Definition (2.7) we can write

$$d\omega(\mathtt{D}_1, \mathtt{D}_2, \ldots, \mathtt{D}_{k+1}) = \sum_{m=1}^{k+1} (-1)^{m-1} \mathtt{D}_m \triangleright \big(\omega(\mathtt{D}_1, \ldots, \hat{\mathtt{D}}_m, \ldots, \mathtt{D}_{k+1})\big). \tag{2.41}$$

The second sum in the formula of Definition (2.7) vanishes because partial derivatives commute $[\mathtt{D}_i, \mathtt{D}_j] = 0$. Next

$$\omega(\mathtt{D}_1, \ldots, \hat{\mathtt{D}}_m, \ldots, \mathtt{D}_{k+1}) = f\, dx^{i_1} \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_k}(\mathtt{D}_1, \ldots, \hat{\mathtt{D}}_m, \ldots, \mathtt{D}_{k+1})$$

The value of a k-form $dx^{i_1} \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_k}$ on basic vector fields $\mathtt{D}_i, i \neq m$ is a number since according to the definition of the wedge product this value will be equal to the determinant whose elements are numbers $dx^{i_r}(\mathtt{D}_s) = \delta_s^{i_r}$, where $s \neq m$. Thus we have

$$\begin{aligned} d\omega(\mathtt{D}_1, \mathtt{D}_2, \ldots, \mathtt{D}_{k+1}) &= \sum_{m=1}^{k+1} (-1)^{m-1} (\mathtt{D}_m \triangleright f)\, dx^{i_1} \wedge \ldots \wedge dx^{i_k}(\mathtt{D}_1, \ldots, \hat{\mathtt{D}}_m, \ldots, \mathtt{D}_{k+1}) \\ &= \sum_{m=1}^{k+1} (-1)^{m-1} df(\mathtt{D}_m)\, dx^{i_1} \wedge \ldots \wedge dx^{i_k}(\mathtt{D}_1, \ldots, \hat{\mathtt{D}}_m, \ldots, \mathtt{D}_{k+1}) \\ &= df \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_k}(\mathtt{D}_1, \ldots, \mathtt{D}_m, \ldots, \mathtt{D}_{k+1}) \end{aligned}$$

Now we consider the exterior differential of the wedge product of two forms (2.40). Making use of (2.35) we find

$$\begin{aligned} d(\omega \wedge \theta) &= d(f\,h\,dx^{i_1} \wedge \ldots \wedge dx^{i_k} \wedge dx^{j_1} \wedge \ldots \wedge dx^{j_l}) \\ &= d(f\,h) \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_k} \wedge dx^{j_1} \wedge \ldots \wedge dx^{j_l} \\ &= \big(d(f)\,h + f\,d(h)\big) \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_k} \wedge dx^{j_1} \wedge \ldots \wedge dx^{j_l} \\ &= (df \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_k}) \wedge (h\,dx^{j_1} \wedge \ldots \wedge dx^{j_l}) \\ &\qquad + (-1)^k\,(f\,dx^{i_1} \wedge \ldots \wedge dx^{i_k}) \wedge (dh \wedge dx^{j_1} \wedge \ldots \wedge dx^{j_l}) \\ &= d\omega \wedge \theta + (-1)^k\,\omega \wedge d\theta. \end{aligned}$$

Now it is easy to prove that $d^2 = d \circ d = 0$. Previously we proved that this property holds in the case of functions. From previous considerations it follows that it is sufficient to prove $d^2 = 0$ only in the case of forms (2.40). We have

$$\begin{aligned} d^2(\omega) = d(df \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_k}) = d^2(f) \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_k} - df \wedge d^2x^{i_1} \wedge \ldots \wedge dx^{i_k} \\ + \ldots + (-1)^k\, df \wedge dx^{i_1} \wedge \ldots \wedge d^2x^{i_k} = 0. \end{aligned}$$

Inn order to prove formula (2.38) we use the properties of the exterior differential. Hence we have

$$\begin{aligned} d\omega &= d\Big(\sum_{i_1<\ldots<i_k} \omega_{i_1\ldots i_k} dx^{i_1} \wedge \ldots \wedge dx^{i_k} \Big) \\ &= \sum_{i_1<\ldots<i_k} d\omega_{i_1\ldots i_k} \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_k} \\ &= \sum_{i_1<\ldots<i_k} \frac{\partial \omega_{i_1\ldots i_k}}{\partial x^m}\, dx^m \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_k}. \end{aligned} \tag{2.42}$$

Since $d\omega$ is the differential $(k+1)$-form we can write it as follows

$$d\omega = \sum_{j_1<\ldots<j_{k+1}} \omega_{j_1\ldots j_{k+1}} dx^{j_1} \wedge \ldots \wedge dx^{j_{k+1}}. \tag{2.43}$$

Now let us consider the question of how the wedge products of the basic forms will be obtained from formula (2.42). For this purpose we choose some product $dx^{j_1} \wedge \ldots \wedge dx^{j_{k+1}}$. This product will be obtained from products $dx^{i_1} \wedge \ldots \wedge dx^{i_k}$ by multiplying them from the left by dx^m. To find those products $dx^{i_1} \wedge \ldots \wedge dx^{i_k}$ which will give $dx^{j_1} \wedge \ldots \wedge dx^{j_{k+1}}$ we remove successively 1-forms $dx^{j_1}, dx^{j_2}, \ldots, dx^{j_{k+1}}$ from the product $dx^{j_1} \wedge \ldots \wedge dx^{j_{k+1}}$. It is obvious that in this way we obtain all the terms that give the form $dx^{j_1} \wedge \ldots \wedge dx^{j_{k+1}}$.

But since the 1-forms in the product $dx^{j_1} \wedge \ldots \wedge dx^{j_{k+1}}$ are ordered we will have to put the differential 1-form dx^m (standing at the first position) to its proper place. Thus there will appear a sign which depends on a position of a corresponding 1-form. Thus we get

$$d\omega = \sum_{j_1 < \ldots < j_{k+1}} \sum_{m=1}^{k+1} (-1)^{m-1} \frac{\partial \omega_{j_1 \ldots \hat{j}_m \ldots j_{k+1}}}{\partial x^{j_m}} \, dx^{j_1} \wedge \ldots \wedge dx^{j_{k+1}}. \tag{2.44}$$

□

2.5 Covariant derivative

In this section, we will consider the question of how to measure the rate of change of one vector field in the direction determined by another vector field. Assume that $U \subset E^n$ is an open subset, $\mathtt{Y}$ is a vector field defined on U and $\xi : I \to U$ is a parametrized curve. If we restrict a vector field $\mathtt{Y}$ to a parametrized curve ξ then we get the vector field $\mathtt{Y}|_\xi$ along a curve ξ, i.e. a vector field depending on one parameter t. In this case the measure of change of a vector field along a curve ξ is its derivative $(\mathtt{Y}|_\xi)'$ with respect to the parameter t. Note that the derivative $(\mathtt{Y}|_\xi)'$ is a new vector field along a parametrized curve ξ.

Now assume that $p \in U$, $\mathtt{v} = (p; \vec{v}) \in T_pE^n$. Let ξ be a parametrized curve passing through a point p, whose velocity vector is $\mathtt{v}$. Such a parametrized curve is always exists because we can take, for instance, the straight line $p + t\vec{v}$. We have $\xi(t_0) = p, \xi'(t_0) = \mathtt{v}$. According to the construction described above, we can measure the rate of change of a vector field $\mathtt{Y}$ at a point p in the direction of a tangent vector $\mathtt{v} = (p; \vec{v})$ if we take the derivative $(\mathtt{Y}|_\xi)'$ of the restriction of a vector field $\mathtt{Y}$ to ξ calculated at a point p as a measure of this change. This derivative of a vector field $\mathtt{Y}$ is called *a covariant derivative of a vector field* $\mathtt{Y}$ at a point p and is denoted by $\nabla_\mathtt{v}\mathtt{Y}$. Hence

$$\nabla_\mathtt{v}\mathtt{Y} = (\mathtt{Y}|_\xi)'\big|_{t=t_0} = \frac{d}{dt}(\mathtt{Y}|_\xi)\big|_{t=t_0}, \quad \xi'(t_0) = \mathtt{v}. \tag{2.45}$$

It follows from this definition that covariant derivative of a vector field at a point of U is a vector at this point.

We can extend the notion of a covariant derivative at a point to the whole open subset U as follows. Let $\mathtt{X}, \mathtt{Y}$ be two vector fields defined on U. A vector field $\mathtt{X}$ defines the tangent vector $\mathtt{X}_p$ at each point $p \in U$. Therefore we can calculate the covariant derivative $\nabla_{\mathtt{X}_p}\mathtt{Y}$ at each point $p \in U$ and this derivative will determine a vector at each point of an open subset U. Hence we get the new vector field on U which will be denoted by $\nabla_\mathtt{X}\mathtt{Y}$ and called a *covariant derivative of a vector field* $\mathtt{Y}$ *with respect to a vector field* $\mathtt{X}$. Hence a covariant derivative $\nabla_\mathtt{X}\mathtt{Y}$ is a vector field defined at each point p of an open subset U as follows

$$(\nabla_\mathtt{X}\mathtt{Y})_p = \nabla_{\mathtt{X}_p}\mathtt{Y}. \tag{2.46}$$

Later we will see that covariant derivative $\nabla_\mathtt{X}\mathtt{Y}$ is a smooth vector field provided that $\mathtt{X}, \mathtt{Y}$ are smooth vector fields.

Now our aim is to find a formula for a covariant derivative in coordinates. We will do this first of all in affine coordinates and then in an arbitrary curvilinear coordinate system. Let us assume that we have an affine coordinate system $x^1, x^2, \ldots, x^n$ defined in U. This means that we have an affine frame $\{O; \vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}$ and the vector fields

$$\mathtt{U}_i = \frac{\partial}{\partial x^i}, \quad (\mathtt{U}_i)_p = \mathtt{e}_{p,i} = (p; \vec{e}_i), \quad p \in U.$$

Now any vector field can be expressed in the terms of basic vector fields as follows $\mathtt{Y} = Y^i\,\mathtt{U}_i$, where Y^i are components of a vector field $\mathtt{Y}$, i.e. smooth functions. It is important here that the basic vector fields $\mathtt{U}_i$ are constant vector fields. By other words, if we restrict a vector field $\mathtt{Y}$ to a parametrized curve ξ then the components Y^i of a vector field become smooth (composite) functions $Y^i(t)$ of one variable t, but the basic vector fields $\mathtt{U}_i$ remain constant. Thus the change of a vector field $\mathtt{Y}$ will be completely determined by its components $Y^i(t)$ and the basic vector fields $\mathtt{U}_i$ do not make any contribution to this change. Hence in the case of an affine coordinate system we have

$$(\mathtt{Y}|_\xi)' = \frac{d}{dt}(\mathtt{Y}|_\xi) = \frac{dY^i(t)}{dt}\,\mathtt{U}_i + Y^i\,\frac{d}{dt}\Big(\mathtt{U}_i|_\xi\Big). \tag{2.47}$$

The second term at the right-hand side of the above formula vanishes because $\mathtt{U}_i$ is a constant vector field. Hence for any $p \in U$, $\mathtt{v} \in T_pE^n$ and a parametrized curve ξ such that $\xi(t_0) = p, \xi'(t_0) = \mathtt{v}$ we have

$$\nabla_{\mathtt{v}}\mathtt{Y} = \frac{d}{dt}\big(Y^i|_\xi\big)\Big|_{t=t_0}\,\mathtt{e}_{p,i} = (\mathtt{v} \triangleright Y^i)\,\mathtt{e}_{p,i}. \tag{2.48}$$

Thus the covariant derivative of a vector field $\mathtt{Y}$ at a point p with respect to a tangent vector $\mathtt{v}$ is the tangent vector at a point p whose coordinates in an affine frame are the directional derivatives of the components Y^i of a vector field $\mathtt{Y}$ in the direction of a tangent vector $\mathtt{v}$. Particularly this formula shows that covariant derivative depends linearly on a vector $\mathtt{v}$ and this property does not depend on a coordinate system. Thus we have

$$\nabla_{\mathtt{v}_1+\mathtt{v}_2}\mathtt{Y} = \nabla_{\mathtt{v}_1}\mathtt{Y} + \nabla_{\mathtt{v}_2}\mathtt{Y}. \tag{2.49}$$

From (2.48) it follows that if $\mathtt{X} = X^i\,\mathtt{U}_i$ is a vector field then

$$\nabla_{\mathtt{X}}\mathtt{Y} = (\mathtt{X} \triangleright Y^i)\,\mathtt{U}_i = X^j\frac{\partial Y^i}{\partial x^j}\,\frac{\partial}{\partial x^i}. \tag{2.50}$$

It follows from (2.49) that for any vector fields $\mathtt{X}_1, \mathtt{X}_2, \mathtt{Y}$ it holds

$$\nabla_{\mathtt{X}_1+\mathtt{X}_2}\mathtt{Y} = \nabla_{\mathtt{X}_1}\mathtt{Y} + \nabla_{\mathtt{X}_2}\mathtt{Y}. \tag{2.51}$$

Formula (2.50) also shows that in affine coordinates the components of the covariant derivative are smooth functions and, consequently, the covariant derivative is a smooth vector field.

Now consider a more general situation, that is, a coordinate system defined on an open set U must not be an affine coordinate system, but can be an arbitrary curvilinear coordinate system. The coordinates of this coordinate system will be denoted by $x^1, x^2, \ldots, x^n$. Then the basic vector fields induced by this coordinate system will be denoted by $\mathtt{D}_1, \mathtt{D}_2, \ldots, \mathtt{D}_n$, where $\mathtt{D}_i = \frac{\partial}{\partial x^i}$. It is important to note that in contrast with affine coordinates now the basic vector fields $\mathtt{D}_i$ are need not to be constant vector fields, that is, they can change from point to point. This leads to an important structure of modern differential geometry, which is called a *connection*. Assume that we are given a vector field $\mathtt{Y}$ and a parametrized curve ξ. Our aim is to find a rate of change of a vector field $\mathtt{Y}$ when a point moves along a curve ξ. We can do this as we did it before, i.e. we restrict a vector field $\mathtt{Y}$ to a parametrized curve ξ and then differentiate this restriction $\mathtt{X}|_\xi$ with respect to a parameter of a parametrized curve. But if we write this derivative in curvilinear coordinates then the right-hand side of formula (2.47) will change. Due to a non-constant nature of the basic vector fields $\mathtt{D}_i$ the second term at the right-hand side of (2.47) does not vanish. Let $\mathtt{Y} = Y^i\,\mathtt{D}_i$. Then we have

$$(\mathtt{Y}|_\xi)' = \frac{d}{dt}(\mathtt{Y}|_\xi) = \frac{dY^i}{dt}\,\mathtt{D}_i + Y^i\,\frac{d}{dt}\Big(\mathtt{D}_i|_\xi\Big) = (\xi' \triangleright Y^i)\,\mathtt{D}_i + Y^i\,\nabla_{\xi'}\mathtt{D}_i. \tag{2.52}$$

It is useful to consider this formula at a fixed point p of a curve ξ. Without loss of generality we can assume that an open interval I of a parametrized curve $\xi; I \to U$ contains 0 and $\xi(0) = p, \xi'(0) = \mathtt{v}$. Then taking $t = 0$ in (2.52) we get

$$\nabla_{\mathtt{v}} \mathtt{Y} = (\mathtt{Y}|_{\xi})'\Big|_{t=0} = (\mathtt{v} \triangleright Y^i)\,(\mathtt{D}_i)_p + Y^i(p)\,\nabla_{\mathtt{v}}\,\mathtt{D}_i \tag{2.53}$$

The covariant derivative $\nabla_{\mathtt{v}}\,\mathtt{D}_i$ is the vector at a point p and it can be expressed as a linear combination of the vectors of basis $(\mathtt{D}_i)_p$, i.e. $\nabla_{\mathtt{v}}\,\mathtt{D}_i = \Gamma_i^j\,(\mathtt{D}_j)_p$, where Γ_i^j are numbers defined at a point p, and, due to the linearity (2.49) of the covariant derivative in respect of a vector $\mathtt{v}$, the coefficients of this combination depend linearly on a vector $\mathtt{v}$.

Proposition 2.1. Let $p \in U$, $\mathtt{v} \in T_pE^n$ and Λ_i^j be numbers defined at a point p by the formula $\nabla_{\mathtt{v}}\,\mathtt{D}_i = \Lambda_i^j\,(\mathtt{D}_j)_p$. Then there exists uniquely determined system of covectors $\varphi_i^j \in T_p^*E^n$ such that $\varphi_i^j(\mathtt{v}) = \Lambda_i^j$.

Proof. First of all let us proof that if $a_1, a_2, \ldots, a_n$ are real numbers defined at a point p and $\mathtt{v}_1, \mathtt{v}_2, \ldots, \mathtt{v}_n$ are linearly independent tangent vectors at a point p then this data uniquely determines a covector $\varphi \in T_p^*E^n$ such that $\varphi(\mathtt{v}_i) = a_i$. We choose a basis $\mathtt{e}_1, \mathtt{e}_2, \ldots, \mathtt{e}_n$ for the tangent space T_pE^n and write each vector $\mathtt{v}_i$ as a linear combination of the vectors of basis $\mathtt{v}_i = v_j^k\,\mathtt{e}_k$. The dual basis will be denoted by $\mathtt{e}^1, \mathtt{e}^2, \ldots, \mathtt{e}^n$. Obviously the determinant of the matrix (v_i^k) is non-zero number. The condition $\varphi(\mathtt{v}_i) = a_i$ leads to the system of linear equations $v_i^k\,\varphi_k = a_i$, where $\varphi(\mathtt{v}_k) = \varphi_k$. The matrix of this system is (v_i^k) and this matrix is regular, i.e. $\mathrm{Det}(v_i^k) \neq 0$. Thus this system of equations has the solution $\varphi_1, \varphi_2, \ldots, \varphi_n$, and this solution is unique. Then $\varphi = \varphi_k\,\mathtt{e}^k$.

Now for every tangent vector $\mathtt{v}_k$ we find the real numbers $(\Lambda_k)_i^j$, where

$$\nabla_{\mathtt{v}_k}\,\mathtt{D}_i = (\Lambda_k)_i^j\,(\mathtt{D}_j)_p.$$

From the previous considerations it follows that there exists uniquely determined system of covectors φ_i^j such that $\varphi_i^j(\mathtt{v}_k) = (\Lambda_k)_i^j$. Let $\mathtt{v} = a^k\,v_k$. Then

$$\nabla_{\mathtt{v}}\,\mathtt{D}_i = \nabla_{a^k\mathtt{v}_k}\,\mathtt{D}_i = a^k\,\nabla_{\mathtt{v}_k}\,\mathtt{D}_i = a^k\,(\Lambda_k)_i^j\,(\mathtt{D}_j)_p.$$

Thus $\Lambda_i^j = a^k\,(\Lambda_k)_i^j$. Hence

$$\varphi_i^j(\mathtt{v}) = \varphi_i^j(a^k\,\mathtt{v}_k) = a^k\,\varphi_i^j(\mathtt{v}_k) = a^k\,(\Lambda_k)_i^j = \Lambda_i^j.$$

□

In the proof of Proposition (2.1) we used an arbitrary basis $\mathtt{v}_k$ for the tangent space T_pE^n at a point p. But curvilinear coordinates induce the basis $\mathtt{D}_k$ and we can use this basis to define the real numbers $(\Lambda_k)_i^j$. In this case they are denoted by $(\Gamma_k)_i^j$. Hence we have

$$\nabla_{\mathtt{D}_k}\,\mathtt{D}_i = (\Gamma_k)_i^j\,(\mathtt{D}_j)_p. \tag{2.54}$$

We can omit the brackets and write $\Gamma_{ki}^j = (\Gamma_k)_i^j$. The real numbers Γ_{ki}^j defined at a point p of an affine space E^n by the formula $\nabla_{\mathtt{D}_k}\,\mathtt{D}_i = \Gamma_{ki}^j\,(\mathtt{D}_j)_p$ are called *Christoffel symbols.*

The following formulas hold at each point $p \in U$

$$\nabla_{\mathtt{v}}\mathtt{D}_i = \varphi_i^j(\mathtt{v})\,\mathtt{D}_j, \quad \nabla_{\mathtt{v}}\mathtt{Y} = (\mathtt{v} \triangleright Y^i)\,(\mathtt{D}_i)_p + \varphi_i^j(\mathtt{v})\,Y^i(p)\,(\mathtt{D}_j)_p. \tag{2.55}$$

Now we extend these formulas to the entire open subset U by replacing a tangent vector $\mathtt{v}$ by a vector field $\mathtt{X}$ and covectors φ_i^j by a differential 1-forms ω_i^j. Then the formulas in (2.55) take on the form

$$\nabla_{\mathtt{X}}\mathtt{D}_i = \omega_i^j(\mathtt{X})\,\mathtt{D}_j, \quad \nabla_{\mathtt{X}}\mathtt{Y} = (\mathtt{X} \triangleright Y^i)\,\mathtt{D}_i + \omega_i^j(\mathtt{X})\,Y^i\,\mathtt{D}_j. \tag{2.56}$$

We see that the covariant derivative of basic vector fields $\mathtt{D}_i$ gives rise to the matrix of differential 1-forms $\omega = (\omega_i^j)$, which is referred to as a *matrix of connection* in an affine space E^n. In what follows we will call the elements of the matrix of connection ω_i^j *connection 1-forms*. If we write $\omega_i^j = \omega_{ki}^j\,dx^k$ then we have the relation $\omega_i^j(\mathtt{D}_k) = \omega_{ki}^j = \Gamma_{ki}^j$. Hence $\omega_i^j = \Gamma_{ki}^j\,dx^k$.

Let us consider the second formula in (2.56). We can write it either by means of Christoffel symbols or connection 1-forms. If we use Christoffel symbols and take $\mathtt{X} = \mathtt{D}_k$ then the second formula in (2.56) takes on the form

$$\nabla_k\mathtt{Y} = \nabla_{\mathtt{D}_k}\mathtt{Y} = (\frac{\partial Y^i}{\partial x^k} + \Gamma_{kj}^i\,Y^j)\,\mathtt{D}_i. \tag{2.57}$$

From this it follows

$$\nabla_{\mathtt{X}}\mathtt{Y} = X^k\,\nabla_k\mathtt{Y} = (\frac{\partial Y^i}{\partial x^k}\,X^k + \Gamma_{kj}^i\,X^k\,Y^j)\,\mathtt{D}_i. \tag{2.58}$$

In the case when we prefer to use the calculus of differential forms we can write the second formula in (2.56) as follows

$$\nabla_{\mathtt{X}}\mathtt{Y} = (dY^i + \omega_j^i\,Y^j)(\mathtt{X})\,\mathtt{D}_i. \tag{2.59}$$

Note that in this formula $\mathtt{X}$ is an arbitrary vector field. Since this vector field is both on the left side of the formula and on the right side, we can formally omit it and the formula will remain true. Hence

$$\nabla\,\mathtt{Y} = (dY^i + \omega_j^i\,Y^j)\,\mathtt{D}_i \tag{2.60}$$

However now we have to explain in what sense we understand the obtained formula. First of all we consider this formula as the definition of a new operator ∇. In order to give a correct definition of this new operator, we should explain how we understand the expression on the right-hand side of formula (2.60). For this purpose we consider two modules $\Omega^1(U), \mathcal{D}(U)$ over the algebra of smooth functions $C^\infty(U)$. Here $\Omega^1(U)$ is the module of differential 1-forms and $\mathcal{D}(U)$ is the module of vector fields. We can form the tensor product of these two modules

$$\Omega^1(U,\mathcal{D}) = \Omega^1(U)\otimes_{C^\infty(U)}\mathcal{D}(U).$$

An element of this tensor product is referred to as a *differential 1-form with values in vector fields*. If we take the basis $\{dx^i\}$ for the module of differential 1-forms $\Omega^1(U)$ and $\{\mathtt{D}_k\}$ for the module of vector fields $\mathcal{D}(U)$ then $\{dx^i\otimes\mathtt{D}_k\}$ is the basis for the tensor product $\Omega^1(U,\mathcal{D})$. Thus any differential 1-form with values in vector fields θ can be expressed in terms of the basis as follows

$$\theta = \theta_i^k\;dx^i\otimes\mathtt{D}_k, \tag{2.61}$$

where θ_i^k are smooth functions. In order to define the value of θ on a vector field $\mathtt{X}$ we write θ in the form

$$\theta = \theta_i^k\;dx^i\otimes\mathtt{D}_k = (\theta_i^k\;dx^i)\otimes\mathtt{D}_k = \theta^k\otimes\mathtt{D}_k.$$

If $\mathtt{X}\in\mathcal{D}(U)$ then we define the value of θ on a vector field $\mathtt{X}$ as follows

$$\theta(\mathtt{X}) = \big(\theta^k(\mathtt{X})\big)\,\mathtt{D}_k.$$

Hence we see that the value of θ on a vector field is the vector field and this justifies the term "differential 1-form with values in vector fields".

Now we can give a correct definition for the expression at the right-hand side of (2.60).

Definition 2.8. Let $\mathtt{Y} = Y^i\,\mathtt{D}_i$ be a vector field. Then the covariant differential of a vector field $\mathtt{Y}$ is a differential 1-form with values in vector fields $\nabla\,\mathtt{Y}$ defined by the formula

$$\nabla\,\mathtt{Y} = (dY^i + \omega^i_j\,Y^j) \otimes \mathtt{D}_i.$$

Let us study in more detail the structure of the matrix of connection form in curvilinear coordinates. First of all, we note that in affine coordinates of the space E^n, the connection matrix is zero. Indeed, in this case the covariant derivative of the basic vector fields $\mathtt{U}_i$ is equal to zero due to the fact these fields are constant. Let us assume that we have two coordinate systems in $U \subset E^n$, where one is the system of affine coordinates $x^1, x^2, \dots, x^n$ and the other is a system of curvilinear coordinates $\tilde{x}^1, \tilde{x}^2, \dots, \tilde{x}^n$. The transition functions from one system of coordinates to another will be denoted by $x^i = x^i(\tilde{x}^1, \tilde{x}^2, \dots, \tilde{x}^n)$ (from curvilinear coordinates to affine coordinates) and $\tilde{x}^i = \tilde{x}^i(x^1, x^2, \dots, x^n)$ (from affine coordinates to curvilinear). In order to simplify notations we will write them in the compact form $x = x(\tilde{x})$ and $\tilde{x} = \tilde{x}(x)$. The Jacobi matrix of the transition $x = x(\tilde{x})$ will be denoted by $A = (A^j_i)$ and the Jacobi matrix of the inverse transition $\tilde{x} = \tilde{x}(x)$ is the reciprocal matrix of A. Hence we have

$$A^j_i = \frac{\partial x^i}{\partial \tilde{x}^j},\ (A^{-1})^j_i = \frac{\partial \tilde{x}^i}{\partial x^j},$$

In these formulas the elements of the transition matrix A are functions of curvilinear coordinates $\tilde{x}^1, \tilde{x}^2, \dots, \tilde{x}^n$, while the elements of the reciprocal matrix A^{-1} are functions of affine coordinates $x^1, x^2, \dots, x^n$. We have

$$\frac{\partial}{\partial \tilde{x}^i} = \frac{\partial x^j}{\partial \tilde{x}^i}\frac{\partial}{\partial x^j} = A^j_i\,\frac{\partial}{\partial x^j},\quad \frac{\partial}{\partial x^i} = \frac{\partial \tilde{x}^j}{\partial x^i}\frac{\partial}{\partial \tilde{x}^j} = (A^{-1})^j_i\,\frac{\partial}{\partial \tilde{x}^j}$$

Making use of the notations $\mathtt{U}_i = \frac{\partial}{\partial x^i}, \mathtt{D}_i = \frac{\partial}{\partial \tilde{x}^i}$ we can write the above formulas as in the form

$$\mathtt{D}_i = A^j_i\,\mathtt{U}_j, \quad \mathtt{U}_i = (A^{-1})^j_i\,\mathtt{D}_j.$$

Now our aim is to find the matrix of connection form in curvilinear coordinates $\tilde{x}^i$. The definition of the matrix of connection in curvilinear coordinates is $\nabla\,\mathtt{D}_i = \omega^j_i\,\mathtt{D}_j$. Substituting $\mathtt{D}_i = A^j_i\,\mathtt{U}_j$ into the left-hand side of the definition of the matrix of connection, we get

$$\nabla\,(A^j_i\,\mathtt{U}_j) = (dA^j_i)\,\mathtt{U}_j + A^j_i\,\nabla\,\mathtt{U}_j.$$

But $\nabla\,\mathtt{U}_j = 0$ and we obtain $\omega^j_i\,A^k_j\,\mathtt{U}_k = dA^k_i\,\mathtt{U}_k$. Hence $A^k_j\,\omega^j_i = dA^k_i$ or, in the matrix form $dA = A\,\omega$. From this we find the matrix of connection

$$\omega^i_j = (A^{-1})^i_k\,dA^k_j \quad \Leftrightarrow \quad \omega = A^{-1}\,dA. \tag{2.62}$$

We mentioned before that the elements of the reciprocal matrix A^{-1} are the functions of affine coordinates x^i. In (2.62) the first factor at the right-hand side is the matrix which is obtained from A^{-1} by substitution $x = x(\tilde{x})$.

The set of dual 1-forms for the set of basic vector fields $\{\mathtt{D}_i\}$ in curvilinear coordinates is $\{d\tilde{x}^j\}$, i.e. $d\tilde{x}^j(\mathtt{D}_i) = \delta^j_i$. Let us denote $\theta^i = d\tilde{x}^i$. We can find the expressions of these 1-forms in affine coordinates. Indeed let $\theta^i = \theta^i_j\,dx^j$. Then

$$\delta^i_j = \theta^i(\mathtt{D}_j) = \theta^i(A^k_j\,\mathtt{U}_k) = A^k_j\,\theta^i_m\,dx^m(\mathtt{U}_k) = A^k_j\,\theta^i_m\,\delta^m_k = \theta^i_k\,A^k_j.$$

Hence $\theta^i_k = (A^{-1})^i_k$ and $\theta^i = (A^{-1})^i_k\, dx^k$. Now we can find the exterior differential of the dual 1-forms θ^i. We have $d\theta^i = d(A^{-1})^i_k \wedge dx^i$. In order to express $d(A^{-1})^i_k$ in the terms of transition matrices we differentiate the both sides of the matrix relation $A^{-1}\, A = I$, where I is the unit matrix. We get

$$d(A^{-1})\, A + A^{-1}\, dA = 0 \ \Rightarrow\ d(A^{-1}) = -A^{-1} dA\, A^{-1}.$$

Thus

$$\begin{aligned} d\theta^i &= -\Big((A^{-1})^i_k\, dA^k_m\, (A^{-1})^m_j\Big) \wedge dx^j \\ &= -\Big((A^{-1})^i_k\, dA^k_m\Big) \wedge \Big((A^{-1})^m_j\, dx^j\Big) \\ &= -\omega^i_m \wedge \theta^m. \end{aligned}$$

The equation

$$d\theta^i = \theta^m \wedge \omega^i_m$$

is called the *first Cartan equation* in n-dimensional affine space E^n.

In n-dimensional affine space E^n we can solve the first Cartan equation. Indeed $\theta^i = d\tilde{x}^i$ and $d\theta^i = d^2\tilde{x}^i = 0$. Hence we have $\theta^m \wedge \omega^i_m = 0$. The dual 1-forms θ^i are independent at each point of U. Hence applying Cartan's lemma we conclude that the connection 1-forms ω^i_m can be expressed in terms of the dual 1-forms θ^m, i.e. we have $\omega^i_m = h^i_{mj}\theta^j$, where h^i_{mj} are the functions of curvilinear coordinates, which are symmetric in subscripts m, j.

Now our aim is to express the exterior differential of the connection 1-forms in terms of dual 1-forms θ^i. We have $\omega^i_j = (A^{-1})^i_k\, dA^k_j$ and differentiating the both sides we obtain

$$\begin{aligned} d\omega^i_j &= -d(A^{-1})^i_k \wedge dA^k_j = -\Big((A^{-1})^i_m\, dA^m_l\, (A^{-1})^l_k\Big) \wedge dA^k_j \\ &= -\Big((A^{-1})^i_m\, dA^m_l\Big) \wedge \Big((A^{-1})^l_k dA^k_j\Big) \\ &= -\omega^i_l \wedge \omega^l_j. \end{aligned}$$

The equation

$$d\omega^i_j = \omega^l_j \wedge \omega^i_l,$$

is called the second Cartan equation in n-dimensional affine space E^n.

The Euclidean structure of the n-dimensional affine space E^n makes it possible to single out a very convenient subclass in the class of curvilinear coordinates. This subclass is called orthogonal curvilinear coordinate systems. In such coordinates the coordinate lines are pairwise orthogonal, which means that $< \mathtt{D}_i, \mathtt{D}_j >= 0$ for every pair $i \neq j$. We can rescale the basic vector fields $\mathtt{D}_i$ introducing new vector fields

$$\mathtt{E}_i = \frac{1}{\|\mathtt{D}_i\|}\,\mathtt{D}_i, \quad \|\mathtt{E}_i\| = 1,$$

which form the orthonormal basis of the tangent space T_pE^n at each point $p \in U$. Let us find what changes in the previous considerations if we use the set of vector fields $\{\mathtt{E}_i\}$ as the basic vector fields. First of all we note that in the case when we use the Euclidean metric of E^n, it makes no sense to distinguish between upper and lower indices. Indeed, in this case the space is canonically isomorphic to its dual space. Thus we will use only subscripts while the superscript will be put on the first place. The matrix $A = (A_{ji})$ in $\mathtt{E}_i = A_{ji}\, \mathtt{U}_j$ is a special orthogonal matrix, i.e. $A^T A = I, \mathrm{Det}\, A = 1$ (we assume that a system of curvilinear coordinates has the same orientation as Cartesian coordinates). The condition of orthogonality can be written in the form $A = (A^{-1})^T$. Differentiating both sides of $A^T\,(A^{-1})^T = I$ we get

$$dA^T\,(A^{-1})^T + A^T\,d(A^{-1})^T = 0 \;\;\Rightarrow\;\; dA^T\,(A^{-1})^T = -A^T\,dA$$

Now we calculate

$$\begin{aligned} \omega + \omega^T &= A^{-1}\,dA + (A^{-1}\,dA)^T = A^{-1}\,dA + dA^T\,(A^{-1})^T \\ &= A^{-1}\,dA - A^T dA = A^{-1}\,dA - A^{-1}\,dA = 0. \end{aligned}$$

Hence the matrix of connection ω calculated in the orthonormal basis $\{\mathtt{E}_i\}$ is a skew-symmetric matrix. If one looks at the derivation of Cartan equations given above then it can be seen that the Cartan equations will have the same form in the case of orthonormal vector fields $\{\mathtt{E}_i\}$.

2.6 Integral curves of a vector field

Assume we are given a vector field $\mathtt{X}$ in an open subset $U \subset E^n$. Fix a point $p \in U$. Now we can put an interesting question of how to find a curve in U, which passes a point p, such that its velocity vector at any point of a curve is the value of a vector field $\mathtt{X}$ at this point. By other words, velocity vector of unknown curve at any of its points is determined by a given vector field $\mathtt{X}$. This curve is called an integral curve of a vector field $\mathtt{X}$ through a point p and in this section we will show that integral curve of any smooth vector field exists and its uniquely determined by a point p. In this section we will use Cartesian coordinates of Euclidean space E^n, but actually the results do not depend on a choice of coordinate system and can be applied in any coordinate system.

Let $X^1, X^2, \ldots, X^n$ be components of a vector field $\mathtt{X}$ in Cartesian coordinates of E^n. We consider the problem of finding a parametrized curve $(\xi = \xi(t), I)$ such that it passes a point p, whose coordinates will be denoted by $p^1, p^2, \ldots, p^n$, and its velocity vector at any point is determined by the value of a vector field $\mathtt{X}$ at this point. We will also assume that I contains the number zero and a parametrized curve $\xi = \xi(t)$ passes a point p, when $t = 0$, i.e. $\xi(0) = p$. In order to write down an equation for unknown parametrized curve $\xi = \xi(t)$, we restrict a vector field $\mathtt{X}$ to a parametrized curve $\xi = \xi(t)$, that is, $\mathtt{X}(t) = \mathtt{X}|_{\xi(t)}$. Clearly, $\mathtt{X}(t)$ is the vector field along a parametrized curve $\xi = \xi(t)$. Now, the condition, that velocity vector of a parametrized curve $\xi = \xi(t)$ is determined by a vector field $\mathtt{X}$, can be written in the form of equation

$$\xi'(t) = \mathtt{X}(t). \tag{2.63}$$

Let $\xi(t) = (\xi^1(t), \xi^2(t), \ldots, \xi^n(t))$. Then the equation (2.63), written in coordinates, leads us to the following system of differential equations of first order

$$\begin{aligned} \frac{d\xi^1}{dt} &= \mathtt{X}^1(\xi^1(t), \xi^2(t), \ldots, \xi^n(t)), \\ \frac{d\xi^2}{dt} &= \mathtt{X}^2(\xi^1(t), \xi^2(t), \ldots, \xi^n(t)), \\ &\;\;\;\ldots\ldots \\ \frac{d\xi^n}{dt} &= \mathtt{X}^n(\xi^1(t), \xi^2(t), \ldots, \xi^n(t)). \end{aligned} \tag{2.64}$$

In this system of differential equations, $\xi^1(t), \xi^2(t), \ldots, \xi^n(t)$ are unknown functions, $\mathtt{X}^i(\xi^1(t), \xi^2(t), \ldots, \xi^n(t))$, where $i = 1, 2, \ldots, n$, are given functions and $\xi^i(0) = p^i$ are initial conditions.

A parametrized curve $(\xi = \xi(t), I)$, which satisfies the equation (2.63) or the system of differential equations of first order (2.64) with initial conditions $\xi^i(0) = p^i$, is called an *integral curve* of a vector field $\mathtt{X}$ through a point p.

It follows from the theory of differential equations [**Pontryagin:1962**] that there exists the solution $(\xi = \xi(t), I)$ of the system of first order equations (2.64), this solution is uniquely determined by initial conditions $\xi^i(0) = p^i$, this solution is smooth if all functions $X^1, X^2, \ldots, X^n$ are smooth functions and it is maximal, that is, if $(\eta = \eta(t), J)$ is also a solution of (2.64) with the same initial conditions, then $J \subset I$ and $\eta(t) = \xi(t)|_J$. Geometrically this means that a vector field $\mathtt{X}$ generates the family of parametrized curves in an open subset U such that through each point of U passes one and only one parametrized curve. We will call this family of parametrized curves a *flow of integral curves generated by a vector field* X.

It is clear that each parametrized curve of this flow, generated by a vector field $\mathtt{X}$, depends not only on parameter t, but also on initial conditions $\xi^i(0) = p^i$. In order to emphasize this dependence, we will denote the solution of (2.64), which satisfies the initial condition $\xi^i(0) = p^i$, by $\xi(p;t) = (\xi^1(p;t), \xi^2(p;t), \ldots, \xi(p;t))$. Thus $\xi^i(p;0) = p^i$. A peculiar property of the flow of integral curves, generated by $\mathtt{X}$, is that it has a structure of a group. Indeed, it follows from the existence and uniqueness of solution $\xi(p;t)$ of the system of differential equations (2.64) that for any point $p \in U$ and $t, s, t+s \in I$, we have

$$\xi\big(\xi(p;t), s\big) = \xi(p; t+s). \tag{2.65}$$

Particularly, for any $t \in I$ such that $-t \in I$, it holds $\xi(\xi(p;t), -t) = \mathrm{id}_U$. Define the family of transformations $g_t : U \to U$ by $g_t(p) = \xi(p;t)$. As any solution of the system (2.64) depends smoothly on initial conditions, a transformation $g_t : U \to U$ for any $t \in I$ is a smooth transformation of U. Then, from (2.65), it follows

$$g_s \circ g_t = g_{t+s}, \quad g_{-t} = g^{-1}(t), \quad g_0 = \mathrm{id}_U. \tag{2.66}$$

The property $g_{-t} = g^{-1}(t)$ shows that each transformation g_t has its inverse, which is also smooth. Thus, g_t is a diffeomorphism of an open subset U for each $t \in I$. Next, the family of diffeomorphisms $g_t : U \to U$, generated by a vector field $\mathtt{X}$, is an Abelian group, where the group operation is a product of diffeomorphisms. This group is called a *one-parameter group of diffeomorphisms of an open subset* U of Euclidean space E^n, generated by a vector field $\mathtt{X}$.

2.7 Connection in Curvilinear Coordinates

In this section, we will elaborate differential approach, which can be used to measure a change of one vector field $\mathtt{Y}$ in the direction determined by another vector field $\mathtt{X}$. This will lead us to such important concepts as the covariant derivative and the covariant differential of a vector field. The peculiarity of the approach in this section is that we will work in curvilinear coordinates. We will first develop this differential calculus at one point of E^n, and then we will extend it to the whole space.

Let us fix an open subset $U \subset E^n$ and two vector fields $\mathtt{X}, \mathtt{Y}$ defined on U. We assume that U is equipped with a coordinate system $(x^i) = (x^1, x^2, \ldots, x^n)$ and this coordinate system is not necessarily Cartesian, that is, it can also be curvilinear. A coordinate system (x^i) induces the set of basic vector fields $\partial/\partial x^i, i = 1, 2, \ldots, n$, which will be denoted by $\mathtt{E}_1, \mathtt{E}_2, \ldots, \mathtt{E}_n$. Let us fix a point $p \in U$ and a parametrized curve $\xi : I \to E^n$, which passes through a point p, i.e. $\xi(t_0) = p, t_0 \in I$ and its velocity vector at this point $\xi'(t_0)$ is the value of a vector field $\mathtt{X}$ at this point, that is, $\xi'(t_0) = \mathtt{X}_p$. In order to simplify notations we will denote this vector by $\mathtt{v}_p$, i.e. $\mathtt{v}_p = \mathtt{X}_p = \xi'(t_0)$. Our goal is to develop a differential approach to measure a change of a vector field $\mathtt{Y}$ at a point p in the direction of a vector field $\mathtt{X}$. Obviously, this change (in infinitesimal) is described by the derivative of a vector field Y along a curve ξ. We can define this derivative in an invariant form, that is, a form that does not use any coordinate system. To do this, we need the increment of a vector field Y when shifting to an infinitely close point along the curve. Thus, we should consider the difference $\mathtt{Y}(\xi(t_0 + \Delta t)) - \mathtt{Y}(\xi(t_0))$. However, this is too straightforward and not correct. The fact is that vector $\mathtt{Y}(\xi(t_0 + \Delta t))$ has its origin at point $\xi(t_0 + \Delta t)$, while a vector $\mathtt{Y}(\xi(t_0))$ has its origin at point $\xi(t_0)$. Thus, vectors are applied to different points in space E^n and we cannot perform addition or subtraction operations on such vectors. We can solve this problem by dropping an initial point of a vector, that is, by going to its vector part, which is a vector-valued function $\vec{\mathtt{Y}}$ (at each point $x \in U$ we have $\mathtt{Y}_x = (x; \vec{\mathtt{Y}}(x))$). In this case, the increment of a vector field Y will be described by the correctly defined difference $\vec{\mathtt{Y}}(\xi(t_0 + \Delta t)) - \vec{\mathtt{Y}}(\xi(t_0))$. However, at this point it is useful to reflect on a geometric essence of what we have done. Indeed, we can interpret dropping an initial point of a vector as a parallel translation of vector $\vec{\mathtt{Y}}(\xi(t_0 + \Delta t))$ from point $\xi(t_0 + \Delta t)$ to point $\xi(t_0)$ (in the sense of Euclidean space E^n) and then subtracting vector $\vec{\mathtt{Y}}(\xi(t_0))$. This interpretation is very important and it indicates a deep connection between a derivative we are developing with parallel translation of vectors in Euclidean space E^n. Since our goal is to apply the resulting derivative to vector fields, we must get a vector at a point. It is easy to do as follows. We will first calculate the derivative of a vector field $\mathtt{Y}$ along a curve ξ (at a point p), and then apply the resulting vector to point p. Let us denote $\vec{\mathtt{Y}}(\xi(t)) = \vec{\mathtt{Y}}(t)$. Then we define a covariant derivative of a vector field $\mathtt{Y}$ at a point p in direction of $\mathtt{X}$ by the formula

$$D_{\mathtt{v}_p} Y = \Big(p; \lim_{\Delta t \to 0} \frac{1}{\Delta t}\big(\vec{\mathtt{Y}}(t_0 + \Delta t) - \vec{\mathtt{Y}}(t_0)\big)\Big) = \Big(p; \frac{d}{dt}\vec{\mathtt{Y}}(t)\Big|_{t=t_0}\Big). \tag{2.67}$$

This definition does not depend on a choice of a coordinate system in U. The next step in the development of a covariant derivative is its calculation in coordinates (x^i).

3 Surfaces in 3-Dimensional Euclidean Space

3.1 How to define a surface?

In this section, we will consider surfaces in three-dimensional Euclidean space E^3. Unlike a curve, a surface has its own internal geometry. Figuratively speaking, we could imagine two-dimensional intellectual beings for whom the surface is their universe and who are unaware of the existence of a third dimension, that is, they do not suspect that their universe is in a space of a higher dimension. These beings measure the lengths of lines on the surface, the angles between the lines, that is, they build the internal geometry of a surface without even knowing that this geometry is induced by the Euclidean geometry of the space enclosing their surface.

As before, we assume that in three-dimensional Euclidean space E^3 we have an orthonormal frame $\{O; \vec{e}_1, \vec{e}_2, \vec{e}_3\}$, which defines the Cartesian coordinate system, whose coordinates will be denoted by x, y, z. How can we define a surface analytically, that is, using an equation? Basically we have two different methods to define a surface by means of equation.

S1) The first one is a method of parametric representation of surfaces by means of two parameters. This method is similar to a parametric representation of a curve, and we used this method in the case of curves in the previous sections.

S2) The second method is a representation of a surface by an equation $f(x, y, z) = c$, where $f(x, y, z)$ is a smooth function and c is a constant real number. This method is called implicit representation of a surface, since in this case the points of a surface are solutions to equation $f(x, y, z) = c$.

We start with a parametric representation of a surface, then describe a method for implicitly representing a surface and point out their relationship. A surface, unlike a curve, which is one-dimensional, is a two-dimensional geometric object and therefore we need two parameters to define a surface parametrically. For this purpose, we will fix a Euclidean plane E^2, which (or part of it) will be a domain of a surface parameters. As before, we assume that Euclidean plane E^2 is equipped with a Cartesian coordinate system, which we will denote by u, v and these coordinates will be a surface parameters. Let $U \subset E^2$ be an open subset of a plane E^2. We will map a subset U to three-dimensional Euclidean space E^3 and in this mapping it will take the shape of a surface. In order to map U into three-dimensional space, we need three functions, let us denote them by $x(u, v), y(u, v), z(u, v)$. We will assume that domain of these functions is an open subset U and these functions are smooth (infinitely differentiable). The mapping determined by functions $x(u, v), y(u, v), z(u, v)$ will be denoted by ϕ, i.e. $\phi : U \to E^3$, where $\phi(u, v) = \big(x(u, v), y(u, v), z(u, v)\big)$. If we use vector notation $\vec{\phi}(u, v)$, then this means that $\vec{\phi}(u, v)$ is the position vector of a point $\phi(u, v)$.

Definition 3.1. A *parametrized* or *local surface* in three-dimensional Euclidean space E^3 is a smooth mapping

$$\phi : (u, v) \in U \to \phi(u, v) = \big(x(u, v), y(u, v), z(u, v)\big) \in E^3,$$

where $U \subset E^2$ is an open subset of a plane E^2 (domain of parameters). The equation $\phi(u,v) = \big(x(u,v), y(u,v), z(u,v)\big)$ will be referred to as a parametric equation of a local surface. If $\phi : U \to E^3$ is an injective mapping then ϕ will be called a *parametrized coordinate surface* or *local coordinate surface*.

Often a parametrized surface will be denoted as a pair (ϕ, U). The assumption that a mapping $\phi(u,v) = \big(x(u,v), y(u,v), z(u,v)\big)$ is smooth not yet a guarantee that the image of a subset U under the mapping ϕ will be a smooth surface in E^3 in the sense that at each point $\phi(u,v) \in E^3$ we will have a tangent plane to a surface. And we will need it, because we will study a surface using the methods of differential calculus and tangent vectors to a surface will be the basis of all other structures and methods. Before we formulate an additional condition that will ensure the existence of a tangent plane to a surface at a given point, we must define a concept of a tangent vector to a parametrized surface. Let $q = (u,v) \in U$ be a point of domain of parameters. Then $p = \phi(u,v) \in E^3$ is the corresponding point of a parametrized surface ϕ in E^3. Let $(\xi = \xi(t), I)$ be a parametrized curve in U such that its parametric equation is $\xi(t) = (u(t), v(t))$ and this parametrized curve passes through a point q, when $t = 0$, that is, $\xi(0) = q$. By applying mapping ϕ to a parametrized curve ξ, we get the parametrized curve $\tilde{\xi} = \phi \circ \xi : I \to E^3$ on a local surface ϕ. Obviously the parametrized curve $\tilde{\xi}(t)$ passes through the point p when $t = 0$. The velocity vector $\tilde{\xi}'|_p$ of the parametrized curve $\tilde{\xi}$ calculated at a point p (i.e. when $t = 0$) will be called a *tangent vector of a parametrized surface* ϕ at a point p. The set of all tangent vectors to the parametrized surface ϕ at the point p will be denoted as $T_p\phi$.

As in the case of curves, we can consider vector fields on a parametrized surface. A mapping which assigns a vector to each point of a surface will be referred to as a *vector field on a surface.* A vector field $\mathtt{X}$ on a parametrized surface ϕ can be defined as follows

$$\mathtt{X}(u,v) = (\phi(u,v); X^1(u,v), X^2(u,v), X^3(u,v)), \tag{3.1}$$

where $u, v \in U \subset E^2$ and for any $i = 1,2,3$ a function $X^i(u,v)$ is smooth function of two variables. If for any point $p = \phi(u,v)$ of a parametrized surface a vector field $\mathtt{X}$ is a tangent (normal) vector, i.e. $\mathtt{X}(p) \in T_p\phi$ ($\mathtt{X}(p) \perp T_p\phi$), then a vector field $\mathtt{X}$ will be called a *tangent (normal) vector field on a parametrized surface.*

As an example, consider curves on a parametrized surface ϕ that are induced by the coordinate lines of plane E^2. Let us fix a value of the second coordinate v and let the first coordinate u change in some sufficient small interval around a value u, that is, $u(t) = u + t$, where $-\delta < t < \delta$. We get the straight line that passes through point q (when $t = 0$) of plane E^2 and is called the u-coordinate line of a plane E^2. Its parametric equation is $\xi_u(t) = (u + t, v)$. Analogously we construct the v-coordinate line, whose parametric equation is $\xi_v(t) = (u, v + t)$. Parametrized curves $\tilde{\xi}_u = \phi \circ \xi_u$ and $\tilde{\xi}_v = \phi \circ \xi_v$ will lie on a local surface ϕ and pass through point p. They are called u-coordinate line and v-coordinate line of a local surface ϕ. Since we can carry out this construction at any point on a parametrized surface ϕ, we can cover the entire parametrized surface ϕ with a grid of coordinate lines, and one u-coordinate line and one v-coordinate line will pass through each point of a parametrized surface ϕ. Bearing in mind the set of coordinate lines of a parametrized surface, we will talk about a (u,v)-coordinate system on a parametrized surface. If at each point of a surface the u-coordinate line is orthogonal to the v-coordinate line, we will call such a coordinate system of a parametrized surface *orthogonal.* Clear the velocity vectors of u-coordinate line and v-coordinate line of a local surface are the tangent vectors of a local surface. In order to find the velocity vector of u-coordinate line we have to differentiate its parametric equation $\tilde{\xi}_u(t) = (x(u+t,v), y(u+t,v), z(u+t,v))$ with respect to t. But in this case it will give us the partial derivatives of the functions

$x(u,v), y(u,v), z(u,v)$ with respect to parameter u. If we denote this velocity vector by ϕ'_u then

$$\phi'_u(u,v) = (\phi(u,v); x'_u(u,v), y'_u(u,v), z'_u(u,v)), \tag{3.2}$$

where x'_u, y'_u, z'_u stand for partial derivatives with respect u. Using similar reasoning, we can show that the velocity vector of the v-coordinate line of a parametrized surface has the form

$$\phi'_v(u,v) = (\phi(u,v); x'_v(u,v), y'_v(u,v), z'_v(u,v)). \tag{3.3}$$

Comparing equations (3.2) and (3.3) with (3.1) we see that tangent vectors of coordinate lines of a parametrized surface can be considered as tangent vector fields on a parametrized surface and we will denote these two tangent vector fields by ϕ'_u and ϕ'_v.

Let us go back to an arbitrary tangent vector $\tilde{\xi}'(t)$ at a point p of a parametrized surface ϕ. We have

$$\tilde{\xi}' = (\phi(u,v); x'_t(u,v), y'_t(u,v), z'_t(u,v)). \tag{3.4}$$

Applying the rule of differentiation of a composite function, we get

$$\begin{aligned} x'_t(u,v) &= x'_u\, u'_t + x'_v\, v'_t, \\ y'_t(u,v) &= y'_u\, u'_t + y'_v\, v'_t, \\ z'_t(u,v) &= z'_u\, u'_t + z'_v\, v'_t. \end{aligned}$$

Substituting these into (3.4) we obtain

$$\tilde{\xi}' = u'_t\, \phi'_u(u,v) + v'_t\, \phi'_v(u,v). \tag{3.5}$$

The previous formula has a very important meaning. It shows that any tangent vector of a parametrized surface ϕ is a linear combination of tangent vectors $\phi'_u(u,v), \phi'_v(u,v)$ and the coefficients of this linear combination are the coordinates of the velocity vector ξ' of a parameterized curve $\xi(t) = (u(t), v(t))$ in a parameters plane E^2. Another very important conclusion follows from this. A parametrized surface will have the tangent plane at a given point of a surface if the tangent vectors $\phi'_u(u,v), \phi'_v(u,v)$ are linearly independent at this point. As it follows from (3.5) in this case the vectors form the basis for the tangent plane. Hence the plane $T_p\,\phi$ spanned by the tangent vectors $\phi'_u(u,v), \phi'_v(u,v)$ will be called the *tangent plane* of a parametrized surface ϕ at a point $p = \phi(u,v)$. A condition for the linear independence of tangent vectors $\phi'_u(u,v), \phi'_v(u,v)$ can be given in the following two equivalent ways:

R1) the tangent vectors $\phi'_u(u,v), \phi'_v(u,v)$ are linearly independent if and only if

$$\phi'_u(u,v) \times \phi'_v(u,v) \neq 0,$$

R2) the tangent vectors $\phi'_u(u,v), \phi'_v(u,v)$ are linearly independent if and only if

$$\operatorname{rank} \begin{pmatrix} x'_u & y'_u & z'_u \\ x'_v & y'_v & z'_v \end{pmatrix} = 2.$$

If at each point of a parametrized surface one of these equivalent conditions is satisfied, we will call such a surface *regular*. In what follows, we will always assume that the regularity condition is satisfied and we will omit the word “regular” simply writing “parametrized surface” or “local surface”.

The first regularity condition has a simple geometric meaning. It shows that at each point a surface has a non-zero normal vector (orthogonal to a tangent plane). Since we assume that a regularity condition for a parametrized surface is satisfied, a parametrized surface ϕ always has the unit normal vector field $\mathtt{N}$ defined by the formula

$$\mathtt{N} = \frac{\phi'_u \times \phi'_v}{||\phi'_u \times \phi'_v||}. \tag{3.6}$$

Example 3.1. As an example of a parameterized surface, we will consider a surface of revolution. Surfaces of revolution form a very important class of surfaces and many well-known surfaces, such as a sphere, torus, catenoid, are surfaces of revolution. The idea of a surface of revolution is that we fix a plane in space, draw a straight line (axis of rotation) on it and take any curve that lies on a plane on one side of a straight line, without crossing or touching it. Then we start to rotate this curve around a straight line. If, with this rotation, the curve left a trace in space, we would see the surface. This is the surface of revolution.

As the aforementioned plane, we can use the xy-coordinate plane, and as the axis of rotation, we can take the x-axis. A curve in the xy-coordinate plane will be represented in a parametrized form $(\xi = \xi(t), I)$ by a parametric equation $\xi(t) = (f(t), h(t))$. Note that we are considering this curve as a curve in xy-coordinate plane, not in space. Therefore, we omit third coordinate in its parametric equation. We assume that curve ξ lies entirely in the upper half-plane of xy-coordinate plane. Consequently $y(t) > 0$ for any $t \in I$, i.e. there is no crossing and touching. It should be recalled here that we assume that the curve regularity condition is satisfied, that is, $(f'(t))^2 + (h'(t))^2 > 0$. As the parameters of a surface of revolution u, v, it is convenient to choose the curve ξ parameter t as parameter u and the angle of rotation as parameter v. Then the parametric equation of a surface of revolution has the form

$$\phi(u,v) = (f(u), h(u)\,\cos v, h(u)\,\sin v), \quad U = I \times [0, 2\pi]. \tag{3.7}$$

This equation is derived from simple considerations. First of all, it is easy to see that when a point of curve $\xi(t) = (f(t), h(t))$ rotates around x-axis, the first coordinate $f(t)$ of a point does not change. Therefore, in the parametric equation (3.7), the first function is $f(u)$. In a plane parallel to the yz-coordinate plane, a point moves along a circle with a radius of $h(u)$. Therefore, the second and third functions in the parametric equation have the form of a parametrization of a circle with a radius of $h(u)$.

We find

$$\begin{aligned}
\vec{\phi}'_u &= \big(f', h'\,\cos v, h'\,\sin v\big),\\
\vec{\phi}'_v &= \big(0, -h\,\sin v, h\,\cos v\big),\\
\vec{\phi}'_u \times \vec{\phi}'_v &= \big(h\,h', -f'\,h\,\cos v, -f'\,h\,\sin v\big),\\
\|\phi'_u \times \phi'_v\| &= h\,\sqrt{(f')^2 + (h')^2}.
\end{aligned}$$

Since $h(t) > 0$ and $(f'(t))^2 + (h'(t))^2 > 0$ it follows that $\|\phi'_u \times \phi'_v\| > 0$ and (3.7) is a regular parametrization of a surface of revolution. Finally we find the unit normal vector field on a surface of revolution

$$\vec{\mathtt{N}} = \frac{1}{\sqrt{(f')^2 + (h')^2}}\,\big(h', -f'\cos v, -f'\sin v\big)$$

Let us now consider the second way of representing a surface in three-dimensional Euclidean space, the so-called implicit way of representing a surface. Although this method is based on equation $f(x,y,z) = c$, the set of solutions of which gives a set of surface points, we start with a more general definition, which later leads to the concept of manifold.

Let $\mathfrak{S} \subset E^3$ be a subset of a three-dimensional Euclidean space E^3.

Definition 3.2. A subset $\mathfrak{S}$ of a three-dimensional Euclidean space E^3 is called a *surface* if for any point $p \in \mathfrak{S}$ there exists a neighborhood V of this point in E^3 and a local coordinate surface (ϕ, U) such that $\phi : U \to \mathfrak{S} \cap V$ is onto mapping.

The geometric meaning of this definition is that a surface consists of several local coordinate surfaces (sometimes called patches), which in some places overlap (places of the so-called gluing) and together they form a surface. This definition says what the structure of a set called a surface should be, but it does not say anything about how a surface in three-dimensional space E^3 can be represented. Therefore, we will consider a method to represent a surface by means of equation $f(x, y, z) = c$ and prove that, under a certain condition on a function f, the set of solutions to this equation forms a surface in the sense of Definition 3.2.

Let $W \subset E^3$ be an open subset, $f \in C^\infty(W)$ be a smooth function and $c \in \mathbb{R}$ be a real number. We consider the equation $f(x, y, z) = c$. The set of all solutions of this equation will be denoted by $f^{-1}(c)$, that is, this set consists of all points in space E^3 at which the value of a function f is equal to c. We will assume that this set is non-empty, i.e. an equation $f(x, y, z) = c$ has at least one solution and call this set *level surface* of a function f. Obviously $f^{-1}(c) \subset W$.

Proposition 3.1. Let $\mathfrak{S} = f^{-1}(c)$ be a level surface of a function f and $p \in \mathfrak{S}$ be a point of this level surface such that the gradient of a function f is non-zero at p, that is, $\nabla f|_p \neq 0$. If $\xi : I \to \mathfrak{S}$ is a parametrized curve on a level surface $\mathfrak{S}$, which passes through a point p, that is, $\xi(t_0) = p, t_0 \in I$ then the velocity vector $\xi'|_p$ of this curve at a point p is orthogonal to the gradient $\nabla f|_p$.

Proof. Let $\xi = \xi(t) = (x(t), y(t), z(t)), \xi(t_0) = p$. Since a parametrized curve $\xi(t)$ lies on a level surface $\mathfrak{S}$ we have $f(\xi(t)) = c$ for any $t \in I$. Differentiating this equation with respect to t and putting $t = t_0$ we get

$$\frac{d}{dt} f(\xi(t))\Big|_{t=t_0} = \frac{\partial f}{\partial x}\Big|_p \frac{dx}{dt}\Big|_{t=t_0} + \frac{\partial f}{\partial y}\Big|_p \frac{dy}{dt}\Big|_{t=t_0} + \frac{\partial f}{\partial z}\Big|_p \frac{dz}{dt}\Big|_{t=t_0} = 0. \tag{3.8}$$

Taking into account that

$$\xi'|_p = (p; \frac{dx}{dt}\Big|_{t=t_0}, \frac{dy}{dt}\Big|_{t=t_0}, \frac{dz}{dt}\Big|_{t=t_0}),$$

is the velocity vector of a curve ξ at a point p we can consider the middle part of (3.8) as a scalar product of two vectors, where first is the vector of gradient

$$\nabla f|_p = (p; \frac{\partial f}{\partial x}\Big|_p, \frac{\partial f}{\partial y}\Big|_p, \frac{\partial f}{\partial z}\Big|_p),$$

and the second is the velocity vector $\xi'|_p$. Hence

$$< \nabla f|_p, \xi'|_p >= 0.$$

Thus $\nabla f|_p \perp \xi'|_p$. □

The proved proposition shows that a tangent vector to a level surface $f^{-1}(c)$ at a point $p \in f^{-1}(c)$ is orthogonal to the gradient of a function f at this point. Thus, the gradient of a function f can be considered as a normal vector to a level surface $f^{-1}(c)$ at some point, and the plane, which is orthogonal to the gradient ∇f at this point, can be considered as a tangent plane to a level surface. We will denote a tangent plane of a level surface $\mathfrak{S} = f^{-1}(c)$ at a point p by $T_p\mathfrak{S}$. Thus

$$T_p\mathfrak{S} = \{\mathtt{v}_p \in T_pE^3 :< \mathtt{v}_p, \nabla f|_p >= 0\}.$$

Thus, we see that the condition for the existence of a tangent plane to a level surface of a function f is that the gradient of this function differs from zero. Therefore, a point on a level surface $f^{-1}(c)$ will be called *regular (nonsingular)* if the gradient ∇f at this point is nonzero. If this condition is satisfied for any point of a level surface, we will call such a level surface *regular.*

Theorem 3.1. A regular level surface is a surface.

Proof. Let $f^{-1}(c)$ be a regular level surface and $p(x_0, y_0, z_0) \in f^{-1}(c)$. This means that $f(x_0, y_0, z_0) = c$. It follows from the assumption that the gradient of a function f is nonzero that at least one of the partial derivatives of this function f'_x, f'_y, f'_z at a point p is nonzero. Without loss of generality, we assume that $f'_z|_p \neq 0$. Now we can apply the implicit function theorem, since all the conditions of this theorem are satisfied. Therefore, there exists in E^3 a neighborhood V of point p, in which the equation $f(x, y, z) = c$ defines z as a single-valued smooth function of the variables x, y, that is, $z = \varphi(x, y)$. This means that if we substitute the function $\varphi(x, y)$ instead of z in a function $f(x, y, z)$, we get the identical equality $f(x, y, \varphi(x, y)) \equiv c$. The value of $\varphi(x, y)$ at a point $q = (x_0, y_0)$ of xy-coordinate plane is z_0, i.e $\varphi(x_0, y_0) = z_0$. Let $\tilde{U}$ denote the intersection of sets V and $f^{-1}(c)$, i.e. $\tilde{U} = V \cap f^{-1}(c)$. Let U be the orthogonal projection of $\tilde{U}$ onto the xy-coordinate plane. Now consider a parametrized surface (ϕ, U), where U is an open subset of the plane (in this case, it is the xy-coordinate plane), and $\phi : U \to E^3$ is a mapping of U into three-dimensional space E^3 defined by $\phi(x, y) = (x, y, \varphi(x, y))$. Obviously, ϕ is a bijective smooth mapping of U onto $\tilde{U} = V \cap f^{-1}(c)$. Thus, (ϕ, U) is a local coordinate surface for $f^{-1}(c)$ and $\phi : U \to \tilde{U} = V \cap f^{-1}(c)$ is onto mapping and the theorem is proved. □

3.2 First fundamental form of a surface

Let us remind that E^3 is a three-dimensional Euclidean space, which means that we have the concept of length of a segment, the concept of angle between straight lines and the formula for calculating the lengths and angles in Cartesian coordinates. A surface $\mathfrak{S}$ is in this three-dimensional space E^3 and, therefore, we can use the concepts of length and angle between straight lines in the case of a surface. But we can not just apply them to surface points, we must first adapt them for use on a surface. In other words, if we have two points on a surface, we can connect them with a straight line and calculate the length of the resulting segment. But this is not very interesting, because our goal is to measure the length of a segment lying on the surface. A surface is generally not a linear object, such as a plane. Therefore, the line segments connecting points of a surface and lying on the surface will be curves. We already know that the calculation of the arc length of a curve should start with the differential of an arc, that is, with an infinitesimal element of an arc, and then, by integration, calculate the finite lengths. Since the concept of a length of a segment comes to our surface from the surrounding Euclidean space E^3, we must start with an infinitesimal length in space E^3.

Let $d\vec{r} = (dx, dy, dz)$ be an infinitesimal vector in E^3. Then the square of its length is $||d\vec{r}||^2 = dx^2 + dy^2 + dz^2$. This expression is often called the infinitesimal metric (or simply the metric) of Euclidean space E^3 in Cartesian coordinates x, y, z. Suppose that a surface

$\mathfrak{S}$ is parametrized in a neighborhood of some of its point p by a local coordinate surface (ϕ, U), where

$$\phi(u,v) = (x(u,v), y(u,v), z(u,v))$$

is a parametric equation of a part $\phi(U) \subset \mathfrak{S}$ of a surface. Then the differentials of coordinates x, y, z on a local coordinate surface (ϕ, U) can be expressed in terms of differentials of parameters u, v as follows

$$\begin{aligned} dx &= x'_u\, du + x'_v\, dv, \\ dy &= y'_u\, du + y'_v\, dv, \\ dz &= z'_u\, du + z'_v\, dv. \end{aligned}$$

Substituting the obtained expressions in $dx^2 + dy^2 + dz^2$, we get

$$dx^2 + dy^2 + dz^2 = ||\phi'_u||^2\, du^2 + 2 < \phi'_u, \phi'_v > du\, dv + ||\phi'_v||^2\, dv^2,$$

where

$$\begin{aligned} ||\phi'_u||^2 &= (x'_u)^2 + (y'_u)^2 + (z'_u)^2, \\ < \phi'_u, \phi'_v > &= x'_u\, x'_v + y'_u\, y'_v + z'_u\, z'_v, \\ ||\phi'_v||^2 &= (x'_v)^2 + (y'_v)^2 + (z'_v)^2. \end{aligned}$$

We will use the following notations introduced by K. F. Gauss

$$E = ||\phi'_u||^2, \quad F =< \phi'_u, \phi'_v >, \quad G = ||\phi'_v||^2. \tag{3.9}$$

Using these notations, we can write the square of the length of an infinitesimal vector $d\vec{r}$ as follows

$$dx^2 + dy^2 + dz^2 = E\, du^2 + 2\, F\, du\, dv + G\, dv^2. \tag{3.10}$$

The expression on the right-hand side of equality (3.10) is called a *first fundamental form of a surface* or *Riemannian metric of a surface*. From an algebraic point of view, this expression is a quadratic form of the differentials du, dv of the surface parameters and we will denote this quadratic form by

$$\Psi_1(du, dv) = E\, du^2 + 2\, F\, du\, dv + G\, dv^2. \tag{3.11}$$

The symmetric matrix of the first fundamental form will be denoted by Ψ_1, i.e.

$$\Psi_1 = \begin{pmatrix} E & F \\ F & G \end{pmatrix}.$$

Let us clarify the geometric meaning of the first fundamental form of a surface. Let $(\xi = \xi(t), I), \xi(t) = (u(t), v(t))$ be a parametrized curve in $U \subset E^2$ and ξ' its velocity vector field. Then $(\tilde{\xi} = \tilde{\xi}(t), I)$, where $\tilde{\xi} = \phi \circ \xi$, is the induced parametrized curve in a parametrized surface (ϕ, U) and $\tilde{\xi}'$ is velocity vector field along $\tilde{\xi}$. Hence

$$\tilde{\xi}' = u'\, \phi'_u + v'\, \phi'_v,$$

and

$$\|\tilde{\xi}'\|^2 = ||\phi'_u||^2\, (u')^2 + 2 < \phi'_u, \phi'_v > u'v' + ||\phi'_v||^2\, (v')^2.$$

Multiplying both sides of this equality by dt^2 and making use of $du = u'\, dt, dv = v'\, dt$, we get

$$\|\tilde{\xi}'\|^2\, dt^2 = E\, du^2 + 2\, F\, du\, dv + G\, dv^2.$$

On the left-hand side of this equality we have the square of the differential of an arc (1.19) of parametrized curve $\tilde{\xi}$ in a surface $\mathfrak{S}$. Thus, we obtain

$$ds^2 = E\,du^2 + 2\,F\,du\,dv + G\,dv^2.$$

Therefore, using integration, i.e.

$$s_{\tilde{\xi}} = \int_{t_0}^{t_1} \sqrt{E\,du^2 + 2\,F\,du\,dv + G\,dv^2}$$

we can calculate the lengths of curves on surface $\mathfrak{S}$. This means that the first fundamental form belongs to the intrinsic geometry of a surface.

The first fundamental form of a surface can be viewed from a slightly different point of view. It is well known that a scalar product in Euclidean space is a positive definite symmetric bilinear form on vectors. If we restrict this bilinear form to a tangent plane to a surface, we get a bilinear form on the tangent vectors and this bilinear form can also be called the first fundamental form of a surface. Thus, in order to compute the value of the first fundamental form on tangent vectors $\mathtt{v}, \mathtt{w} \in T_p\mathfrak{S}$, we must compute their scalar product $< \mathtt{v}, \mathtt{w} >$. If we compute this scalar product in the basis $\{\phi'_u, \phi'_v\}$, where $\mathtt{v} = a_1\phi'_u + a_2\phi'_v, \mathtt{w} = b_1\phi'_u + b_2\phi'_v$, we get

$$< \mathtt{v}, \mathtt{w} >= E\,a_1b_1 + F\,(a_1b_2 + a_2b_1) + G\,a_2b_2.$$

The first fundamental form of a surface can be used to compute the area of a region defined on the surface. Let (ϕ, U) be a parametrized coordinate surface, where the parameters are denoted by u, v. Suppose a certain region $\phi(D)$ is given on this surface, where $D \subset U$ and the boundary of D is a piecewise-smooth, closed, simple curve lying in U.

We cover this region with a grid of coordinate u-lines and v-lines. This grid divides the surface region into curvilinear quadrilaterals, which we assume to be sufficiently small. Each such quadrilateral can be approximated by a corresponding parallelogram formed by the vectors $\phi'_u\,du + \phi'_v\,dv$, where du, dv describe the displacement from the vertex (u, v) of the quadrilateral to its neighboring vertices $(u+du, v)$ and $(u, v+dv)$. These displacements represent the transition from one coordinate line to the adjacent one.

The area of the parallelogram is given by $\|\phi'_u \times \phi'_v\|$, which provides a good approximation of the area of the curvilinear quadrilateral on the surface. Summing up the areas of all these parallelograms and taking the limit as the grid refinement tends to infinity, we obtain the surface area of the region.

Recall the identity:

$$\|\vec{a} \times \vec{b}\|^2 = \|\vec{a}\|^2\|\vec{b}\|^2 - \langle\vec{a}, \vec{b}\rangle^2.$$

Applying this identity, we get:

$$\|\phi'_u \times \phi'_v\|^2 = \|\phi_u\|^2\,\|\phi_v\|^2 - \langle\phi_u, \phi_v\rangle^2 = E\,G - F^2.$$

The limiting process described above leads to a double integral. Thus, the surface area of the region is computed using formula

$$\int\int_D \sqrt{EG - F^2}\,dudv. \tag{3.12}$$

3.3 Shape operator of a surface

An important tool in the study of a local structure of a surface is a shape operator. A shape operator underlies concepts such as a normal curvature of a surface and principal curvatures of a surface. A notion of shape operator is based on the idea of studying the rate of change of a unit normal vector field of a surface along a curve lying on this surface. A logical reasoning in this case is similar to the reasoning in the case of the natural parametrization of a curve, that is, since the length of a vector does not change when moving along a curve, only its direction changes and the rate of change of direction of a vector gives us information about a curvature of a surface along this curve.

Let $\mathfrak{S}$ be a surface, $p \in \mathfrak{S}$ be a point of this surface and (ϕ, U) be a local coordinate surface in a neighborhood of p. Then we have the unit normal field on a parametrized surface (ϕ, U) induced by cross product of tangent vector fields ϕ'_u and ϕ'_v. This unit normal vector field will be denoted by $\mathtt{N}$. The tangent plane of a surface $\mathfrak{S}$ will be denoted by $T_p\mathfrak{S}$. Assume that we have a curve $(\xi = \xi(t), I)$ lying on a local coordinate surface $\phi(U) \subset \mathfrak{S}$ and passing through a point p when $t = 0$. The velocity vector of this curve at a point p will be denoted by $\mathtt{v}$. Hence $\xi'(t)|_{t=0} = \mathtt{v}$ and $\mathtt{v} \in T_p\mathfrak{S}$. At each point of a curve ξ, we have a unit normal vector of surface $\mathfrak{S}$. The restriction of the unit normal vector field $\mathtt{N}$ to a curve ξ is denoted by $\mathtt{N}(t)$, that is, $\mathtt{N}|_{\xi(t)} = \mathtt{N}(t)$. Obviously $\mathtt{N}(t)$ is a vector field along a curve ξ. Now our goal is to investigate how fast the vector field $\mathtt{N}(t)$ changes in a neighborhood of a point p along a curve ξ. To do this, we use the derivative of vector field $\mathtt{N}(t)$ with respect to parameter t. The derivative $\mathtt{N}'(t)$ is also a vector field along a curve ξ. As $\mathtt{N}(t)$ is a unit vector field it follows from Proposition 1.4 that $\mathtt{N}'(t)$ is orthogonal to $\mathtt{N}(t)$. This means that $\mathtt{N}'(t)$ is a tangent vector field to a surface $\mathfrak{S}$. The derivative $\mathtt{N}'(t)$ calculated at a point p, i.e. when $t = 0$, will give us a characteristic of instantaneous rate of change of direction of the normal vector field at a point p in the direction of tangent vector $\mathtt{v}$. This derivative is referred to as a *covariant derivative* of a normal vector field $\mathtt{N}$ at a point p in the direction of a tangent vector $\mathtt{v}$. We will denote covariant derivative by $\nabla_{\mathtt{v}}\mathtt{N}$. Hence

$$\nabla_{\mathtt{v}}\mathtt{N} = \frac{d}{dt}\Big(\mathtt{N}(t)\Big)\Big|_{t=0}.$$

There are two important points to note here. The first is that $\nabla_{\mathtt{v}}\mathtt{N}$ is a vector, and the second is that it is a tangent vector to surface $\mathfrak{S}$ at a point p, i.e. $\nabla_{\mathtt{v}}\mathtt{N} \in T_p\mathfrak{S}$.

Thus, the covariant derivative of the normal vector field of a surface $\mathfrak{S}$ determines the transformation of the tangent plane

$$\mathtt{v} \in T_p\mathfrak{S} \;\rightarrow\; \nabla_{\mathtt{v}}\mathtt{N} \in T_p\mathfrak{S}.$$

First of all, let us show that this transformation is linear. Let

$$(\xi = \xi(t), I), \xi(t) = (u(t), v(t)), \xi(I) \subset U,$$

be a parametrized curve in parameters plane E^2. Assume this curve passes a point $q = \xi(t_0), t_0 \in I$ and $\mathtt{w} = \xi'(t)|_{t=t_0} = (w_1, w_2)$, where $w_1 = u'(t)|_{t=t_0}, w_2 = v'(t)|_{t=t_0}$, is the velocity vector of this curve at a point q. Denote by $(\tilde{\xi} = \tilde{\xi}(t), I)$, where $\tilde{\xi} = \phi \circ \xi$, induced parametrized curve on a surface $\mathfrak{S}$ and $p = \phi(q)$. Then the coordinates of the velocity vector $\mathtt{v} = \tilde{\xi}'(t)|_{t=t_0}$ of induced curve $\tilde{\xi}$ at the point in the basis $\{\phi'_u|_p, \phi'_v|_p\}$ p are w_1, w_2, i.e.

$$\mathtt{v} = w_1\, \phi'_u|_p + w_2\, \phi'_v|_p.$$

The covariant derivative can be expressed as follows

$$\nabla_{\mathtt{v}}\, \mathtt{N} = w_1\, \mathtt{N}'_u|_p + w_2\, \mathtt{N}'_v|_p.$$

The above formula shows that the covariant derivative $\nabla_{\mathtt{v}}\mathtt{N}$ of the normal vector field $\mathtt{N}$ in the direction of a tangent vector $\mathtt{v}$ linearly depends on the coordinates w_1, w_2 of vector $\mathtt{v}$ and therefore it has the property

$$\nabla_{a\mathtt{v}+\mathtt{w}}\mathtt{N} = a\,\nabla_{\mathtt{v}}\mathtt{N} + \nabla_{\mathtt{w}}\mathtt{N},$$

where $a \in \mathbb{R}$ and $\mathtt{v}, \mathtt{w} \in T_p\mathfrak{S}$.

Definition 3.3. The linear transformation of tangent plane $T_p\mathfrak{S}$ of a surface

$$S : \mathtt{v} \in T_p\mathfrak{S} \ \to\ S(\mathtt{v}) = -\nabla_{\mathtt{v}}\mathtt{N} \in T_p\mathfrak{S}$$

is called a *shape operator* of a surface $\mathfrak{S}$ at a point p.

Theorem 3.2. If $\mathtt{v}, \mathtt{w} \in T_p\mathfrak{S}$ are tangent vector to a surface $\mathfrak{S}$ at a point p then

$$< S(\mathtt{v}), \mathtt{w} > = < \mathtt{v}, S(\mathtt{w}) >,$$

i.e. the shape operator is symmetric.

Proof. We parametrize a surface $\mathfrak{S}$ in a neighborhood of a point p using a local coordinate surface (ϕ, U). Then the tangent vectors ϕ'_u, ϕ'_v to a surface $\mathfrak{S}$ at a point p form the basis for the tangent plane $T_p\mathfrak{S}$. Hence tangent vectors $\mathtt{v}, \mathtt{w}$ at a point p is a linear combination of the vectors of basis $\mathtt{v} = a\,\phi'_u + b\,\phi'_v, \mathtt{w} = c\,\phi'_u + d\,\phi'_v$, where a, b, c, d are real numbers. One side we have

$$\begin{aligned} < S(\mathtt{v}), \mathtt{w} > &= < S(a\,\phi'_u + b\,\phi'_v), c\,\phi'_u + d\,\phi'_v > \\ &= ac\, < S(\phi'_u), \phi'_u > +ad\, < S(\phi'_u), \phi'_v > \\ &\qquad\qquad +bc\, < S(\phi'_v), \phi'_u > +bd\, < S(\phi'_v), \phi'_v > . \end{aligned}$$

On the other side we have

$$\begin{aligned} < \mathtt{v}, S(\mathtt{w}) > &= < a\,\phi'_u + b\,\phi'_v, S(c\,\phi'_u + d\,\phi'_v) > \\ &= ac\, < \phi'_u, S(\phi'_u) > +ad\, < \phi'_u, S(\phi'_v) > \\ &\qquad\qquad +bc\, < \phi'_v, S(\phi'_u) > +bd\, < \phi'_v, S(\phi'_v) > . \end{aligned}$$

Subtracting the second from the first, we get

$$< S(\mathtt{v}), \mathtt{w} > - < \mathtt{v}, S(\mathtt{w}) > = (ad - bc)\big(< S(\phi'_u), \phi'_v > - < \phi'_u, S(\phi'_v) > \big).$$

Thus, the assertion of the theorem will be proved if we show that

$$< S(\phi'_u), \phi'_v > = < \phi'_u, S(\phi'_v) > .$$

Consider the left-hand side of this formula. It is easy to see that $S(\phi'_u) = -\mathtt{N}_u$. Indeed, by the definition of the shape operator, we should calculate the derivative of the normal vector field $\mathtt{N}$ in the direction of the tangent vector ϕ'_u, that is, along the u-coordinate line. But this will be the partial derivative of the normal vector field $\mathtt{N}$ with respect to parameter u. Similarly $S(\phi'_v) = -\mathtt{N}_v$. Since the normal vector field is orthogonal to a tangent plane we have $< \mathtt{N}, \phi'_v) > = 0$. Differentiating both sides with respect to u, we get

$$< \mathtt{N}_u, \phi'_v) > + < \mathtt{N}, \phi''_{vu}) > = 0.$$

So, from the formulas $-<\mathtt{N}_u, \phi'_v)>=<\mathtt{N}, \phi'_{vu})>=$ and $S(\phi'_u) = -\mathtt{N}_u$ it follows that

$$<S(\phi'_u), \phi'_v>=-<\mathtt{N}_u, \phi'_v>=<\mathtt{N}, \phi'_{vu}> . \tag{3.13}$$

Analogously differentiating $<\mathtt{N}, \phi'_u)>=0$ with respect to v we obtain

$$<\mathtt{N}_v, \phi'_u> + <\mathtt{N}, \phi'_{uv}>=0.$$

Hence $-<\mathtt{N}_v, \phi'_u)>=<\mathtt{N}, \phi'_{uv})>=$ and $S(\phi'_v) = -\mathtt{N}_v$ give

$$<\phi'_u, S(\phi'_v)>=-<\phi'_u, \mathtt{N}_v>=<\mathtt{N}, \phi'_{uv}> . \tag{3.14}$$

Comparing (3.13) with (3.14) and taking into account that the result of differentiation does not depend on the order in which the partial derivatives are taken, we get

$$<S(\phi'_u), \phi'_v>=<\phi'_u, S(\phi'_v)> .$$

□

3.4 Principal curvatures of a surface

At the beginning of the previous section, we noted that a shape operator of a surface refers to a curvature of a surface in some direction. However, it is important to have a numerical characteristic of the curvature. First of all, note that the covariant derivative $\nabla_{\mathtt{v}}\mathtt{N}$ is a vector that lies in the tangent plane of the surface $T_p\mathfrak{S}$. It can be decomposed into two components, one of which will be an orthogonal projection to the direction of vector $\mathtt{v}$, and the other to an orthogonal direction $\mathtt{v}^{\perp}$. The projection to the orthogonal direction $\mathtt{v}^{\perp}$ has no relation to a curvature of a surface in the direction of vector $\mathtt{v}$ and we can discard this component. The second component is directly related to a curvature of a surface in direction $\mathtt{v}$ and the length of this projection is a numerical characteristic of a curvature of a surface. An orthogonal projection is easiest to calculate if we have a unit vector of a given direction, then a scalar product with such a vector will give an orthogonal projection onto that direction. This prompts us to consider the set of unit vectors in the tangent plane $T_p\mathfrak{S}$. Obviously, the end points of these vectors will lie on a unit circle centered at point p. We will denote this unit circle by $\mathbb{S}^1_p$. Thus

$$\mathbb{S}^1_p = \{\mathtt{v} \in T_p\mathfrak{S} : ||v|| = 1\}.$$

Definition 3.4. A *normal curvature* of a surface $\mathfrak{S}$ at a point p in the direction of a tangent vector $\mathtt{v}$ is called $\kappa(\mathtt{v}) =<S(\mathtt{v}), \mathtt{v}>$, where $\mathtt{v} \in \mathbb{S}^1_p$.

A normal curvature of a surface $\mathfrak{S}$ at point p can be viewed as a function on the unit circle $\mathbb{S}^1_p$, that is, $\kappa : \mathbb{S}^1 \to \mathbb{R}$. Geometrically, the meaning of this function is quite clear, it calculates the projection of the covariant derivative of the normal vector field $\nabla_{\mathtt{v}}\mathtt{N}$ (taken with minus) onto a tangent vector $\mathtt{v}$. Since this function is defined using a linear transformation and scalar product, this function is smooth. In particular, it is continuous. But a continuous function on a compact set, in this case the unit circle $\mathbb{S}^1_p$, reaches its maximum and minimum value.

Definition 3.5. Those unit vectors in $\mathbb{S}^1_p$ on which function κ reaches its maximum or minimum value are called *principal directions* at a point p of a surface $\mathfrak{S}$. The values of the function κ on these vectors (or at the points of the circle $\mathbb{S}^1_p$ determined by the endpoints of these vectors) are called the *principal curvatures* of a surface $\mathfrak{S}$ at a point p. In the case when a normal curvature of a surface $\mathfrak{S}$ at a point p is constant, that is, does not depend on a tangent vector $\mathtt{v}$, then point p is called an *umbilical point.*

It should be noted here that a normal curvature has the property

$$\kappa(-\mathtt{v}) =< S(-\mathtt{v}), -\mathtt{v} >=< S(\mathtt{v}), \mathtt{v} >= \kappa(\mathtt{v}). \tag{3.15}$$

Therefore, a normal curvature of a surface does not depend on a tangent vector $\mathtt{v}$, but on the straight line it defines in a tangent plane. It is for this reason that the term principal direction is used. The definition of the principal curvatures of a surface has a transparent geometric meaning, which consists in the fact that the principal curvatures give the maximal and minimal values of a normal curvature of a surface at some point. All other values of a normal curvature of a surface lie between these two values. However, this definition is difficult to use to calculate the principal curvatures of a surface. Therefore, in the next theorem, we will prove that the principal curvatures are the eigenvalues of the shape operator of a surface and this will provide a practical method for calculating the principal curvatures.

Theorem 3.3. Let $\mathfrak{S}$ be a surface, p a point of this surface. Then the principal curvatures and the principal directions of a surface $\mathfrak{S}$ at a point p are the eigenvalues and the eigenvectors of the shape operator $S : T_p\mathfrak{S} \to T_p\mathfrak{S}$ respectively.

Proof. In the proof, it is convenient to use the identification of vectors of a tangent plane $T_p\mathfrak{S}$ with complex numbers. First of all, we will construct the axis of the Cartesian coordinate system on a tangent plane $T_p\mathfrak{S}$. As x-axis, we choose one of the principal directions in a tangent plane $T_p\mathfrak{S}$, namely, the one on which the value of the normal curvature κ is minimal. The unit directional vector of x-axis will be denoted by $\mathtt{e}_1$ and $\kappa(\mathtt{e}_1) = \kappa_1$ is the minimal value of normal curvature. Hence κ_1 is one of principal curvatures and for any $\mathtt{v} \in \mathbb{S}^1_p$ it holds $\kappa(\mathtt{v}) \geq \kappa_1$. As the y-axis, we take a straight line which passes through a point p and whose directional vector is $\mathtt{e}_2 = J(\mathtt{e}_1)$, where J is the complex structure of a tangent plane. Now, to each vector $\mathtt{v} = x\,\mathtt{e}_1 + y\,\mathtt{e}_2$ of a tangent plane with the origin at point p, we associate a complex number $z = x + i\,y$. Obviously this mapping $\mathtt{v} \to z$ is an isomorphism of vector spaces. If $\mathtt{v}_1 \to z_1 = x_1 + i\,y_1$ and $\mathtt{v}_2 \to z_2 = x_2 + i\,y_2$ then the scalar product $< \mathtt{v}_1, \mathtt{v}_2 >$ can be expressed in terms of corresponding complex numbers as follows

$$z_1 \cdot z_2 = \frac{1}{2}(z_1\,\bar{z}_2 + \bar{z}_1\,z_2).$$

The shape operator S of a surface $\mathfrak{S}$ at a point p can be given in terms of complex numbers as follows

$$S(z) = \zeta\,z - \theta\,\bar{z}, \tag{3.16}$$

where $\zeta, \theta \in \mathbb{C}$ are constant complex numbers. The matrix of the shape operator S (we denote it by the same letter S) in the basis $\{\mathtt{e}_1, \mathtt{e}_2\}$ has the form

$$S = \begin{pmatrix} \Re(\zeta - \theta) & -\Im(\zeta + \theta) \\ \Im(\zeta - \theta) & \Re(\zeta + \theta) \end{pmatrix}, \tag{3.17}$$

where $\Re, \Im$ stand for the real and imaginary components of a complex number. As the shape operator is symmetric one can easily show that this leads to $\Im\,\zeta = 0$, that is, ζ in the case of a symmetric operator is real number $\zeta \in \mathbb{R}$. Thus the matrix of the shape operator takes on the form

$$S = \begin{pmatrix} \zeta - \Re\,\theta & -\Im\,\theta \\ -\Im\,\theta & \zeta + \Re\,\theta \end{pmatrix}. \tag{3.18}$$

Using a complex coordinate of a plane makes it easy to parametrize the unit circle $\mathbb{S}^1_p$ as follows $z = e^{i\varphi}$. With this parametrization of the unit circle $\mathbb{S}^1_p$, the normal curvature becomes an explicit function of φ of the form

$$\begin{aligned}\kappa_p(\varphi) &= < S(e^{i\varphi}), e^{i\varphi} >= \frac{1}{2}\big((\bar{\zeta}\, e^{-i\varphi} - \bar{\theta}\, e^{i\varphi})\, e^{i\varphi} + (\zeta\, e^{i\varphi} - \theta\, e^{-i\varphi})\, e^{-i\varphi}\big) \\ &= \frac{1}{2}(\zeta + \bar{\zeta}) - \frac{1}{2}(\bar{\theta}\, e^{2i\varphi} + \theta\, e^{-2i\varphi}) = \zeta - \Re(\theta\, e^{-2i\varphi}).\end{aligned}$$

Due to the property (3.15) of normal curvature, it suffices to consider this function on the segment $0 \le \varphi < \pi$. Recall that, according to the assumption made at the beginning of the proof, at point $\varphi = 0$ the function $\kappa_p(\varphi)$ reaches its minimum on the segment $[0, \pi)$. Hence the derivative of the function $\kappa_p(\varphi)$ at this point should be zero. We have

$$\kappa_p'(\varphi) = 2\,\Re(i\,\theta\, e^{-2i\varphi}), \quad \kappa_p'(\varphi)\big|_{\varphi=0} = 2\,\Re(i\,\theta).$$

From the equality to zero of the derivative $\kappa_p'(\varphi)$ at the point $\varphi = 0$ it follows that $\Re(i\,\theta) = 0$. Consequently the imaginary part of the number θ must be equal to zero, that is, $\Im\theta = 0$. Hence $\theta \in \mathbb{R}$ is a real number. Taking into account the obtained result, we see that the matrix of the shape operator (3.28) becomes diagonal, i.e.

$$S = \begin{pmatrix} \zeta - \theta & 0 \\ 0 & \zeta + \theta \end{pmatrix}.$$

This means that the vectors $\mathsf{e}_1, \mathsf{e}_2$ of the tangent plane $T_p\mathfrak{S}$ are eigenvectors of the shape operator operator S. Hence the principal direction e_1 is an eigenvector of the shape operator S. From the form of matrix of the shape operator it also follows that the eigenvalue of the shape operator in the case of eigenvector e_1 is equal to $\zeta - \theta$. Hence $\kappa_1 =< S(\mathsf{e}_1), \mathsf{e}_1 >= (\zeta - \theta)\|\mathsf{e}_1\| = \zeta - \theta$ and this is the minimum of the function $\kappa(\varphi)$ on the segment $[0, \pi)$. Taking into account that θ is a real number, we get that function $\kappa(\varphi)$ has the form

$$\kappa(\varphi) = \zeta - \theta\,\Im(e^{-2i\varphi}) = \zeta - \theta\,\cos 2\varphi, 0 \le \varphi < \pi. \tag{3.19}$$

It is easy to show that $\theta > 0$. Indeed, $\kappa_1 = \zeta - \theta$ is a minimum value of $\kappa(\varphi)$ when $\varphi = 0$, hence $\zeta - \theta\,\cos 2\varphi > \zeta - \theta$ for any $\varphi \in (0, \pi)$ and, consequently, $\theta(1 - \cos 2\varphi) > 0$. Making use of $1 - \cos 2\varphi = 2\,\sin^2\varphi$, we get $\theta \sin^2\varphi > 0$, thus $\theta > 0$. Taking this into account, as well as the graph of function $\cos 2\varphi$, we see that function $\kappa(\varphi)$ has a maximum at point $\varphi = \pi/2$ of the open segment $(0, \pi)$ and it is equal to $\zeta + \theta$. But point $\varphi = \pi/2$ corresponds to the second basis vector e_2. This means that this is the second principal direction of the shape operator and at the same time it is the eigenvector of the shape operator. Thus $\kappa_2 = \zeta + \theta$ is the second principal curvature (maximal). $\square$

In the proof of Theorem (3.3) we obtained the formula for normal curvature function $\kappa(\varphi) = \zeta - \theta\,\cos 2\varphi$. We write this function as

$$\kappa(\varphi) = \zeta\,(\cos^2\varphi + \sin^2\varphi) - \theta\,(\cos^2\varphi - \sin^2\varphi) = (\zeta - \theta)\cos^2\varphi + (\zeta + \theta)\sin^2\varphi.$$

Taking into account that $\kappa_1 = \zeta - \theta, \kappa_2 = \zeta + \theta$ are principal curvatures we get

$$\kappa(\varphi) = \kappa_1 \cos^2\varphi + \kappa_2 \sin^2\varphi, \tag{3.20}$$

and this is called *Euler curvature formula.* Thus, the normal curvature $\kappa(\varphi)$ of a surface $\mathfrak{S}$ in the direction that is determined by an angle ϕ, counted counterclockwise from the principal direction corresponding to the minimal value of normal curvature, is expressed by formula (3.20).

Exercises

- **Exercise 3.4.1** Find the principal directions and principal curvatures of the right helicoid $\phi(u,v) = (u\cos v,\ u\sin v,\ av)$, where $a > 0$ is a constant. See solution on page 103.

3.5 Second fundamental form of a surface.

Definition 3.6. Let $\mathfrak{S}$ be a surface, p be a point on this surface, $T_p\mathfrak{S}$ be the tangent plane of $\mathfrak{S}$ at a point p and $\mathtt{v}, \mathtt{w}$ be two tangent vectors. Second fundamental form of a surface $\mathfrak{S}$ at a point p is a bilinear form $\Psi_2 : T_p\mathfrak{S} \times T_p\mathfrak{S} \to \mathbb{R}$ defined by

$$\Psi_2(\mathtt{v}, \mathtt{w}) = < S(\mathtt{v}), \mathtt{w} > .$$

Let us derive the form of Ψ_2 in local parameters u, v of a surface $\mathfrak{S}$, that is, we assume that in a neighborhood of a point p a surface $\mathfrak{S}$ is parametrized with the help of a local coordinate surface (ϕ, U). To express the second fundamental form Ψ_2 in terms of the local parameters u, v and their differentials du, dv, we use an infinitesimal tangent vector

$$d\phi = du\,\phi'_u + dv\,\phi'_v.$$

We have

$$\begin{aligned} \Psi_2(d\phi, d\phi) &= < S(du\,\phi'_u + dv\,\phi'_v), du\,\phi'_u + dv\,\phi'_v > \\ &= < du\,S(\phi'_u) + dv\,S(\phi'_v), du\,\phi'_u + dv\,\phi'_v > \\ &= du^2\ < S(\phi'_u), \phi'_u > + 2\,dudv\ < S(\phi'_u), \phi'_v > + < S(\phi'_v), \phi'_v > . \end{aligned}$$

Let us introduce the notations

$$l = < S(\phi'_u), \phi'_u >,\ m = < S(\phi'_u), \phi'_v >,\ n = < S(\phi'_v), \phi'_v > . \tag{3.21}$$

Using these notation, the second fundamental form can be written as follows

$$\Psi_2(du, dv) = l\,du^2 + 2m\,dudv + n\,dv^2. \tag{3.22}$$

The symmetric matrix of the second fundamental form will be denoted by Ψ_2, i.e.

$$\Psi_2 = \begin{pmatrix} l & m \\ m & n \end{pmatrix} \tag{3.23}$$

3.6 The Weingarten equations

In this section, we will calculate the matrix of the shape operator of a surface in the basis $\{\phi'_u, \phi'_v\}$ of a tangent plane of a surface. The corresponding formulas are called the Weingarten equations.

Let $\mathfrak{S}$ be a surface, $p \in \mathfrak{S}$ be a point of this surface, (ϕ, U) be a local coordinate surface in a neighborhood of this point. Then tangent vectors ϕ'_u, ϕ'_v form the basis for a tangent plane $T_p\mathfrak{S}$. Our aim is to calculate the matrix of the shape operator in the basis $\{\phi'_u, \phi'_v\}$. We have

$$S(\phi'_u) = S_{11}\,\phi'_u + S_{21}\,\phi'_v, \tag{3.24}$$
$$S(\phi'_v) = S_{12}\,\phi'_u + S_{22}\,\phi'_v, \tag{3.25}$$

where

$$S = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix},$$

is the matrix of the shape operator of a surface. Let us multiply, using scalar multiplication, the first equation (3.24) first by vector ϕ'_u, and then by vector ϕ'_v. We get

$$\begin{aligned} < S(\phi'_u), \phi'_u > &= S_{11} < \phi'_u, \phi'_u > + S_{21} < \phi'_v, \phi'_u >, \\ < S(\phi'_u), \phi'_v > &= S_{11} < \phi'_u, \phi'_v > + S_{21} < \phi'_v, \phi'_v > . \end{aligned}$$

Making use of the definitions (3.9)(3.21) of the coefficients of first and second fundamental form of a surface we get the system of linear equations

$$\begin{aligned} E\,S_{11} + F\,S_{21} &= l, \\ F\,S_{11} + G\,S_{21} &= m, \end{aligned}$$

where S_{11}, S_{21} are unknowns. Solving this system by means of determinants we get

$$S_{11} = \frac{\begin{vmatrix} l & F \\ m & G \end{vmatrix}}{\begin{vmatrix} E & F \\ F & G \end{vmatrix}} = \frac{lG - mF}{EG - F^2}, \; S_{21} = \frac{\begin{vmatrix} E & l \\ F & m \end{vmatrix}}{\begin{vmatrix} E & F \\ F & G \end{vmatrix}} = \frac{mE - lF}{EG - F^2}. \tag{3.26}$$

Similarly, multiplying (with the help of scalar multiplication) the second equation (3.25) alternately by vectors ϕ'_u and ϕ'_v, then using the definitions of the coefficients of the first and second fundamental forms and solving the resulting system of linear equations, we get

$$S_{12} = \frac{mG - nF}{EG - F^2}, \; S_{22} = \frac{nE - mF}{EG - F^2}. \tag{3.27}$$

Thus, we have found the matrix of the shape operator of a surface in the basis $\{\phi'_u, \phi'_v\}$ of the tangent plane $T_p\mathfrak{S}$

$$S = \frac{1}{EG - F^2} \begin{pmatrix} lG - mF & mG - nF \\ mE - lF & nE - mF \end{pmatrix} \tag{3.28}$$

Considering that $S(\phi'_u) = -N'_u, S(\phi'_v) = -N'_v$, we can write equations (3.24), (3.25) in the form

$$N'_u = \frac{mF - lG}{EG - F^2}\,\phi'_u + \frac{lF - mE}{EG - F^2}\,\phi'_v, \tag{3.29}$$

$$N'_v = \frac{nF - mG}{EG - F^2}\,\phi'_u + \frac{mF - nE}{EG - F^2}\,\phi'_v. \tag{3.30}$$

Equations (3.29), (3.30) are called the Weingarten equations of a surface.

3.7 Gauss and mean curvatures

In this section we will consider two important characteristics of a surface, which are a mean and Gaussian curvature of a surface. For this we use the shape operator of a surface. From the theory of linear operators, we know that by choosing a basis of a finite-dimensional linear space, we can associate each linear operator with its matrix. Obviously, this matrix depends on the choice of the basis; when passing to another basis, the matrix of the operator will change. However, there are real numbers associated with a matrix that do not change when a basis is changed. In the case of a second-order matrix, these are the determinant and the trace of a matrix. Recall that the trace $\operatorname{Tr} A$ of an nth order square matrix A is the sum of its elements on the main diagonal of a matrix, that is, $\operatorname{Tr} A = \sum_{i=1}^{n} a_{ii}$. Since the determinant and the trace of a matrix are invariants of a transformation of a basis, we can talk about the determinant and trace of a linear operator.

Definition 3.7. The Gaussian curvature K of a surface is the determinant of the shape operator of a surface, that is, $K = \operatorname{Det} S$ The mean curvature H of a surface is the half of the trace of the shape operator, that is, $H = \frac{1}{2}\operatorname{Tr} S$.

We can easily find formulae to express the Gaussian K and mean H curvatures of a surface in terms of the coefficients of the first and second fundamental forms. Indeed, the Weingarten equations give us the matrix of the shape operator of a surface and, calculating the determinant and trace of this matrix, we obtain formulae for the Gaussian and mean curvatures. Hence for Gaussian curvature we get

$$\begin{aligned}
K &= \frac{1}{(EG-F^2)^2}\Big((lG-mF)(nE-mF)-(mG-nF)(mE-lF)\Big) \\
&= \frac{lnEG-\cancel{lmFG}-\cancel{mnEF}+m^2F^2-m^2EG+\cancel{lmFG}+\cancel{mnEF}-lnF^2}{(EG-F^2)^2} \\
&= \frac{ln(EG-F^2)-m^2(EG-F^2)}{(EG-F^2)^2} = \frac{ln-m^2}{EG-F^2}.
\end{aligned}$$

Making use of the matrices of the first and second fundamental forms of a surface we can write

$$K = \frac{\operatorname{Det}\Psi_2}{\operatorname{Det}\Psi_1}. \tag{3.31}$$

Analogously calculating the trace of the matrix (3.28) of the shape operator we find the mean curvature of a surface

$$H = \frac{lG-2\,mF+nE}{2(EG-F^2)}. \tag{3.32}$$

3.8 Derivational Formulas

In the theory of curves in three-dimensional Euclidean space, we made use of a moving frame, known as the *Frenet–Serret frame*. We studied how the Frenet–Serret frame evolves as a point moves along a curve. In other words, we derived the *Frenet–Serret formulas*, which express the derivatives of the vector fields forming the frame in terms of the same frame. The coefficients in this decomposition — the *curvature* and *torsion* of the curve — are the primary differential-geometric invariants of a space curve.

We can apply the same method to the study of surfaces. What would be the analogue of the Frenet–Serret frame in the case of a surface? It is a frame in the ambient Euclidean space E^3 that is *adapted* to the given surface.

Let $\{\mathtt{E}_1, \mathtt{E}_2, \mathtt{E}_3\}$ be three smooth vector fields defined either on an open subset of Euclidean space E^3 containing the surface, or directly on an open region of the surface itself. We assume that, at each point p in the domain of definition, the vectors $\{\mathtt{V}_1(p), \mathtt{V}_2(p), \mathtt{N}(p)\}$ are linearly independent. Moreover, if the point p lies on the surface, then the vectors $\mathtt{V}_1(p)$ and $\mathtt{V}_2(p)$ are tangent to the surface at p, while the vector $\mathtt{N}(p)$ is a unit vector orthogonal to the tangent plane — that is, $\mathtt{N}(p)$ is the *unit normal vector* to the surface at p. To be specific, we shall assume that the triple $\{\mathtt{V}_1(p), \mathtt{V}_2(p), \mathtt{N}(p)\}$ is *positively oriented* at each point p. A triple of vector fields satisfying these conditions will be called a *frame field adapted to the surface*. If the adapted frame consists of orthonormal vector fields — that is, if $\mathtt{V}_1(p)$ and $\mathtt{V}_2(p)$ are unit and mutually orthogonal vectors at each point — we shall denote them by $\mathtt{E}_1$ and $\mathtt{E}_2$ respectively.

Let $\mathfrak{S}$ be a surface and let p be a point on $\mathfrak{S}$. By the definition of a surface, we can parametrize a neighborhood of p using a coordinate chart (ϕ, U), where U is a domain in the parameter space with coordinates (u, v). Then, the triple $\{\phi_u, \phi_v, \mathtt{N}\}$, where

$$\mathtt{N} = \frac{\phi_u \times \phi_v}{\|\phi_u \times \phi_v\|},$$

forms a frame field adapted to the surface, defined on the subset $\phi(U) \subset \mathfrak{S}$. In general, this adapted frame is not orthonormal.

We begin our study of surfaces using this adapted frame field, since historically this was the original approach to the differential geometry of surfaces. Later, we will adopt an orthonormal frame field, and make use of differential forms and Cartan's structure equations developed in the previous chapters.

In this section, index notation proves to be convenient, as it yields more compact expressions and elucidates the underlying structure of the formulas. Let $\tau^1 = u, \tau^2 = v$ and $\phi_1 = \phi_{\tau^1} = \phi_u, \phi_2 = \phi_{\tau^2} = \phi_v$. As indices, we shall use Greek letters such as $\alpha, \beta, \gamma, \mu$, each ranging over the values 1 and 2. The coefficients of the first fundamental form of the surface will be denoted as follows: $E = g_{11}$, $F = g_{12} = g_{21}$, and $G = g_{22}$, that is, $g_{\alpha\beta} = \langle \phi_\alpha, \phi_\beta \rangle$.Then the first fundamental form of the surface can be written as follows

$$\Psi_1(d\tau^1, d\tau^2) = ds^2 = g_{\alpha\beta}\, d\tau^\alpha\, d\tau^\beta, \;\; g_{\alpha\beta} = g_{\beta\alpha}.$$

The coefficients of the second fundamental form of the surface will be denoted as follows:

$$\begin{aligned} l &=< S(\phi_u), \phi_u >=< \phi_{uu}, \mathtt{N} >=< \phi_{11}, \mathtt{N} >= h_{11}, \\ m &=< S(\phi_u), \phi_v >=< \phi_{uv}, \mathtt{N} >=< \phi_{12}, \mathtt{N} >= h_{12} = h_{21}, \\ n &=< S(\phi_v), \phi_v >=< \phi_{vv}, \mathtt{N} >=< \phi_{22}, \mathtt{N} >= h_{22}. \end{aligned}$$

Hence

$$\Psi_2(d\tau^1, d\tau^2) = h_{\alpha\beta}\, d\tau^\alpha\, d\tau^\beta, \;\; h_{\alpha\beta} =< \phi_{\alpha\beta}, \mathtt{N} > .$$

In the theory of surfaces, an important role is played by the derivational formulas. These formulas describe how the partial derivatives of the vector fields of the adapted frame $\phi_1 = \phi_u$, $\phi_2 = \phi_v$, and $\mathtt{N}$ with respect to the parameters τ^1, τ^2 can be expressed in terms of the frame fields themselves. We begin with partial derivatives of tangent vector fields. We can write

$$\phi_{\alpha\beta} = \Gamma^1_{\alpha\beta}\, \phi_1 + \Gamma^2_{\alpha\beta}\, \phi_2 + x_{\alpha\beta}\, \mathtt{N}, \tag{3.33}$$

where the quantities $\Gamma^1_{\alpha\beta}, \Gamma^2_{\alpha\beta}, x_{\alpha\beta}$ are functions of the surface parameters τ^1, τ^2, and our goal is to determine them. The functions $\Gamma^1_{\alpha\beta}, \Gamma^2_{\alpha\beta}$ are called the Christoffel symbols of the surface. It is clear that these symbols are symmetric in the lower indices, which follows immediately from the equality $\phi_{\alpha\beta} = \phi_{\beta\alpha}$ and the uniqueness of the decomposition (3.33). It is easy to find the functions $x_{\alpha\beta}$. To proceed, take the scalar product of both sides of the vector equation (3.33) with $\mathtt{N}$, taking into account that $\mathtt{N}$ is orthogonal to the tangent vectors ϕ_1 and ϕ_2. We get $x_{\alpha\beta} =< \phi_{\alpha\beta}, \mathtt{N} >$. But $< \phi_{\alpha\beta}, \mathtt{N} >= h_{\alpha\beta}$. Hence $x_{\alpha\beta} = h_{\alpha\beta}$, that is, the functions $x_{\alpha\beta}$ are the coefficients of the second fundamental form of the surface.

In order to find Christoffel symbols of the surface, we take the scalar product of both sides of (3.33) with ϕ_1, ϕ_2. We get

$$\langle \phi_1, \phi_{\alpha\beta} \rangle = \Gamma^1_{\alpha\beta}\, g_{11} + \Gamma^2_{\alpha\beta}\, g_{21}, \tag{3.34}$$

$$\langle \phi_2, \phi_{\alpha\beta} \rangle = \Gamma^1_{\alpha\beta}\, g_{12} + \Gamma^2_{\alpha\beta}\, g_{22}. \tag{3.35}$$

Thus, the scalar products $\langle \phi_\gamma, \phi_{\alpha\beta} \rangle$ are expressed in terms of the Christoffel symbols and the coefficients of the first fundamental form. Since our goal is to compute the Christoffel symbols, we will solve the inverse problem — that is, we will express the Christoffel symbols in terms of the scalar products $\langle \phi_\gamma, \phi_{\alpha\beta} \rangle$.

Let us denote $\Gamma_{\gamma,\alpha\beta} =< \phi_\gamma, \phi_{\alpha\beta} >$. Now both formulas (3.34) and (3.35) can be written in a compact form $\Gamma_{\gamma,\alpha\beta} = g_{\gamma\mu}\Gamma^\mu_{\alpha\beta}$. To solve this equation for the Christoffel symbols $\Gamma^\mu_{\alpha\beta}$, we use the matrix inverse of the matrix of the first fundamental form $g = (g_{\gamma\mu})$. We denote the inverse matrix as $g^{-1} = (g^{\mu\nu})$. The elements of the inverse matrix satisfy the relations

$$g_{\alpha\mu}\, g^{\mu\gamma} = \delta^\gamma_\alpha \quad \text{and} \quad g^{\alpha\mu}\, g_{\mu\gamma} = \delta^\alpha_\gamma. \tag{3.36}$$

Multiplying both sides of the equation $\Gamma_{\gamma,\alpha\beta} = g_{\gamma\mu}\Gamma^\mu_{\alpha\beta}$ by $g^{\nu\gamma}$, summing over the index γ and making use of the relations (3.36), we obtain the expressions for the Christoffel symbols:

$$g^{\nu\gamma}\Gamma_{\gamma,\alpha\beta} = g^{\nu\gamma}g_{\gamma\mu}\Gamma^\mu_{\alpha\beta} = \delta^\nu_\mu\Gamma^\mu_{\alpha\beta} = \Gamma^\nu_{\alpha\beta} \quad\Rightarrow\quad \Gamma^\nu_{\alpha\beta} = g^{\nu\gamma}\,\Gamma_{\gamma,\alpha\beta}. \tag{3.37}$$

To find the elements of the inverse matrix $g^{-1} = (g^{\alpha\beta})$, we use the standard formula for the inverse of a 2×2 matrix. Given that the first fundamental form is represented by the matrix

$$g = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix},$$

the inverse matrix is

$$g^{-1} = \frac{1}{g_{11}g_{22} - g_{12}^2}\begin{pmatrix} g_{22} & -g_{12} \\ -g_{12} & g_{22} \end{pmatrix}.$$

Thus, the elements of the inverse matrix are

$$g^{11} = \frac{g_{22}}{g_{11}g_{22} - g_{12}^2}, \quad g^{12} = g^{21} = \frac{-g_{12}}{g_{11}g_{22} - g_{12}^2}, \quad g^{22} = \frac{g_{11}}{g_{11}g_{22} - g_{12}^2}.$$

The formula (3.37) now gives

$$\Gamma^1_{\alpha\beta} = g^{11}\,\Gamma_{1,\alpha\beta} + g^{12}\,\Gamma_{2,\alpha\beta} = \frac{g_{22}\,\Gamma_{1,\alpha\beta} - g_{12}\,\Gamma_{2,\alpha\beta}}{g_{11}g_{22} - g_{12}^2}, \tag{3.38}$$

$$\Gamma^2_{\alpha\beta} = g^{21}\,\Gamma_{1,\alpha\beta} + g^{22}\,\Gamma_{2,\alpha\beta} = \frac{-g_{12}\,\Gamma_{1,\alpha\beta} + g_{11}\,\Gamma_{2,\alpha\beta}}{g_{11}g_{22} - g_{12}^2}, \tag{3.39}$$

Let us consider three integers α, β, γ, each taking values in the set $\{1, 2\}$. Then for any such triple, the following equalities hold:

$$\langle \phi_\alpha, \phi_\beta \rangle = g_{\alpha\beta}, \quad \langle \phi_\beta, \phi_\gamma \rangle = g_{\beta\gamma}, \quad \langle \phi_\gamma, \phi_\alpha \rangle = g_{\gamma\alpha}.$$

Take the partial derivative of the first equation with respect to τ^γ, the partial derivative of the second with respect to τ^α, and the partial derivative of the third with respect to τ^β. We obtain

$$\underline{\langle \phi_{\alpha\gamma}, \phi_\beta \rangle} + \underbrace{\langle \phi_\alpha, \phi_{\beta\gamma} \rangle} = \frac{\partial g_{\alpha\beta}}{\partial \tau^\gamma},$$

$$\langle \phi_{\beta\alpha}, \phi_\gamma \rangle + \underline{\langle \phi_\beta, \phi_{\gamma\alpha} \rangle} = \frac{\partial g_{\beta\gamma}}{\partial \tau^\alpha},$$

$$\underbrace{\langle \phi_{\gamma\beta}, \phi_\alpha \rangle} + \langle \phi_\gamma, \phi_{\alpha\beta} \rangle = \frac{\partial g_{\gamma\alpha}}{\partial \tau^\beta},$$

In the resulting equations, the equal terms on the left-hand sides are indicated by underlines. Thus, by adding the second equation to the third and subtracting the first (counting from top to bottom), we obtain

$$\Gamma_{\gamma,\alpha\beta} = \frac{1}{2}\Big(\frac{\partial g_{\gamma\alpha}}{\partial \tau^{\beta}} + \frac{\partial g_{\beta\gamma}}{\partial \tau^{\alpha}} - \frac{\partial g_{\alpha\beta}}{\partial \tau^{\gamma}}\Big). \tag{3.40}$$

The resulting formula plays an important role in the theory of surfaces and may be regarded as one of the fundamental formulas. It shows that the scalar products $\langle \phi_\gamma, \phi_{\alpha\beta}\rangle$ can be expressed in terms of the first-order partial derivatives of the coefficients of the first fundamental form of the surface. We can now write down the explicit expressions for the Christoffel symbols of the surface in terms of the classical notation for the coefficients of the first fundamental form, E, F, G, and their derivatives. Making use of the formula (3.40) we find

$$\begin{aligned} &\Gamma_{1,11} = \frac{1}{2}\,E'_u, && \Gamma_{1,12} = \frac{1}{2}\,E'_v, && \Gamma_{1,22} = F'_v - \frac{1}{2}\,G'_u, \\ &\Gamma_{2,11} = F'_u - \frac{1}{2}\,E'_v, && \Gamma_{2,12} = \frac{1}{2}\,G'_u, && \Gamma_{2,22} = \frac{1}{2}\,G'_v. \end{aligned} \tag{3.41}$$

Substituting these expressions into the formulas (3.38), (3.39), we get

$$\Gamma^1_{11} = \frac{G\,E'_u - 2\,F\,F'_u + F\,E'_v}{2(E\,G - F^2)}, \quad \Gamma^2_{11} = \frac{-E\,E'_v + 2\,E\,F'_u - F\,E'_u}{2(E\,G - F^2)}, \tag{3.42}$$

$$\Gamma^1_{12} = \frac{G\,E'_v - F\,G'_u}{2(E\,G - F^2)}, \quad \Gamma^2_{12} = \frac{E\,G'_u - F\,E'_v}{2(E\,G - F^2)}, \tag{3.43}$$

$$\Gamma^1_{22} = \frac{-G\,G'_u + 2\,G\,F'_v - F\,G'_v}{2(E\,G - F^2)}, \quad \Gamma^2_{22} = \frac{E\,G'_v - 2\,F\,F'_v + F\,G'_u}{2(E\,G - F^2)}. \tag{3.44}$$

The equations

$$\phi_{\alpha\beta} = \Gamma^1_{\alpha\beta}\,\phi_1 + \Gamma^2_{\alpha\beta}\,\phi_2 + h_{\alpha\beta}\,\mathtt{N}, \tag{3.45}$$

where the Christoffel symbols $\Gamma^\gamma_{\alpha\beta}$ are given by the expressions (3.42),(3.43) and (3.44), are known as the *derivational formulas.*

As we can see, the use of index notation significantly simplifies the appearance of formulas, making them more compact. Let us now rewrite the previously derived Weingarten equations (3.29), (3.30) using index notation. Let us denote $\mathtt{N}_\alpha = \partial\mathtt{N}/\partial\tau^\alpha$. Then the right-hand side of the first Weingarten equation can be rewritten as follows:

$$\begin{aligned} \mathtt{N}_1 &= \frac{mF - lG}{EG - F^2}\,\phi'_u + \frac{lF - mE}{EG - F^2}\,\phi'_v \\ &= \big(-l\,\frac{g_{22}}{EG - F^2} + m\,\frac{g_{12}}{EG - F^2}\big)\,\phi_1 + \big(l\,\frac{g_{12}}{EG - F^2} - m\,\frac{g_{11}}{EG - F^2}\big)\,\phi_2 \\ &= (-h_{11}\,g^{11} - h_{12}\,g^{21})\,\phi_1 + (-h_{11}\,g^{12} - h_{12}\,g^{22})\,\phi_2 = -h_{1\alpha}g^{\alpha\beta}\phi_\beta. \end{aligned}$$

The second Weingarten equation can be rewritten in a similar way. Thus, we arrive at the conclusion that both Weingarten equations can be expressed, using index notation, as a single unified equation

$$\mathtt{N}_\alpha = -h_{\alpha\beta}\,g^{\beta\gamma}\,\phi_\gamma. \tag{3.46}$$

3.9 Gauss Theorem

The goal of this section is to prove Gauss's theorem. Gauss was so impressed by the elegance of this result that he gave it a Latin name—*Theorema Egregium*, which translates as *Remarkable Theorem*. The essence of this theorem is closely connected with the concept of the bending of a surface. By *bending*, we mean a type of deformation that does not alter the lengths of curves lying on the surface.

Imagine that the surface is made from a very thin and flexible, yet inextensible and incompressible film. If we draw a curve on this film and then bend it, it is clear that the length of the curve remains unchanged. The geometric properties of the surface that are preserved under such bending constitute the *intrinsic geometry of the surface*.

The intuitive description of the bending of a surface provides a helpful conceptual understanding of the idea. Let us now give a formal mathematical definition of this concept.

Definition 3.8. Let $\mathfrak{S}_1$ and $\mathfrak{S}_2$ be two surfaces. A diffeomorphism $f : \mathfrak{S}_1 \to \mathfrak{S}_2$ is called an isometry if the length of any curve ξ on the surface $\mathfrak{S}_1$ is equal to the length of the curve $f(\xi)$ on the surface $\mathfrak{S}_2$. If there exists such an isometry between two surfaces $\mathfrak{S}_1$ and $\mathfrak{S}_2$, then the surfaces are said to be isometric, or that one is obtained from the other by bending.

The length of a curve on a surface is computed via the integral of the arc length differential, whose square equals the first fundamental form of the surface evaluated along the curve. Assume that the surface is locally parametrized by parameters u, v. This means that each point on the surface (within this local region) corresponds to a unique pair of parameter values u, v. Under a bending of the surface, a point p on the surface is generally mapped to another point $f(p)$; however, we retain the same parameter values u, v for the corresponding point $f(p)$. This is possible because a bending f is a smooth bijection. In this case, the first fundamental form of the surface remains unchanged, since the square of the arc length differential—that is, the squared length of infinitesimal arcs—remains the same. We conclude that any geometric quantity of the surface that can be expressed in terms of the coefficients of the first fundamental form, $E = g_{11}, F = g_{12} = g_{21}, G = g_{22}$, or their derivatives, belongs to the intrinsic geometry of the surface and is invariant under bending.

Theorem 3.4. The Gaussian curvature of a surface is an intrinsic geometric quantity. If $f : \mathfrak{S}_1 \to \mathfrak{S}_2$ is an isometry from surface $\mathfrak{S}_1$ onto surface $\mathfrak{S}_2$, then the Gaussian curvature of $\mathfrak{S}_1$ at a point p is equal to the Gaussian curvature of $\mathfrak{S}_2$ at the point $f(p)$; that is,

$$K_p = K_{f(p)}.$$

Proof. To prove the theorem, we begin with the formula for the Gaussian curvature of a surface:

$$K = \frac{h_{11}h_{22} - h_{12}^2}{g_{11}g_{22} - g_{12}^2}. \tag{3.47}$$

Our goal is to show that the Gaussian curvature is an intrinsic quantity; in other words, that it can be expressed in terms of the coefficients of the first fundamental form $g_{\alpha\beta}$ and their first and second partial derivatives.

The denominator of the right-hand side (3.47) is the determinant of the first fundamental form and is thus already expressed in terms of the coefficients $g_{\alpha\beta}$. Therefore, to complete the proof, it remains to show that the determinant of the second fundamental form, appearing in the numerator of equation (3.47), can also be expressed in terms of the coefficients of the first fundamental form and their derivatives.

We now write the derivation formulas for specific values of the indices α, β

$$\begin{aligned}
\phi_{11} &= \Gamma_{11}^1\phi_1 + \Gamma_{11}^2\phi_2 + h_{11}\,\mathtt{N},\\
\phi_{22} &= \Gamma_{22}^1\phi_1 + \Gamma_{22}^2\phi_2 + h_{22}\,\mathtt{N},\\
\phi_{12} &= \Gamma_{12}^1\phi_1 + \Gamma_{12}^2\phi_2 + h_{12}\,\mathtt{N}.
\end{aligned}$$

Making use of these formulas, we compute the scalar products $\langle \phi_{11}, \phi_{22}\rangle$ and $\|\phi_{12}\|^2 = \langle \phi_{12}, \phi_{12}\rangle$. We get

$$\begin{aligned}
\langle \phi_{11}, \phi_{22}\rangle &= \Gamma^1_{11}\Gamma^1_{22}\, g_{11} + \left(\Gamma^1_{11}\Gamma^2_{22} + \Gamma^2_{11}\Gamma^1_{22}\right) g_{12} + \Gamma^2_{11}\Gamma^2_{22}\, g_{22} + h_{11}h_{22},\\
\langle \phi_{12}, \phi_{12}\rangle &= (\Gamma^1_{12})^2\, g_{11} + (\Gamma^2_{12})^2\, g_{12} + 2\,\Gamma^1_{12}\Gamma^2_{12}\, g_{22} + h^2_{12}.
\end{aligned}$$

Subtracting the second equation from the first and expressing the determinant of the second fundamental form of the surface in terms of the remaining terms of the equations, we obtain equation

$$\begin{aligned}
h_{11}h_{22} - h^2_{12} = \langle \phi_{11}, \phi_{22}\rangle - \langle \phi_{12}, \phi_{12}\rangle + \left((\Gamma^1_{12})^2 - \Gamma^1_{11}\Gamma^1_{22}\right) g_{11}\\
+\left((\Gamma^2_{12})^2 - \Gamma^2_{11}\Gamma^2_{22}\right) g_{22} + \left(2\,\Gamma^1_{12}\Gamma^2_{12} - \Gamma^1_{11}\Gamma^2_{22} - \Gamma^2_{11}\Gamma^1_{22}\right) g_{12}.
\end{aligned} \tag{3.48}$$

In the right-hand side of the resulting formula, all terms except for the first two can be expressed in terms of the coefficients of the first fundamental form and their first-order partial derivatives. This follows from the results of the previous section, where it was shown that the Christoffel symbols can be written in terms of the coefficients of the first fundamental form and their derivatives (3.42),(3.43),(3.44). Therefore, it remains to show that the first two terms can also be expressed through the coefficients of the first fundamental form and their derivatives. To this end, we make use of $\langle \phi_\gamma, \phi_{\alpha\beta}\rangle = \Gamma_{\gamma,\alpha\beta}$ and (3.40) when $\gamma = 1, \alpha = \beta = 2$ and $\gamma = 1, \alpha = 1, \beta = 2$. We get

$$\langle \phi_1, \phi_{22}\rangle = \frac{\partial g_{12}}{\partial \tau^2} - \frac{1}{2}\frac{\partial g_{22}}{\partial \tau^1}, \quad \langle \phi_1, \phi_{12}\rangle = \frac{1}{2}\frac{\partial g_{22}}{\partial \tau^2}.$$

Differentiating both sides of the first equation with respect to τ^1 and both sides of the second equation with respect to τ^2, we obtain equations

$$\begin{aligned}
\langle \phi_{11}, \phi_{22}\rangle + \langle \phi_1, \phi_{221}\rangle &= \frac{\partial^2 g_{12}}{\partial \tau^2 \partial \tau^1} - \frac{1}{2}\frac{\partial^2 g_{22}}{\partial \tau^1 \partial \tau^1},\\
\langle \phi_{12}, \phi_{12}\rangle + \langle \phi_1, \phi_{122}\rangle &= \frac{1}{2}\frac{\partial^2 g_{11}}{\partial \tau^2 \partial \tau^2}.
\end{aligned}$$

Subtracting second equation from the first and taking into account that $\phi_{221} = \phi_{122}$, we get

$$\langle \phi_{11}, \phi_{22}\rangle - \langle \phi_{12}, \phi_{12}\rangle = \frac{\partial^2 g_{12}}{\partial \tau^1 \partial \tau^2} - \frac{1}{2}\frac{\partial^2 g_{11}}{\partial \tau^2 \partial \tau^2} - \frac{1}{2}\frac{\partial^2 g_{22}}{\partial \tau^1 \partial \tau^1}. \tag{3.49}$$

Thus, we have shown that the first two terms in formula (3.48) can be expressed in terms of the second-order partial derivatives of the coefficients of the first fundamental form. This completes the proof of Gauss's Theorem. □

In the course of proving Gauss's Theorem, we derived equation

$$\begin{aligned}
h_{11}h_{22} - h^2_{12} = \frac{\partial^2 g_{12}}{\partial \tau^1 \partial \tau^2} - \frac{1}{2}\frac{\partial^2 g_{11}}{\partial \tau^2 \partial \tau^2} - \frac{1}{2}\frac{\partial^2 g_{22}}{\partial \tau^1 \partial \tau^1} + \left((\Gamma^1_{12})^2 - \Gamma^1_{11}\Gamma^1_{22}\right) g_{11}\\
+\left((\Gamma^2_{12})^2 - \Gamma^2_{11}\Gamma^2_{22}\right) g_{22} + \left(2\,\Gamma^1_{12}\Gamma^2_{12} - \Gamma^1_{11}\Gamma^2_{22} - \Gamma^2_{11}\Gamma^1_{22}\right) g_{12}.
\end{aligned} \tag{3.50}$$

From now on, we will refer to this equation as the Gauss equation.

This equation plays a fundamental role in the theory of surfaces. First, as previously noted, it shows that the determinant of the second fundamental form can be expressed in terms of the coefficients of the first fundamental form and their first and second order partial derivatives. This leads to Gauss's theorem on the invariance of the Gaussian curvature at a point under surface bending.

However, the Gauss equation also has a second important aspect. A surface possesses two fundamental forms. This naturally raises the question: is there any relation between these two forms, or can they be specified independently of one another? The Gauss equation reveals that such a relation does exist. If the first fundamental form of a surface is given, then the second fundamental form cannot be arbitrary—its determinant must satisfy the Gauss equation.

Tracing back the derivation of the Gauss equation suggests that it is related to the integrability conditions of the derivational equations. Indeed, let us examine the derivational formulas from the perspective of the theory of differential equations. On the left-hand sides of the derivational equations, we encounter the second derivatives $\phi_{11}, \phi_{12}, \phi_{22}$. Thus, we are led to consider two integrability conditions of the system:

$$\phi_{112} - \phi_{121} = 0, \quad \phi_{221} - \phi_{212} = 0.$$

These are vector equations. When expressed in coordinates, they yield six scalar conditions. Furthermore, the derivational formulas also include the Weingarten equations. This gives rise to another integrability condition:

$$\mathtt{N}_{12} - \mathtt{N}_{21} = 0, \;\; \mathtt{N}_{\alpha\beta} = \frac{\partial^2 \mathtt{N}}{\partial \tau^\alpha \partial \tau^\beta}, \tag{3.51}$$

which, in coordinates, provides three additional scalar conditions. In total, this leads us to expect nine scalar integrability conditions for the system of derivational equations.

However, it turns out that only three of these are essential, one of which is the Gauss equation. We now proceed to identify the remaining two. We have

$$h_{\beta 1} = -\langle \phi_\beta, \mathtt{N}_1 \rangle, \quad h_{\beta 2} = -\langle \phi_\beta, \mathtt{N}_2 . \rangle$$

Differentiating the first relation with respect to τ^2, the second with respect to τ^1 and then subtracting one from the other, we get

$$\frac{\partial h_{\beta 1}}{\partial \tau^2} - \frac{\partial h_{\beta 2}}{\partial \tau^1} = -\langle \phi_{\beta 1}, \mathtt{N}_2 \rangle - \langle \phi_\beta, \mathtt{N}_{21} \rangle + \langle \phi_{\beta 2}, \mathtt{N}_1 \rangle + \langle \phi_\beta, \mathtt{N}_{12} \rangle.$$

Taking into account the condition of integrability (3.51), we obtain

$$\frac{\partial h_{\beta 1}}{\partial \tau^2} - \frac{\partial h_{\beta 2}}{\partial \tau^1} = -\langle \phi_{\beta 1}, \mathtt{N}_2 \rangle + \langle \phi_{\beta 2}, \mathtt{N}_1 \rangle.$$

Taking the scalar product of the both sides of the equations

$$\phi_{\beta 1} = \Gamma^1_{\beta 1} \phi_1 + \Gamma^2_{\beta 1} \phi_2 + h_{\beta 1} \, \mathtt{N}, \quad \phi_{\beta 2} = \Gamma^1_{\beta 2} \phi_1 + \Gamma^2_{\beta 2} \phi_2 + h_{\beta 2} \, \mathtt{N},$$

with $\mathtt{N}_2$ and $\mathtt{N}_1$ respectively and taking into account $\langle \mathtt{N}, \mathtt{N}_1 \rangle = \langle \mathtt{N}, \mathtt{N}_2 \rangle = 0$, we get

$$\frac{\partial h_{\beta 1}}{\partial \tau^2} - \Gamma^1_{\beta 2} \, h_{11} - \Gamma^2_{\beta 2} \, h_{12} = \frac{\partial h_{\beta 2}}{\partial \tau^1} - \Gamma^1_{\beta 1} \, h_{12} - \Gamma^2_{\beta 1} \, h_{22}. \tag{3.52}$$

The two equations obtained are known as the Peterson–Mainardi–Codazzi equations. These represent two additional compatibility conditions (integrability conditions) for the derivational formulas.

3.10 Gauss-Bonnet Theorem

In this section, we study the important question of parallel transport of a tangent vector along a curve lying on a surface. The notion of parallel transport of a vector along a curve relies on the concept of the covariant derivative of a vector field tangent to the surface. We define the notion of the covariant derivative of a tangent vector field at a point and along a curve.

Let $\mathfrak{S}$ be a surface with a local parametrization $\phi : U \to \mathfrak{S}$ in a neighborhood of a point p. Let $\mathtt{N}$ denote a unit normal vector field on $\phi(U)$, that is, an orientation of the surface. We assume that the surface is oriented in such a way that, at each point, the triple $\{\phi_u, \phi_v, \mathtt{N}\}$ forms a positively oriented (right-handed) frame. Let $\mathtt{X}$ be a vector field defined in a neighborhood $\phi(U) \subset \mathfrak{S}$, and assume that $\mathtt{X}$ is tangent to the surface, that is, $\mathtt{X}_q \in T_q\mathfrak{S}$ for each $q \in \phi(U)$.

The purpose of the covariant derivative on the surface is to measure the rate of change of a tangent vector field along a given direction on the surface. Thus, the covariant derivative of a vector field $\mathtt{X}$ must itself be a tangent vector field. Let v be a tangent vector at the point p. Choose a curve $\xi : I \to \phi(U)$ on the surface such that $\xi(0) = p$ and $\xi'(0) = v$. The covariant derivative is defined by

$$D_v\,\mathtt{X} = \frac{d}{dt}\mathtt{X}(\xi(t))\bigg|_{t=0} - \left\langle \frac{d}{dt}\mathtt{X}(\xi(t))\bigg|_{t=0}, \mathtt{N}_p \right\rangle \mathtt{N}_p.$$

On the right-hand side of this formula, the first term is the derivative of the vector field $\mathtt{X}$ with respect to the parameter t at the point p. This means that the first term, so to speak, measures the rate of change of the vector field in the direction of the tangent vector v, but the resulting vector is generally not tangent to the surface. Therefore, the right-hand side includes a second term, which is the projection of the first term onto the normal vector to the surface, and by subtracting it, we obtain the tangent component of the rate of change of the vector field $\mathtt{X}$.

Similarly, the covariant derivative of a vector field tangent to the surface and defined along a curve lying on the surface is introduced. Let $\xi : I \to \mathfrak{S}$ be a curve on the surface, and let $\mathtt{X}$ be a vector field along ξ, tangent to the surface. The covariant derivative of the vector field $\mathtt{X}$ is defined by the formula

$$\frac{D\mathtt{X}}{dt} = \frac{d\mathtt{X}}{dt} - \langle \frac{d\mathtt{X}}{dt}, \mathtt{N}\rangle\, \mathtt{N}.$$

Thus, the covariant derivative of a vector field tangent to the surface and defined along a curve is itself a vector field along the curve, tangent to the surface.

Suppose we are given a curve $\xi : [a, b] \to E^2$ on the plane, with $\xi(a) = p$ and $\xi(b) = q$ as its initial and terminal points, respectively. Let a vector $\vec{v}$ be given at the point p. We seek to transport $\vec{v}$ parallelly from p to q along the curve ξ. To do so, we assign to each point of the curve ξ a vector that is parallel to the initial vector $\vec{v}$ at p. In this way, we obtain a vector field along ξ, and its value at the endpoint q is naturally called the vector $\vec{v}$ transported parallelly along ξ. Clearly, the resulting vector field is constant; that is, its value at each point of ξ is equal to $\vec{v}$. This means that the derivative of this vector field with respect to the curve parameter t is zero.

We now define the parallel transport of a vector tangent to a surface along a curve lying on the surface in a similar manner. Let $\xi : [a, b] \to \mathfrak{S}$ be a curve lying on a surface $\mathfrak{S}$. Let a tangent vector $v \in T_p\mathfrak{S}$ be given at the initial point $p = \xi(a)$, and suppose we wish to transport it parallelly (with respect to the surface $\mathfrak{S}$) along ξ from p to $q = \xi(b)$. To achieve this, we construct a tangent vector field $\mathtt{X}(t)$ along the curve ξ as the solution to the differential equation

$$\frac{D\mathtt{X}(t)}{dt} = 0.$$

The initial condition for this system is $X(a) = v$. Then, the value of the vector field $\mathtt{X}$ at the endpoint $q = \xi(b)$ is called the vector v transported parallelly from p to q along the curve ξ on the surface $\mathfrak{S}$.

Now suppose that the curve ξ is closed, meaning that its initial and terminal points coincide. Furthermore, we assume that ξ is parameterized by arc length, so the parameter s varies from 0 to l, where l is the total length of the curve. Thus, the condition that the curve is closed can be written as $\xi(0) = \xi(l)$. We also assume that ξ is a simple curve, meaning it has no self-intersections, and that the region of the surface it bounds is simply connected. We denote the portion of the surface bounded by ξ as V, so that ξ is the boundary of this region, which we write as $\partial V = \xi$.

Orientation plays an important role in the Gauss–Bonnet theorem. In this context, we assume that the curve ξ and the region V it bounds lie within a coordinate chart (ϕ, U) of the surface, where the coordinates are denoted, as before, by u and v. Then, over the patch of the surface $\phi(U)$, we have a frame field $\{\phi_u, \phi_v, \mathtt{N}\}$, where ϕ_u and ϕ_v are tangent vector fields to the surface, and $\mathtt{N}$ is a normal vector field collinear at each point of $\phi(U)$ with the cross product $\phi_u \times \phi_v$.

The normal vector field $\mathtt{N}$ defines the orientation of the surface patch $\phi(U)$ in the sense that traversal of the closed curve ξ is considered to be in the positive direction if, when viewed from the tip of the normal vector $\mathtt{N}$, the motion appears counterclockwise. Likewise, the sign of angles measured from a given vector is defined accordingly. It is appropriate here to clarify what we mean by the region V bounded by the curve ξ. Consider the vector cross product $\mathtt{N} \times \xi'$, where ξ' is the tangent vector to the curve ξ. Then, the region V lies on the side of the tangent vector ξ' toward which the vector $\mathtt{N} \times \xi'$ points.

Note that we may also make use of the complex structure of the tangent plane to the surface, provided we agree that the origin of the coordinate system in the tangent plane is the point of contact, and that positive rotation is counterclockwise when viewed from the tip of the normal vector $\mathtt{N}$. In this case, we have $\mathtt{N} \times \xi' = J(\xi')$, where J denotes the operation of rotation by $\frac{\pi}{2}$ in the positive (counterclockwise) direction.

Let p denote the point on the curve ξ corresponding to the initial value of the natural parameter, $s = 0$. Let $\mathtt{r}_0$ be a unit tangent vector to the surface at the point p. Now, using parallel transport on the surface, we transport the vector $\mathtt{r}_0$ along the curve ξ. Due to the closedness of the curve, we return to the original point p. Let the vector obtained by parallel transport at the point p be denoted by $\mathtt{r}_1$. Let us denote the angle between the vectors $\mathtt{r}_0$ and $\mathtt{r}_1$ by $\Delta\varphi = \angle(\mathtt{r}_0, \mathtt{r}_1)$. In this notation, we mean that the angle is measured from the vector $\mathtt{r}_0$ to the vector $\mathtt{r}_1$ in the positive (counterclockwise) direction. The parallel transport of the vector $\mathtt{r}_0$ is realized by means of a vector field along the curve ξ. We denote this vector field by $\mathtt{R}$. By the definition of parallel transport, this vector field satisfies the conditions

$$\mathtt{R}(0) = \mathtt{r}_0, \quad \mathtt{R}(l) = \mathtt{r}_1, \quad \frac{D\mathtt{R}}{ds} = 0. \tag{3.53}$$

Moreover, since the length of a vector remains unchanged under parallel transport, the vector field $\mathtt{R}$ along the curve ξ is a unit (tangent to the surface) vector field.

Let $\mathtt{E}$ be a unit tangent vector field on the surface, defined on $\phi(U)$. Define

$$\varphi(s) = \angle(\mathtt{E}(s), \mathtt{R}(s)),$$

where $\mathtt{E}(s)$ is the restriction of the vector field $\mathtt{E}$ to the curve $\xi(s)$. Clearly, $\varphi(s)$ is a smooth function of the arc-length parameter s. If we set

$$\varphi_0 = \angle(\mathtt{E}_p, \mathtt{r}_0), \quad \varphi_1 = \angle(\mathtt{E}_p, \mathtt{r}_1),$$

then

$$\varphi_0 = \varphi(0), \quad \varphi_1 = \varphi(l), \quad \Delta\varphi = \varphi_1 - \varphi_0.$$

Hence

$$\Delta\varphi = \int_\xi d\varphi. \tag{3.54}$$

From $\varphi = \angle(\mathtt{E}, \mathtt{R})$ it follows that $\cos\varphi = \langle \mathtt{E}, \mathtt{R}\rangle$. Differentiating the both sides of this equation, we get

$$-\sin\varphi \, \frac{d\varphi}{ds} = \langle \frac{D\mathtt{E}}{ds}, \mathtt{R}\rangle + \langle \mathtt{E}, \frac{D\mathtt{R}}{ds}\rangle. \tag{3.55}$$

Due to the properties (3.53) of the vector field $\mathtt{R}$, the second term on the right-hand side of the previous equation vanishes and we get

$$-\sin\varphi \, \frac{d\varphi}{ds} = \langle \frac{D\mathtt{E}}{ds}, \mathtt{R}\rangle. \tag{3.56}$$

As noted earlier, the orientation of the surface given by $\mathtt{N}$ allows us to use the complex structure J on the tangent plane to the surface. Consequently, $J\mathtt{E}$ is a unit vector field tangent to the surface and orthogonal to $\mathtt{E}$. Restricting $J\mathtt{E}$ to the curve ξ gives a vector field along ξ, which will be denoted by $E^\perp$.

Now, suppose that instead of the vector $\mathtt{r}_0$, we take another unit tangent vector $\tilde{\mathtt{r}}_0$ at the point p. Repeating the construction described above, we obtain a vector field $\tilde{\mathtt{R}}$ along the curve ξ and a function $\tilde{\varphi}(s)$. Clearly,

$$\tilde{\varphi}(s) - \varphi(s) = \angle(\mathtt{R}(s), \tilde{\mathtt{R}}(s)).$$

Since both $\mathtt{R}(s)$ and $\tilde{\mathtt{R}}(s)$ are parallel vector fields along ξ, the angle between them remains constant as the point moves along the curve. Therefore,

$$\tilde{\varphi}(s) - \varphi(s) = \text{const} \quad \text{and} \quad \frac{d\tilde{\varphi}}{ds} = \frac{d\varphi}{ds}.$$

Let now $\xi(s)$ be an arbitrary point on the curve ξ. At this point, we have the vector $E^\perp(s)$ and it forms the angle of $\frac{\pi}{2}$ with the vector $\mathtt{E}(s)$. Since parallel transport of tangent vectors on the surface defines a bijective linear map between tangent planes at different points, there exists a vector $\tilde{\mathtt{r}}_0$ at point p on the surface such that its parallel transport along the curve ξ from p to the point $\xi(s)$ coincides with the vector $E^\perp(s)$. In this case, at the point $\xi(s)$ we have $\tilde{\varphi}(s) = \frac{\pi}{2}$, and therefore,

$$-\frac{d\tilde{\varphi}}{ds} = \left\langle \frac{D\mathtt{E}}{ds}, E^\perp \right\rangle.$$

But since $d\tilde{\varphi}(s) = d\varphi(s)$, we obtain the formula:

$$-\frac{d\varphi}{ds} = \left\langle \frac{D\mathtt{E}}{ds}, E^\perp \right\rangle.$$

As the point $\xi(s)$ was chosen arbitrarily, it follows that this formula holds at every point along the curve ξ. Note that in the right-hand side of this formula, the symbol for covariant differentiation D can be replaced by the ordinary derivative d, so the previous formula can be written in the equivalent form

$$-\frac{d\varphi}{ds} = \left\langle \frac{d\mathtt{E}}{ds}, E^{\perp} \right\rangle. \tag{3.57}$$

Let $\eta(s) = (u(s), v(s)) \in U$ be the closed curve in the parameter plane (u, v) of the surface $\mathfrak{S}$, which under the mapping $\phi : U \to \mathfrak{S}$ is mapped to the curve $\xi(s)$ on the surface, that is,

$$\phi(\eta(s)) = \xi(s).$$

Let V denote the region in the uv-plane for which the given curve is the boundary. Then differentiation with respect to s gives $\frac{dE}{ds} = E_u\,\frac{du}{ds} + E_v\,\frac{dv}{ds}$. Substituting this expression into (3.57) and omitting ds, we obtain

$$\Delta\varphi = -\int\limits_{\xi} \langle E_u, \mathtt{E}^{\perp}\rangle\, du + \langle E_v, \mathtt{E}^{\perp}\rangle\, dv.$$

Applying Green's formula, we can transform this line integral into a double integral

$$\begin{aligned} \Delta\varphi &= -\iint\limits_{V} \big(\langle E_{vu}, \mathtt{E}^{\perp}\rangle + \langle E_v, \mathtt{E}^{\perp}_u\rangle - \langle E_{uv}, \mathtt{E}^{\perp}\rangle - \langle E_u, \mathtt{E}^{\perp}_v\rangle\big)\, du\, dv \\ &= \iint\limits_{V} \big(\langle E_u, \mathtt{E}^{\perp}_v\rangle - \langle E_v, \mathtt{E}^{\perp}_u\rangle\big)\, du\, dv. \end{aligned} \tag{3.58}$$

For further calculations, we will use the frame $\{E, \mathtt{E}^{\perp}, N\}$ of the three-dimensional Euclidean space, which is defined at each point of the region $\phi(U) \subset \mathfrak{S}$. We have

$$E_u \perp E, E_v \perp E, \mathtt{E}^{\perp}_u \perp \mathtt{E}^{\perp}, \mathtt{E}^{\perp}_v \perp \mathtt{E}^{\perp}, N_u \perp N, N_v \perp N.$$

Hence

$$E_u = y_1\, \mathtt{E}^{\perp} + z_1\, N, \quad E_v = y_2\, \mathtt{E}^{\perp} + z_2\, N, \tag{3.59}$$
$$\mathtt{E}^{\perp}_u = x_1\, E + z_3\, N, \quad \mathtt{E}^{\perp}_v = x_2\, E + z_4\, N, \tag{3.60}$$
$$N_u = x_3\, E + y_3\, \mathtt{E}^{\perp}, \quad N_v = x_4\, E + y_4\, \mathtt{E}^{\perp}, \tag{3.61}$$

where x_i, y_j, z_k are functions of variables u, v. We have $\langle E, \mathtt{E}^{\perp}\rangle = 0$, and differentiating this equality once with respect to u and then with respect to v, we respectively obtain

$$\langle E_u, \mathtt{E}^{\perp}\rangle + \langle E, \mathtt{E}^{\perp}_u\rangle = 0, \quad \langle E_v, \mathtt{E}^{\perp}\rangle + \langle E, \mathtt{E}^{\perp}_v\rangle = 0. \tag{3.62}$$

Taking the scalar product of both sides of equations (3.59) with $\mathtt{E}^{\perp}$, we obtain $\langle E_u, \mathtt{E}^{\perp}\rangle = y_1, \langle E_v, \mathtt{E}^{\perp}\rangle = y_2$. Taking the scalar product of both sides of equations (3.60) with E, we obtain $\langle \mathtt{E}^{\perp}_u, E\rangle = x_1, \langle \mathtt{E}^{\perp}_v, E\rangle = x_2$. Making use of the relations (3.62) we obtain $y_1 = -x_1, y_2 = -x_2$.

Using formulas (3.59) and (3.60), we transform the integrand in (3.58) into the form

$$\langle E_u, \mathtt{E}^{\perp}_v\rangle - \langle E_v, \mathtt{E}^{\perp}_u\rangle = z_1\, z_4 - z_2\, z_3. \tag{3.63}$$

Our goal now is to relate the obtained expression to the Gaussian curvature of the surface. To this end, we use the Weingarten equations (3.29),(3.30). From these equations it follows that

$$\mathtt{N}_u \times \mathtt{N}_v = K\phi_u \times \phi_v,$$

where K is the Gaussian curvature of the surface. If we now multiply and divide the right-hand side of this formula by the length of the cross product $\|\phi_u \times \phi_v\| = \sqrt{g_{11}\, g_{22} - g_{12}^2}$, we obtain equation

$$\mathtt{N}_u \times \mathtt{N}_v = K \sqrt{g_{11}\, g_{22} - g_{12}^2}\, \mathtt{N}.$$

Taking scalar product of both sides of this equation with $\mathtt{N}$ and replacing $\mathtt{N} = \mathtt{E} \times \mathtt{E}^{\perp}$, we get

$$\langle \mathtt{N}_u \times \mathtt{N}_v, \mathtt{E} \times \mathtt{E}^{\perp} \rangle = K \sqrt{g_{11}\, g_{22} - g_{12}^2}.$$

Applying Lagrange's identity to the left-hand side of this equation, we obtain

$$\begin{vmatrix} \langle \mathtt{N}_u, \mathtt{E} \rangle & \langle \mathtt{N}_u, \mathtt{E}^{\perp} \rangle \\ \langle \mathtt{N}_v, \mathtt{E} \rangle & \langle \mathtt{N}_v, \mathtt{E}^{\perp} \rangle \end{vmatrix} = K \sqrt{g_{11}\, g_{22} - g_{12}^2}. \tag{3.64}$$

Differentiating the relation $\langle \mathtt{N}, \mathtt{E} \rangle = 0$ with respect to u, we get $\langle \mathtt{N}_u, \mathtt{E} \rangle = -\langle \mathtt{N}, \mathtt{E}_u \rangle$ and, making use of (3.59), we obtain $\langle \mathtt{N}_u, \mathtt{E} \rangle = -z_1$. Analogously

$$\langle \mathtt{N}_u, \mathtt{E}^{\perp} \rangle = -z_3, \langle \mathtt{N}_v, \mathtt{E} \rangle = -z_2, \langle \mathtt{N}_v, \mathtt{E}^{\perp} \rangle = -z_4.$$

Hence the determinant at the left-hand side of (3.64) is equal to $z_1 z_4 - z_2 z_3$ and, taking into account (3.63), we finally obtain

$$\langle E_u, \mathtt{E}_v^{\perp} \rangle - \langle E_v, \mathtt{E}_u^{\perp} \rangle = K \sqrt{g_{11}\, g_{22} - g_{12}^2}, \tag{3.65}$$

or

$$\Delta\varphi = \iint_V K \sqrt{g_{11}\, g_{22} - g_{12}^2}\, du\, dv. \tag{3.66}$$

We will refer to this formula as the formula for *the parallel transport of a unit tangent vector to the surface along a closed curve.*

In the derivation of the previous formula, we assumed that the vector is transported parallelly along a smooth closed curve on the surface. We now consider the case when the closed curve has the form of a curvilinear triangle, i.e., it is piecewise smooth. The reasoning and computations presented below can easily be extended to the case of a curvilinear convex polygon with n vertices.

We will need the notion of geodesic curvature of a curve lying on a surface. Let $\xi\colon [a, b] \to \mathfrak{S}$ be a smooth curve on the surface $\mathfrak{S}$, parametrized by the natural parameter s (i.e., arc length). We assume the curve lies entirely within a parametrized patch (ϕ, U) of the surface, where an orientation $\mathtt{N}$ of the surface has been chosen.

At each point of the curve we have the unit tangent vector $\xi'(s)$ and it defines a unit tangent vector field along the curve, denoted by $\mathtt{T}$. The acceleration vector $\xi''(s)$ is called the curvature vector of the curve ξ, and its magnitude is the curvature of the curve at that point: $\kappa(s) = \|\xi''(s)\|$. Let $\mathtt{N}_\xi$ denote the unit principal normal vector field of the curve. Then,

$$\mathtt{T}' = \kappa\, \mathtt{N}_\xi.$$

Now consider the vector product $\mathtt{N} \times \mathtt{T}$. Since the acceleration vector is perpendicular to the velocity vector, i.e., $\xi'' \perp \xi'$, the acceleration vector lies in the plane spanned by the vectors $\mathtt{N} \times \mathtt{T}$ and $\mathtt{N}$. Therefore, we can write:

$$\xi'' = \kappa_g\, (\mathtt{N} \times \mathtt{T}) + \kappa_n\, \mathtt{N}, \tag{3.67}$$

where κ_n is the normal curvature of the surface in the direction of the unit tangent vector ξ', and κ_g is called the geodesic curvature of the curve on the surface. Let us show that κ_n is a normal curvature of the surface $\mathfrak{S}$. From the equation (3.69) we find $\kappa_n = \langle \xi'', \mathtt{N}\rangle$. Differentiating both sides of $\langle \xi', \mathtt{N}\rangle = 0$, we obtain $\langle \xi'', \mathtt{N}\rangle = -\langle \xi', \mathtt{N}'\rangle$. Hence

$$k_n = \langle \xi'', \mathtt{N}\rangle = -\langle \xi', \mathtt{N}'\rangle = \langle \xi', -\mathtt{N}'\rangle = \langle \xi', S(\xi')\rangle = \kappa(\xi'). \tag{3.68}$$

In order to find the geodesic curvature κ_g we write the formula (3.69) in the form

$$\kappa\, \mathtt{N}_\xi = \kappa_g\, (\mathtt{N} \times \mathtt{T}) + \kappa_n\, \mathtt{N}.$$

Taking the scalar product of the both sides of this equation with the vector field $\mathtt{N} \times \mathtt{T}$ we get

$$\kappa_g = \kappa\, \langle \mathtt{N}_\xi, \mathtt{N} \times \mathtt{T}\rangle. \tag{3.69}$$

Note that from the definition of the geodesic curvature of a curve on a surface, an important property follows. The geodesic curvature has a sign and if the direction of traversal along the curve is reversed, the sign of the geodesic curvature changes to the opposite. Furthermore, in the special case where the curve is a geodesic, its geodesic curvature is zero.

Thus, the geodesic curvature measures how much a given curve deviates from being a geodesic. If geodesics on a surface are considered analogous to straight lines in the plane, then the geodesic curvature quantifies how far a curve on the surface deviates from a straight line.

Let us return to the case of a curvilinear triangle on a surface. Let its vertices be denoted by p_1, p_2, p_3. Denote the curve forming the boundary of the curvilinear triangle, as before, by $\xi : [0, l] \to \mathfrak{S}$. We assume that the parameter of the curve ξ is the arc length s, so that l is the total length of the curve. Furthermore, the interval $[0, l]$ is divided into three subintervals: $[0, l_1]$, $[l_1, l_2]$, $[l_2, l]$. The restriction of the curve ξ to each of these subintervals defines smooth curves $\xi_1 = \xi|_{[0,l_1]}$, $\xi_2 = \xi|_{[l_1,l_2]}$, $\xi_3 = \xi|_{[l_2,l]}$, whose lengths are l_1, $l_2 - l_1$, and $l - l_2$, respectively. We assume that at the points $0, l_1, l_2, l$ of the interval $[0, l]$, the vector-valued function $\vec{\xi}(s)$ possesses one-sided derivatives, and these derivatives define the tangent vectors of the corresponding arcs, which are non-zero vectors or not collinear. Thus, we have six tangent vectors

$$\begin{aligned}
\vec{\xi}_1'(l_1) &= \lim_{s \uparrow l_1} \frac{\vec{\xi}(l_1) - \vec{\xi}(s)}{l_1 - s}, \quad \vec{\xi}_2'(l_1) = \lim_{s \downarrow l_1} \frac{\vec{\xi}(s) - \vec{\xi}(l_1)}{s - l_1},\\
\vec{\xi}_2'(l_2) &= \lim_{s \uparrow l_2} \frac{\vec{\xi}(l_2) - \vec{\xi}(s)}{l_2 - s}, \quad \vec{\xi}_3'(l_2) = \lim_{s \downarrow l_2} \frac{\vec{\xi}(s) - \vec{\xi}(l_2)}{s - l_2},\\
\vec{\xi}_3'(l) &= \lim_{s \uparrow l} \frac{\vec{\xi}(l) - \vec{\xi}(s)}{l - s}, \quad \vec{\xi}_1'(0) = \lim_{s \downarrow 0} \frac{\vec{\xi}(s) - \vec{\xi}(0)}{s},
\end{aligned} \tag{3.70}$$

Let us denote the angles between these tangent vectors as follows

$$\beta_1 = \angle(\vec{\xi}_3'(l), \vec{\xi}_1'(0)), \quad \beta_2 = \angle(\vec{\xi}_1'(l_1), \vec{\xi}_2'(l_1)), \quad \beta_3 = \angle(\vec{\xi}_2'(l_2), \vec{\xi}_3'(l_2)).$$

To derive the Gauss–Bonnet formula, we will use the tangent vector of the piecewise smooth curve ξ. We denote it by $\mathtt{T}$ and consider it as a vector field along ξ, meaning that in each point of the curve, $\mathtt{T}$ defines the unit tangent vector to ξ. Introduce two angles: $\varphi = \angle(\mathtt{E}, \mathtt{R})$, and $\psi = \angle(\mathtt{R}, \mathtt{T})$. We first study the variation of these angles along the smooth arcs ξ_1, ξ_2, ξ_3, temporarily leaving aside the question of the corner points of the curvilinear triangle.

At each point of the curve, the first angle represents the angle of rotation from the vector $\mathtt{E}$ to the vector $\mathtt{R}$ (the parallel transported vector $\mathtt{r}_0$). Thus, subtracting the initial value of this angle, $\varphi - \varphi_0$, where $\varphi_0 = \angle(\mathtt{E}(0), \mathtt{r}_0)$, we obtain the total angle $\Delta\varphi$ by which the parallel transported vector $\mathtt{r}_0$ has rotated. On the smooth arcs of the curve, we compute this angle using the previously obtained formula ((3.66).

Consider the second angle $\psi = \angle(\mathtt{R}, \mathtt{T})$. Clearly, $-\psi = \angle(\mathtt{T}, \mathtt{R})$. Thus, this situation is analogous to the one described before formula (3.57). The only difference is that instead of the vector field $\mathtt{E}$, we now use the tangent vector field $\mathtt{T}$. Therefore, the derivative of the angle $-\psi$ with respect to the parameter s satisfies the equation

$$\frac{d(-\psi)}{ds} = -\langle \frac{dT}{ds}, \mathtt{T}^{\perp} \rangle,$$

where $\mathtt{T}^{\perp} = J\mathtt{T} = \mathtt{N} \times \mathtt{T}$. Applying the Frenet formula $\mathtt{T}' = \kappa\, \mathtt{N}$ and making use of (3.69), we obtain

$$\frac{d\psi}{ds} = \langle \frac{dT}{ds}, \mathtt{T}^{\perp} \rangle = \langle \kappa\, \mathtt{N}_{\xi}, \mathtt{T}^{\perp} \rangle = \kappa\, \langle \mathtt{N}_{\xi}, \mathtt{N} \times \mathtt{T} \rangle = \kappa_g.$$

Thus, the increment of the angle ψ along the smooth arcs of the curve ξ is given by the integral of the geodesic curvature

$$\Delta\psi = \int\limits_{\xi} \kappa_g\, ds. \tag{3.71}$$

In this formula, we mean that the integral is equal to the sum of three integrals taken over the smooth arcs ξ_1, ξ_2, ξ_3, respectively.

It remains to account for the contribution to the change in the angle of rotation of the parallel transported vector coming from the corner points, that is, the vertices of the curvilinear triangle. At the vertices of the triangle, the tangent vector of the curve ξ undergoes a discontinuous change in direction. The contribution of the vertices of the curvilinear triangle to the variation of the angle ψ is equal to the sum $\beta_1 + \beta_2 + \beta_3$. Thus, after the parallel transport of the vector $\mathtt{r}_0$ along the curvilinear triangle the angle $\psi + \varphi$, taken at the initial point p, undergoes an increase

$$\iint\limits_{V} K\, \sqrt{g_{11}\, g_{22} - g_{12}^2}\, du\, dv + \int\limits_{\xi} \kappa_g\, ds + \sum_{i=1}^{3} \beta_i.$$

Now observe that the angle $\psi + \varphi$ is the angle between the vector fields $\mathtt{E}, \mathtt{T}$, i.e., $\psi + \varphi = \angle(\mathtt{E}, \mathtt{T})$. However, the vectors $\mathtt{E}(p), \mathtt{T}(p)$, after being transported around the curvilinear triangle, return to their original positions. Therefore, the angle $\psi + \varphi$ must change by a quantity equal to $2\pi k$, where k is an integer. It can be shown that in this case $k = 1$. We demonstrate this through an argument that is not a rigorous proof but illustrates the geometric essence of the structure. We can draw an analogy with a continuous map from one unit circle (of radius 1) to another. Such continuous maps are, up to continuous deformation, classified by an integer, which equals the number of times one circle winds around the other. Moreover, if one map has m windings and another has n, and $m \neq n$, then it is impossible to deform one map into the other continuously. Following this reasoning, we continuously deform the portion of the surface containing the curvilinear triangle into a portion of the plane. Then we continuously deform the curvilinear triangle into a circle, and the vector field $\mathtt{E}$ into a constant vector field along the circle. The change in the angle

$\psi + \varphi$ remains the same as before, but now it becomes easy to compute. Transporting the tangent vector to the circle along the circle results in an increase of 2π. Hence

$$\iint_V K \sqrt{g_{11}\, g_{22} - g_{12}^2}\, du\, dv + \int_\xi \kappa_g\, ds + \sum_{i=1}^{3} \beta_i = 2\,\pi. \tag{3.72}$$

This formula will be referred to as *Gauss-Bonnet formula.*

A *triangulation of a surface* is a covering of the surface by curvilinear triangles $\{\triangle_i\}_{i\in\mathcal{I}}$. It consists of a family of curvilinear triangles. The sides of these curvilinear triangles will be called edges, and the curvilinear triangles themselves will be called faces. Thus, a triangulation consists of a set of vertices, edges, and faces. A triangulation must satisfy the following conditions [1].

1. The union of all curvilinear triangles in the covering must equal the entire surface, that is, $\bigcup_{i\in\mathcal{I}} \triangle_i = \mathfrak{S}$.

2. Any two curvilinear triangles have no interior points in common.

3. Each edge of triangulation is shared exactly by two curvilinear triangles.

4. If p is a vertex of a curvilinear triangle $\triangle_i$, then there exist m curvilinear triangles $\triangle_{i_1}, \triangle_{i_2}, \ldots, \triangle_{i_m}$, where m is an integer, such that each triangle shares p as a common vertex, with $\triangle_{i_1} = \triangle_i$, and each consecutive pair shares a common edge: $\triangle_{i_1}$ shares an edge with $\triangle_{i_2}$, $\triangle_{i_2}$ shares an edge with $\triangle_{i_3}$, and so on, with $\triangle_{i_m}$ sharing one edge with $\triangle_{i_{m-1}}$ and the other with $\triangle_{i_1} = \triangle_i$.

If the number of curvilinear triangles in the triangulation is finite, the triangulation is said to be finite. In surface topology, it is established that every compact surface admits a finite triangulation. Assuming that triangulation is finite we define an integer by the following formula $v - e + f$, where v is the number of vertices, e is the number of edges and f is the number of faces (curvilinear triangles) of triangulation. It can be shown that the integer $v - e + f$, does not depend on the particular triangulation and is determined by topology of the surface. This integer is called the *Euler characteristic of a surface* and is denoted by $\chi(\mathfrak{S})$.

Now suppose that a compact surface $\mathfrak{S}$ is equipped with a finite triangulation $\{\triangle_i\}_{i=1}^m$, where m is a positive integer. Moreover, without loss of generality, we may assume that each curvilinear triangle of this triangulation lies entirely within some local coordinate chart (ϕ, U) of the surface $\mathfrak{S}$, with local coordinates u, v. Then, we define the integral of the Gaussian curvature over the entire surface $\mathfrak{S}$ as the sum of the integrals of the Gaussian curvature over all the curvilinear triangles of the given triangulation, that is,

$$\int_{\mathfrak{S}} K\, d\mathcal{S} = \sum_{i=1}^{m} \iint_{\triangle_i} K \sqrt{g_{11} g_{22} - g_{12}^2}\, du\, dv, \tag{3.73}$$

where we denote by $d\mathcal{S}$ the surface area element, which, in the local coordinates u, v of a curvilinear triangle, takes the form $\sqrt{g_{11} g_{22} - g_{12}^2}\, du\, dv$. The integral (3.73) is referred to as *the total Gaussian curvature of a surface.*

Theorem 3.5. (Gauss-Bonnet for compact surfaces) If $\mathfrak{S}$ is a compact orientable surface, then the total Gaussian curvature of $\mathfrak{S}$ multiplied by $\frac{1}{2\pi}$ is equal to the Euler characteristic of $\mathfrak{S}$, that is,

$$\frac{1}{2\pi} \int_{\mathfrak{S}} K\, d\mathcal{S} = \chi(\mathfrak{S}).$$

Proof. At the heart of the proof of this theorem lies formula (3.72). Consider some curvilinear triangle $\triangle_i$ of a triangulation $\{\triangle_i\}_{i=1}^{m}$. Then we have the formula (3.72). Now we move the second and third terms from the left-hand side to the right-hand side with a minus sign. We obtain

$$\iint\limits_{\triangle_i} K\,\sqrt{g_{11}\,g_{22}-g_{12}^2}\,du\,dv = 2\,\pi - \int\limits_{\partial\triangle_i} \kappa_g\,ds - \sum_{a=1}^{3} \beta_a^{(i)}, \tag{3.74}$$

where $\partial\triangle_i$ is the boundary of curvilinear triangle $\triangle_i$, i.e., the edges of $\triangle_i$, and $\beta_1^{(i)}, \beta_2^{(i)}, \beta_3^{(i)}$ are the exterior angles of $\triangle_i$. Now sum both the left-hand side and the right-hand side of the equation over all curvilinear triangles of the given triangulation. Obviously, on the left-hand side, we obtain — according to formula (3.73) — the total Gaussian curvature of the surface $\mathfrak{S}$. On the right-hand side of the equation, the first term, that is, the number 2π, yields $2\pi f$, where f is the number of faces in the triangulation, or the number of curvilinear triangles. The second term, namely the integral of the geodesic curvature, after summing over all the edges of the triangulation (which is the same as summing over all the sides of the curvilinear triangles), gives zero. Indeed, according to the third requirement of the triangulation, each edge of the triangulation belongs to two curvilinear triangles. However, when traversing one triangle, we move along this edge in one direction, and when traversing the other, we move in the opposite direction. Since the geodesic curvature changes sign when the direction of motion along the curve is reversed, the total sum is zero.

The last term on the right-hand side, that is, the sum of the exterior angles of the i-th curvilinear triangle, can be written as

$$\sum_{a=1}^{3} \beta_a^{(i)} = 3\pi - \sum_{a=1}^{3} \alpha_a^{(i)},$$

where $\alpha_1^{(i)}, \alpha_2^{(i)}, \alpha_3^{(i)}$ are the interior angles of the i-th curvilinear triangle. The term 3π, when summed over all curvilinear triangles, yields $3\pi\,f$. We sum the interior angles according to the fourth condition of the triangulation: starting from an arbitrary vertex, we sum all interior angles for which this point serves as the vertex of the angle, obtaining 2π. Repeating this for all vertices, we obtain a total of $2\pi\,v$, where v is the total number of vertices in the triangulation. Hence, after summation over all curvilinear triangles, the formula (3.74) takes on the form

$$\int\limits_{\mathfrak{S}} K\,d\mathcal{S} = 2\pi\,v - \pi\,f. \tag{3.75}$$

Each curvilinear triangle has three sides, but the third condition of the triangulation states that each side belongs to exactly two curvilinear triangles. Therefore, if we multiply the number of triangles by three, we obtain twice the number of edges, that is, $3f = 2e$. This equality can be rewritten as $-f = -2e + 2f$. Substituting this relation into the right-hand side of the formula (3.75) and dividing then both sides by 2π, we obtain

$$\frac{1}{2\pi}\int\limits_{\mathfrak{S}} K\,d\mathcal{S} = v - e + f = \chi(\mathfrak{S}). \tag{3.76}$$

$\square$

3.11 Euler Characteristic

In the previous section, the Gauss–Bonnet formula was derived and the Gauss–Bonnet theorem was proven. These results lead to a number of important geometric consequences. Moreover, the Euler characteristic plays a central role in the Gauss–Bonnet theorem. In the previous section, a definition of this important topological invariant of a surface was given. However, to fully understand the nature of this integer, it is essential to compute the Euler characteristic for specific surfaces. This is one of the main objectives of the present section.

Let $\mathfrak{S}$ be a surface containing a curvilinear triangle whose sides are arc-length parameterized curves. Then, for this curvilinear triangle, the Gauss–Bonnet formula (3.72) holds. In the proof of the Gauss–Bonnet theorem, using the interior angles $\alpha_1, \alpha_2, \alpha_3$ of the curvilinear triangle, we expressed this formula in the form

$$\alpha_1 + \alpha_2 + \alpha_3 = \pi + \iint\limits_V K\, d\mathcal{S} + \int\limits_{\xi} \kappa_g\, ds. \tag{3.77}$$

Now suppose that the sides of the curvilinear triangle are arcs of geodesic curves on the surface $\mathfrak{S}$. In this case, the last integral on the right-hand side of formula (1) vanishes, since the geodesic curvature of a geodesic curve is zero. Thus, for a geodesic triangle on the surface $\mathfrak{S}$, we have

$$\alpha_1 + \alpha_2 + \alpha_3 = \pi + \iint\limits_V K\, d\mathcal{S}. \tag{3.78}$$

If we assume that the Gaussian curvature is zero at every point of the surface, we obtain a surface whose intrinsic geometry coincides with that of the Euclidean plane. In this case, formula (1) becomes the well-known property of a triangle in the Euclidean plane — namely, that the sum of the interior angles of a triangle is equal to π. If the Gaussian curvature is positive at every point of a geodesic curvilinear triangle, that is, $K > 0$, then the sum of the interior angles of the curvilinear triangle is greater than π, meaning $\alpha_1 + \alpha_2 + \alpha_3 > \pi$. Thus, if we take a sphere of radius R as our surface and consider a curvilinear triangle on this sphere formed by arcs of great circles (geodesics), the sum of its interior angles will exceed π, with the excess over π equal to the area of the geodesic triangle multiplied by $\frac{1}{R^2}$. This fact is well known in spherical geometry. Similarly, if the Gaussian curvature of the surface is negative, $K < 0$, as is the case, for example, with a catenoid, then the sum of the interior angles of a geodesic triangle will be less than π. This property of triangles holds in hyperbolic geometry.

Let us now proceed to compute the Euler characteristic of compact, orientable surfaces. First, let us examine how broad the class of orientable compact (boundaryless) two-dimensional surfaces is. It turns out that, up to homeomorphism, this class of surfaces consists of the following types. The simplest orientable and compact surface is the sphere. If we cut two disks out of a sphere and attach a cylinder in such a way that the bases of the cylinder (two circles) match the boundaries of the removed disks, we obtain a so-called sphere with one handle. Clearly, a sphere with one handle is homeomorphic to a torus. If we repeat this operation, that is, attach another handle, we get a sphere with two handles, which is homeomorphic to the surface shown in Figure (3.1). By continuing this process, we obtain a sphere with g handles. The number of attached handles is called the *genus of the surface*. If we denote the sphere with g handles by $\mathscr{T}_g$, then the genus of this surface is g. Naturally, if $g = 0$, the surface $\mathscr{T}_0$ is a sphere.

Theorem 3.6. If a surface is an orientable compact 2-dimensional surface in 3-dimensional Euclidean space, it is homeomorphic to one of the surfaces $\mathscr{T}_g$.

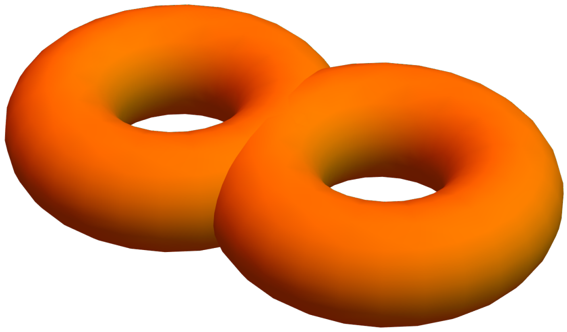

Figure 3.1: Double torus or surface of genus 2

This theorem provides a clear understanding of how broad the class of orientable compact surfaces in three-dimensional Euclidean space is.

Since homeomorphic surfaces have the same Euler characteristic, it is sufficient to compute the Euler characteristic for a surface of genus g. Let us start with the sphere. We inscribe a tetrahedron into the sphere and consider an arbitrary point on the surface of this tetrahedron, excluding its vertices. Connect this point with the center of the sphere and extend the resulting line until it intersects the surface of the sphere. In this way, we obtain a mapping of the points on the surface of the tetrahedron to points on the surface of the sphere. Note that the vertices of the tetrahedron already lie on the sphere, since the tetrahedron is inscribed in it. Under this mapping, the faces of the tetrahedron are transformed into curvilinear triangles on the sphere, and we obtain a triangulation of the sphere, which contains 4 vertices, 6 edges, and 4 faces. Thus, the Euler characteristic of the sphere is

$$\chi(\mathscr{T}_0) = 4 - 6 + 4 = 2.$$

To compute the Euler characteristic of the torus $\mathscr{T}_1$, we use a topological model of this surface. By a topological model, we mean a rectangle with opposite sides identified. We subdivide this rectangle into triangles as shown in Figure (3.2). This subdivision is a triangulation of the torus. The triangulation of the torus contains 9 vertices, 27 edges,

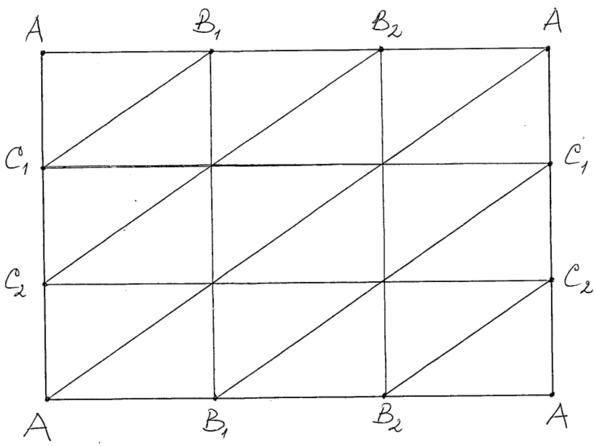

Figure 3.2: Triangulation of a torus

and 18 faces (triangles). Thus, the Euler characteristic of the torus is 9-27+18=0, i.e., $\chi(\mathscr{T}_1) = 0$. Thus, we see that adding one more handle decreases the Euler characteristic by 2. We arrive at the conclusion that

$$\chi(\mathscr{T}_g) = 2 - 2g.$$

3.12 Riemannian Geometry

In this section, we describe the central idea of Riemannian geometry and illustrate it with several simple examples. In the previous sections of this chapter, the main object of study was surfaces. Let us take a closer look at the notion of a surface. We defined a surface $\mathfrak{S}$ as a subset of three-dimensional space that can locally be parametrized by coordinate charts (ϕ, U), where $\phi : U \to \mathfrak{S}$ is a bijective map from an open subset U of the (u, v)-plane to a portion of the surface $\phi(U) \subset \mathfrak{S}$. If we introduce rectangular coordinates in the ambient three-dimensional space via an orthonormal frame, a local chart can be written in coordinates as $x = x(u, v),\ y = y(u, v),\ z = z(u, v)$, where $(u, v) \in U$. Since the local chart is now described by three functions of two variables, we can use the notion of smoothness, i.e., we require these functions to be continuously differentiable of any order. Thus, we arrive at the notion of a coordinate chart on a surface as a smooth bijective map from U to $\mathfrak{S}$.

Which element of this construction can we discard in order to obtain a more general concept than that of a surface in three-dimensional space? Observe that the intrinsic geometry of a surface plays a fundamental role in its study. However, in the case of a surface embedded in three-dimensional Euclidean space, its intrinsic geometry is induced by the surrounding ambient space. Indeed, to compute the coefficients of the first fundamental form of the surface, $g_{11} = E,\ g_{12} = g_{21} = F,\ g_{22} = G$, we calculate scalar products of the basis vectors of the tangent plane, $\|\phi_u\|^2, \langle \phi_u, \phi_v \rangle, \|\phi_v\|^2$, in the ambient three-dimensional Euclidean space. In other words, the Euclidean geometry of the surrounding space generates the intrinsic geometry of the surface.

Thus, if we wish to investigate the various possible intrinsic geometries of a surface, we must abandon the assumption that the surface is embedded in some ambient space. What, then, should we retain from the previously described structure? Clearly, we must keep the requirement that the set under consideration is a topological space M. In the case of a surface, its topology is induced from the topology of the ambient three-dimensional space. Once we remove this ambient space from the definition, we explicitly require the surface to possess a topology, and to ensure that the resulting model closely resembles a surface, we impose additional conditions: our topological space M must satisfy the Hausdorff separation axiom and possess a countable basis. An essential part of the structure of a surface is its local parametrizations (ϕ, U). Since it is not fundamentally important whether we consider the mapping $\phi : U \to \mathfrak{S}$ or its inverse, we express the property of local parametrization as follows: our topological space M is locally homeomorphic to an open subset of $\mathbb{R}^2$. More precisely, for any point $p \in M$, there exists a neighborhood V — an open subset containing that point — that is homeomorphic to an open subset of $U \subset \mathbb{R}^2$, i.e. $\phi : V \subset M \to U \subset \mathbb{R}^2$, where ϕ is a homeomorphism. As in the case of surfaces, we call the pair (ϕ, V) a local chart.

However, at this stage of generalization, we encounter a problem: the issue of smoothness. We need a notion of smoothness in order to study the geometry of M using the methods of differential and integral calculus. Naturally, one defines the smoothness of a function f defined on M by requiring it to be smooth in every local chart (ϕ, V). Let u, v be the corresponding local coordinates. Then, by restricting the function f to the subset $V \subset M$, we may consider it as a function of two variables: $f(u,v) = (f \circ \phi^{-1})(u,v)$, and in this case, the notion of smoothness is well defined.

However, if two charts (ϕ, V) and (ψ, W) overlap, that is, if $V \cap W \neq \emptyset$, then on the intersection of these charts, the function f transitions from one set of coordinates to another. This transition is described by the mapping $\psi \circ \phi^{-1}$ (or equivalently $\phi \circ \psi^{-1}$), which is continuous. The problem is that a continuous change of variables does not necessarily preserve smoothness. Therefore, there may be an inconsistency in the smoothness of the function f across different local charts. If we require that the transition maps $\psi \circ \phi^{-1}$ and $\phi \circ \psi^{-1}$ be smooth, however, then smoothness becomes a consistent and well-defined property across all charts.

Definition 3.9. A topological space M is called a differentiable two-dimensional manifold if the following conditions are satisfied:

- M is a Hausdorff space with a countable basis;
- M is locally homeomorphic to open subsets of $\mathbb{R}^2$; that is, for every point $p \in M$, there exists a pair (ϕ, V), called a local chart of M, where $V \subset M$ is an open neighborhood of p and $\phi : V \to U \subset \mathbb{R}^2$ is a homeomorphism of V onto U;
- If two local charts (ϕ, V) and (ψ, W) overlap (i.e., $V \cap W \neq \emptyset$), then the transition map $\phi \circ \psi^{-1}$ is a diffeomorphism.

It should be noted that by replacing the two-dimensional space $\mathbb{R}^2$ with the n-dimensional space $\mathbb{R}^n$, the given definition naturally extends to the concept of a differentiable n-dimensional manifold.

The first fundamental structure in the differential geometry of surfaces is the notion of a tangent vector and the tangent plane. In the case of a manifold, we adopt the approach developed in Chapter 2. According to Definition 3.9, we can locally identify the manifold M with an open subset of $\mathbb{R}^2$. This means that we associate a point $p \in M$ with the point $\phi(p) = (u,v) \in \mathbb{R}^2$, and the tangent space T_pM at p with the tangent plane to $\mathbb{R}^2$ at the point $\phi(p)$.

As seen in Chapter 2, the tangent plane to $\mathbb{R}^2$ at the point (u,v) is spanned by the vectors $\partial/\partial u$ and $\partial/\partial v$, which we denote ∂_u and ∂_v. Accordingly, the two-dimensional tangent space T_pM in local coordinates u, v is considered to be spanned by the basis vectors ∂_u, ∂_v. When changing from one coordinate chart (u,v) to another $(\tilde{u}, \tilde{v})$, where $\tilde{u} = \tilde{u}(u,v), \tilde{v} = \tilde{v}(u,v)$, the basis vectors transform according to the rule

$$\partial_u = \frac{\partial \tilde{u}}{\partial u}\,\partial_{\tilde{u}} + \frac{\partial \tilde{v}}{\partial u}\,\partial_{\tilde{v}}, \quad \partial_v = \frac{\partial \tilde{u}}{\partial v}\,\partial_{\tilde{u}} + \frac{\partial \tilde{v}}{\partial v}\,\partial_{\tilde{v}}.$$

We now turn to the central idea of Riemannian geometry. It is easy to see that a manifold M does not inherently possess a geometric structure that would allow us to measure lengths of curves or angles between them. In the case of a surface, such measurements are made using the arc length differential or the first fundamental form of the surface, which, as previously noted, is induced by the scalar product in the ambient three-dimensional space in which the surface is embedded.

Thus, in order to construct a geometry on a manifold, we must introduce an additional structure—a scalar product on the tangent vectors at each point of the manifold. From a general point of view, a scalar product is a real-valued, symmetric, positive-definite bilinear form on a vector space. Therefore, to define a geometry on the manifold, we introduce this additional structure, which we shall refer to as a metric on the manifold.

Definition 3.10. A two-dimensional manifold M is called a Riemannian manifold if, at every point $p \in M$, there is defined a symmetric, positive-definite bilinear form

$$g_p : T_pM \times T_pM \to \mathbb{R}$$

that depends smoothly on the local coordinates of the manifold in any coordinate chart (ϕ, V).

We denote a Riemannian manifold by (M, g), where g is called the Riemannian metric of the manifold. In local coordinates u, v, the metric is completely determined by its values on the basis vectors:

$$g_{11} = g(\partial_u, \partial_u), \quad g_{12} = g_{21} = g(\partial_u, \partial_v), \quad g_{22} = g(\partial_v, \partial_v).$$

Then the Riemannian metric can be written in terms of differentials of local coordinates as follows

$$g = g_{11}\, du^2 + 2\, g_{12}\, du\, dv + g_{22}\, dv^2. \tag{3.79}$$

It is convenient to represent the components of the Riemannian metric as a second-order matrix

$$(g_{\alpha\beta}) = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix},$$

where $\alpha, \beta = 1, 2$. Now the computation of curve length on the Riemannian manifold becomes possible and should be carried out using the same formulas as in the case of curves on a surface. For example, if a curve is given in local coordinates of the manifold $u^1 = u, u^2 = v$ by $\xi : u^1 = u^1(t),\ u^2 = u^2(t)$, where $a \le t \le b$, then its length is calculated by the formula

$$\ell = \int_a^b \sqrt{g_{\alpha\beta}(\xi(t))\, \frac{du^\alpha}{dt} \frac{du^\beta}{dt}}\, dt.$$

However, not all formulas from the theory of surfaces carry over to the case of a Riemannian manifold. We can no longer use concepts or formulas that rely on the existence of a normal vector to the surface. The normal vector is orthogonal to the tangent plane of a surface and points in the third dimension of the ambient space in which the surface is embedded. However, in the case of a two-dimensional manifold, there is no such third dimension transverse to the manifold. Hence, the shape operator and the second fundamental form of a surface are no longer applicable in the case of a manifold. It follows that we can only use those formulas from surface theory which involve solely the components of the metric, or which can be reduced to expressions depending only on these components. First of all, we observe that the Christoffel symbols can be computed entirely in terms of the metric (3.42). Thus, our first step is to compute the Christoffel symbols of the Riemannian manifold. Then, using the Gauss equation (3.50), we can compute the numerator of the fraction appearing in the formula for Gaussian curvature (3.31) or (3.47). The next step is to calculate the Gaussian curvature of the Riemannian manifold. We will illustrate how this procedure works through specific examples.

Example 3.2. In this example, the manifold is the plane, and we construct a Riemannian metric on it using the stereographic projection of the unit sphere onto the plane. We assume that in three-dimensional Euclidean space with Cartesian coordinates x, y, z, a unit sphere $\mathscr{S}^2$ is given, centered at the origin. The points $N(0, 0, 1)$ and $S(0, 0, -1)$ are called the north and south poles of the sphere, respectively. The plane $\mathfrak{P}$, on which we construct the Riemannian metric, is the tangent plane to the sphere $\mathscr{S}^2$ at the south pole, i.e., at the point S. We assume that this plane is equipped with a Cartesian coordinate system u, v, with the origin at the point of tangency S, and such that the u-axis is parallel to the x-axis and the v-axis is parallel to the y-axis of the three-dimensional space. Remove the north pole N from the sphere and denote the resulting surface by $\mathscr{S}_0$. Then $\mathscr{S}_0$ can be mapped bijectively onto the plane $\mathfrak{P}$ as follows. Let $p \in \mathscr{S}_0$. Draw a ray from the north pole through the point p until it intersects the plane $\mathfrak{P}$. Denote the point of intersection by q. Clearly, the mapping $p \mapsto q$ defines a one-to-one correspondence between $\mathscr{S}_0$ and $\mathfrak{P}$. This mapping is called the *stereographic projection* of the sphere onto the plane. If (x, y, z) are the coordinates of the point p on the sphere and (u, v) are the coordinates of the point q on the plane, then each set of coordinates can be expressed in terms of the other as follows

$$\begin{aligned} u &= \frac{2x}{1-z}, \quad v = \frac{2y}{1-z}, \\ x &= \frac{4u}{4+u^2+v^2}, \quad y = \frac{4v}{4+u^2+v^2}, \quad z = \frac{-4+u^2+v^2}{4+u^2+v^2}, \end{aligned} \tag{3.80}$$

where $-1 \leq z < 1$. It follows from these formulas that the stereographic projection is a diffeomorphism.

On the sphere, we have the standard metric—its first fundamental form—when viewed as a surface embedded in three-dimensional space. This metric is induced by the Euclidean metric $dx^2 + dy^2 + dz^2$ of $\mathbb{R}^3$. Our next step is to transfer the spherical metric onto the plane via the stereographic projection; in other words, we aim to express the metric of the sphere in terms of the planar coordinates u, v. In doing so, we obtain a new metric on the plane, distinct from the Euclidean metric $du^2 + dv^2$, and the plane, equipped with this new metric, becomes a Riemannian manifold.

We combine the coordinates of the plane $\mathfrak{P}$ into a single complex variable $\zeta = u + i\, v$. Making use of (3.80), we get

$$\zeta = \frac{2x}{1-z} + i\, \frac{2y}{1-z}, \quad \bar{\zeta} = \frac{2x}{1-z} - i\, \frac{2y}{1-z}$$

Then

$$\begin{aligned} d\zeta &= 2\,\Big(\frac{(1-z)\,dx + x\,dz}{(1-z)^2} + i\,\frac{(1-z)\,dy + y\,dz}{(1-z)^2}\Big), \\ d\bar{\zeta} &= 2\,\Big(\frac{(1-z)\,dx + x\,dz}{(1-z)^2} - i\,\frac{(1-z)\,dy + y\,dz}{(1-z)^2}\Big). \end{aligned}$$

Thus

$$\begin{aligned} d\zeta\, d\bar{\zeta} &= 4\Big(\big(\frac{(1-z)\,dx + x\,dz}{(1-z)^2}\big)^2 + \big(\frac{(1-z)\,dy + y\,dz}{(1-z)^2}\big)^2\Big) \\ &= 4\,\frac{\big((1-z)\,dx + xdz\big)^2 + \big((1-z)\,dy + ydz\big)^2}{(1-z)^4} \\ &= 4\,\frac{(1-z)^2(dx^2+dy^2) + (x^2+y^2)\,dz^2 + 2(1-z)(x\,dx + y\,dy)\,dz}{(1-z)^4} \end{aligned}$$

Substituting $x^2 + y^2 = -1$ and $x\,dx + y\,dy = -z\,dz$, we obtain

$$d\zeta\, d\bar{\zeta} = 4\,\frac{dx^2 + dy^2 + dz^2}{(1-z)^2}. \tag{3.81}$$

The spherical metric can be now expressed in terms of plane coordinates as follows

$$dx^2 + dy^2 + dz^2 = \frac{(1-z)^2}{4}\, d\zeta\, d\bar{\zeta} = \frac{(1-z)^2}{4}\,(du^2 + dv^2). \tag{3.82}$$

To eliminate z in the resulting expression and rewrite it in terms of the coordinates u, v, we compute

$$|\zeta|^2 + 4 = \frac{4x^2}{(1-z)^2} + \frac{4y^2}{(1-z)^2} = 4\left(\frac{1+z}{1-z} + 1\right) = \frac{8}{1-z}. \tag{3.83}$$

Expressing $1-z$ and substituting into (3.82), we finally obtain

$$g = \frac{16}{(4+u^2+v^2)^2}\left(du^2 + dv^2\right). \tag{3.84}$$

The plane equipped with the metric (3.84) is a Riemannian manifold known as the *spherical plane*. Note that the metric (3.84) differs from the standard Euclidean metric on the plane, $du^2 + dv^2$, by a factor λ^2, where

$$\lambda = \frac{4}{4+u^2+v^2}.$$

Hence $g = \lambda^2\,\left(du^2 + dv^2\right)$ and Riemannian metrics of this form are called *conformal*.

As noted above, the spherical plane is constructed by transferring the spherical metric onto the plane via stereographic projection. This means that the stereographic projection is an isometry between $\mathscr{S}_0$ (the sphere with the north pole removed) and $\mathfrak{P}$ (the plane). Since isometries preserve Gaussian curvature, it follows that the Gaussian curvature of the spherical plane is equal to that of the unit sphere, i.e., $K = 1$.

4 Solutions

Shape operator of a surface

- **3.4.1** We compute the matrix of the shape operator of the helicoid and then find its eigenvalues and eigenvectors, that is, the principal curvatures and principal directions. As a basis of the tangent plane to the surface, we take the tangent vectors ϕ_u and ϕ_v. We compute

$$\phi_u = (\cos v,\ \sin v,\ 0), \quad \phi_v = (-u \sin v,\ u \cos v,\ a).$$

We compute the normal vector of the helicoid using the normalized cross product of the vectors ϕ_u and ϕ_v. We get

$$\phi_u \times \phi_v = (a\ \sin v, -a\ \cos v, u), \ \ \|\phi_u \times \phi_v\| = \sqrt{u^2 + a^2},$$

and

$$\mathtt{N} = (\frac{a\ \sin v}{\sqrt{u^2+a^2}}, -\frac{a\ \cos v}{\sqrt{u^2+a^2}}, \frac{u}{\sqrt{u^2+a^2}}).$$

To compute the matrix of the shape operator S, we use the definition $S(\phi_u) = -\mathsf{N}_u$, $S(\phi_v) = -\mathsf{N}_v$, and decompose the resulting vectors with respect to the basis ϕ_u, ϕ_v. We get

$$\begin{aligned} \mathtt{N}_u &= \frac{a}{(u^2+a^2)^{3/2}}(-u \sin v,\ u \cos v,\ a) = \frac{a}{(u^2+a^2)^{3/2}}\ \phi_v, \\ \mathtt{N}_v &= \frac{a}{(u^2+a^2)^{1/2}}(\cos v,\ \sin v,\ 0) = \frac{a}{(u^2+a^2)^{1/2}}\ \phi_u. \end{aligned}$$

Hence the matrix of the shape operator (we denote it by the same letter S) has the form

$$S = \begin{pmatrix} 0 & -\frac{a}{(u^2+a^2)^{1/2}} \\ -\frac{a}{(u^2+a^2)^{3/2}} & 0 \end{pmatrix}.$$

The characteristic equation $\det(S - \lambda E) = 0$ for this matrix has the form

$$\lambda^2 = \frac{a^2}{(u^2+a^2)^2}.$$

Thus we get two eigenvalues

$$\lambda_{1,2} = \pm\ \frac{a}{u^2+a^2}.$$

We have found the principal curvatures of the right helicoid

$$\kappa_1 = \frac{a}{u^2+a^2}, \quad \kappa_2 = -\frac{a}{u^2+a^2}.$$

We now find the eigenvectors of the matrix of the shape operator. We denote an eigenvector by (du, dv), understanding it as a tangent vector to the (u, v)-plane, attached to the point (u, v). In the case of κ_1 we have

$$\begin{pmatrix} 0 & -\frac{a}{(u^2+a^2)^{1/2}} \\ -\frac{a}{(u^2+a^2)^{3/2}} & 0 \end{pmatrix} \begin{pmatrix} du \\ dv \end{pmatrix} = \pm \frac{a}{u^2+a^2} \begin{pmatrix} du \\ dv \end{pmatrix}.$$

From this we obtain

$$\frac{du}{dv} = \mp\sqrt{u^2+a^2}. \tag{4.1}$$

This equation determines two principal directions of the right helicoid in the (u, v)-plane. To visualize these directions, one should draw two lines through the point (u, v) in the plane with slopes $+\sqrt{u^2+a^2}$ and $-\sqrt{u^2+a^2}$. To obtain the principal directions in the tangent plane to the helicoid at the point $\phi(u, v)$, we lift the tangent vector (du, dv) to the surface, resulting in the tangent vector $du\,\phi_u + dv\,\phi_v$ to the surface. Using equation (4.1), we obtain two vectors

$$\mathtt{v}_1 = \sqrt{u^2+a^2}\,\phi_u + \phi_v, \quad \mathtt{v}_2 = -\sqrt{u^2+a^2}\,\phi_u + \phi_v.$$

that serve as vectors of the principal directions. It is easy to verify that these vectors are perpendicular to each other. ← back to exercise