1 Geometry of Manifolds

1.1 Smooth manifold

A powerful tool for studying the geometry of space is the coordinate method proposed by Descartes. The effectiveness of this method is based on the fact that a coordinate system in space allows us to consider, instead of a point in space, an ordered set of numbers, i.e. the coordinates of the point, and then apply the methods of algebra, differential, and integral calculus. For example, by introducing affine coordinates in an affine space with the help of an affine frame, we can study the geometric properties of lines and planes through the corresponding equations. By extending the class of coordinate systems, we can investigate the geometric properties of affine space by means of curvilinear coordinates defined in some open subset of the affine space.

In this chapter, we give the definition of a manifold, which historically represented the next stage in the development of the concept of space. A prototype of a manifold may be taken as a surface in three-dimensional Euclidean space. A surface, in the neighborhood of each of its points, can be parametrized by means of two parameters, which makes it possible to identify it with an open subset of the plane $\mathbb{R}^2$. However, the surface lies in three-dimensional Euclidean space, and the geometry of this ambient space influences the geometry of the surface. Taking as a basis the local property of the surface and eliminating the surrounding space, we arrive at the concept of a manifold. We begin with the definition of a topological manifold.

Definition 1.1. An n-dimensional topological manifold is a topological space M^n that satisfies the following requirements:

- M^n is a Hausdorff space with a countable basis of topology,
- for any point $x \in M^n$ there exists an open subset $U \subset M^n$ containing it such that U is homeomorphic to an open subset $U' \subset \mathbb{R}^n$, that is, $\phi : U \to \phi(U) = U' \subset \mathbb{R}^n$, where ϕ is a homeomorphism.

From the given definition it follows that locally a topological manifold looks like an n-dimensional Euclidean space $\mathbb{R}^n$. A pair (ϕ, U), where U is an open subset of M^n and ϕ is a homeomorphism, is called a *local chart* of the topological manifold M^n. A local chart defines a coordinate system on the open subset U, since we can identify a point $x \in U$ with an ordered set of real numbers $\phi(x) = (x^1, x^2, \ldots, x^n) \in U' \subset \mathbb{R}^n$. A collection of local charts (ϕ_α, U_α), where α runs through some index set $\mathcal{A}$, is called an atlas of the topological manifold M^n if the open subsets $U_\alpha, \alpha \in \mathcal{A}$ form an open covering of the manifold M^n, that is, $\cup_{\alpha\in\mathcal{A}} U_\alpha = M^n$. An atlas is called *maximal* if it contains all possible local charts of the given topological manifold.

The class of topological manifolds is very broad. Within this class we can study the topological properties of a manifold. In order to be able to study the structure of a manifold using the methods of differential and integral calculus, we must have the notion of differentiability. We can introduce this notion by imposing additional requirements on the structure of a topological manifold.

Let $(\phi, U), (\psi, V)$ be two local charts of a topological manifold M^n, with $U \cap V \neq \emptyset$. Each local chart defines a coordinate system. Thus, on the intersection of subsets $U \cap V$ we have two coordinate systems, that is, if $x \in U \cap V$, then in the coordinate system (ϕ, U) this point has coordinates $\phi(x) = (x^1, x^2, \dots, x^n)$, and in the coordinate system (ψ, V) its coordinates are denoted by $\psi(x) = (\tilde{x}^1, \tilde{x}^2, \dots, \tilde{x}^n)$. One set of coordinates can be expressed in terms of the other by means of the mutually inverse mappings

$$\psi \circ \phi^{-1} : \phi(U \cap V) \subset \mathbb{R}^n \to \psi(U \cap V) \subset \mathbb{R}^n, \tag{1.1}$$
$$\phi \circ \psi^{-1} : \psi(U \cap V) \subset \mathbb{R}^n \to \phi(U \cap V) \subset \mathbb{R}^n. \tag{1.2}$$

Indeed, a mapping from $\mathbb{R}^n$ to $\mathbb{R}^n$ consists of an ordered set of components, which are ordinary real-valued functions of n variables. Let us introduce the notation $\psi \circ \phi^{-1} = (f^1, f^2, \dots, f^n), \phi \circ \psi^{-1} = (g^1, g^2, \dots, g^n)$. Then we can write

$$\tilde{x}^i = f^i(x^1, x^2, \dots, x^n), \quad x^i = g^i(\tilde{x}^1, \tilde{x}^2, \dots, \tilde{x}^n), \ \ i = 1, 2, \dots, n. \tag{1.3}$$

The mappings (1.1), (1.2) are called the *transition functions* from the local chart (ϕ, U) to the local chart (ψ, V) of the topological manifold M^n. It is obvious that these mappings are continuous and, moreover, they are homeomorphisms.

As already noted above, the transition functions from one local chart of a manifold to another are mappings from $\mathbb{R}^n$ to $\mathbb{R}^n$ and, therefore, consist of an ordered set of real-valued functions of n variables. Thus, we can impose an additional requirement on the transition functions, namely, that they be smooth. Recall that a function of n variables is called *smooth* if in its domain of definition there exist partial derivatives of all orders and they are continuous functions. A smooth function is also called infinitely differentiable. A mapping $\varphi : U \subset \mathbb{R}^n \to \varphi(U) = U' \subset \mathbb{R}^n$, where $\varphi = (\varphi^1, \varphi^2, \dots, \varphi^n)$, is called smooth if all its components φ^i are smooth functions. A mapping φ is called a *diffeomorphism* if it is bijective, smooth, and its inverse is also smooth. Thus, the notion of diffeomorphism is an analogue of the notion of homeomorphism.

Definition 1.2. A smooth structure on a topological manifold M^n is called a maximal atlas $\mathfrak{A} = \{(\phi\_\alpha, U\_\alpha), \alpha \in \mathcal{A}\}$ such that the transition functions for any two local charts of this atlas are diffeomorphisms. By the maximality of the atlas $\mathfrak{A}$ we mean the following: if (ϕ, U) is a local chart of the topological manifold M^n such that the transition functions $\phi\_\alpha \circ \phi^{-1}, ; \phi \circ \phi\_\alpha^{-1}$ are smooth for any $\alpha \in \mathcal{A}$, then (ϕ, U) belongs to the atlas $\mathfrak{A}$. A topological manifold M^n endowed with a smooth structure is called a *smooth manifold.*

In what follows we will consider only smooth manifolds. Therefore, we will often omit the word "smooth" and call a smooth manifold simply a manifold. In the above definition of a smooth manifold we used the class of smooth functions. In the theory of real functions one also considers the class of analytic functions, that is, functions which in a neighborhood of each point of their domain can be represented by convergent power series. If in the definition of a smooth manifold we replace the requirement of smoothness of the transition functions by the requirement of analyticity, we obtain the notion of a *real analytic manifold.* If in the definition of a smooth manifold we replace the real n-dimensional space $\mathbb{R}^n$ by the n-dimensional complex space $\mathbb{C}^n$ and require that all transition functions of the maximal atlas be holomorphic, we arrive at the notion of an n-dimensional *complex analytic manifold.* A one-dimensional complex analytic manifold is called a *Riemann surface.*

Example 1.1. As the first example of an n-dimensional smooth manifold, let us consider the n-dimensional affine space. Let $\{O; \vec{e}_1, \vec{e}_2, \dots, \vec{e}_n\}$ be a frame of the n-dimensional affine space. A frame determines a system of affine coordinates, where the coordinates of a point P of the affine space are defined as the coordinates of its radius vector $\overrightarrow{OP} = x^i\,\vec{e}_i$ (with summation over the index i from 1 to n). It is obvious that the mapping of the affine space to $\mathbb{R}^n$, associating to each point its coordinates, is a bijection. A subset of the affine space is called open if its image under the mapping to the coordinate space $\mathbb{R}^n$ is an open subset of $\mathbb{R}^n$. It is clear that in such a topology the mapping of the affine space to $\mathbb{R}^n$ is a homeomorphism. Thus, we have constructed a coordinate chart that covers the whole affine space, i.e. it is a global coordinate system. This global coordinate chart defines the standard smooth structure on the affine space. Indeed, to obtain the maximal atlas we must adjoin to the constructed affine coordinate system all other coordinate systems on the affine space or its open subsets that are related to the above coordinate system by smooth transition functions. This means that all affine coordinate systems will belong to this atlas, since in this case the transition functions have the form

$$x^i = A^i_j\,\tilde{x}^j + c^i, \tag{1.4}$$

where $\tilde{x}^i$ are new affine coordinates, $A = (A^i_j)$ is the matrix of transition to another basis $\vec{e}_i{}' = A^j_i\,\vec{e}_j$, and $(c^1, c^2, \ldots, c^n)$ are the coordinates of the new origin. Thus, the affine transformation (1.4) defines a mapping from $\mathbb{R}^n$ to $\mathbb{R}^n$, which is a bijection (non-degeneracy of the transition matrix A), smooth (linear functions), and its inverse is also smooth. Therefore, (1.4) is a diffeomorphism. It follows that the standard smooth structure does not depend on the choice of the frame. In addition to affine coordinate systems, the maximal atlas will also contain all possible curvilinear coordinate systems. Recall that the curvilinear coordinates $\acute{x}^1, \acute{x}^2, \ldots, \acute{x}^n$ in the affine space are defined by the formulas

$$\acute{x}^i = f^i(x^1, x^2, \ldots, x^n), \quad i = 1, 2, \ldots, n, \tag{1.5}$$

where x^i are affine coordinates, and f^i are smooth functions of n variables defining a diffeomorphism of the domain of definition of these functions onto the range of their values. Since affine space is a smooth manifold, we may with full justification say that the concept of a smooth manifold is a generalization of affine space.

Example 1.2. The $(n-1)$-dimensional unit sphere S^{n-1} is defined as the set of points in $\mathbb{R}^n$ whose coordinates satisfy the equation

$$(x^1)^2 + (x^2)^2 + \ldots + (x^n)^2 = 1.$$

The unit sphere S^{n-1} is a subset of the topological space $\mathbb{R}^n$. Therefore, we can endow it with the topology induced by the topology of $\mathbb{R}^n$, that is, we call a subset of the sphere open if it is the intersection of the sphere with an open subset of $\mathbb{R}^n$. Clearly, the induced topology inherits all the properties of the topology of $\mathbb{R}^n$, that is, it satisfies the Hausdorff separation axiom and has a countable basis. Let N, S be the points of the sphere with coordinates $(0, 0, \ldots, 0, 1)$ and $(0, 0, \ldots, 0, -1)$, respectively. We call them the north and south poles of the sphere, respectively. We construct an atlas of the sphere consisting of two local charts using the *stereographic projection*. To construct the first local chart, we use the north pole N. Let P be a point of the sphere different from the north pole. From the north pole we draw a ray passing through the point P until it intersects the hyperplane $x^n = -1$. Note that this hyperplane passes through the south pole and is tangent to the sphere at that point. Denote the obtained intersection point by Q. Thus, we have constructed a mapping of the sphere S^{n-1} onto an $(n-1)$-dimensional hyperplane, which can be identified with the space $\mathbb{R}^{n-1}$. This mapping is called the stereographic projection of the sphere from the point N onto the plane $x^n = -1$. This mapping is a bijection, it is continuous and its inverse is also continuous, i.e. we have a homeomorphism. Thus, the first coordinate chart covers the entire sphere except the north pole, i.e. it is a local chart. Similarly, we construct the second local chart, where we project the point P from the south pole S onto the hyperplane $x^n = 1$ and denote the projection by R. The union of the charts covers the entire sphere, i.e. it is an atlas. To show that the constructed atlas is smooth, we compute the transition functions. First, we compute the coordinates of Q. Here we mean that we take as the coordinates of the points Q, R their coordinates as points of the hyperplanes $x^n = -1$ and $x^n = 1$, respectively, but we do not write out the last coordinate x^n, since this coordinate is constant and equal either to -1 or to 1. Let $(x_0^1, x_0^2, \ldots, x_0^n)$ be the coordinates of the point P, $(\xi^1, \xi^2, \ldots, \xi^{n-1})$ the coordinates of the point Q, and $(\eta^1, \eta^2, \ldots, \eta^{n-1})$ those of the point R. The parametric equation of the line passing through the points N and P has the form

$$x^1 = x_0^1\, t,\ x^2 = x_0^2\, t,\ \ldots,\ x^{n-1} = x_0^{n-1}\, t,\ x^n = (x_0^n - 1)\, t + 1.$$

Since Q is the point of intersection of this line with the hyperplane $x^n = -1$, solving the system of equations we find

$$\xi^i = \frac{2\, x_0^i}{1 - x_0^n}.$$

Similarly, in the case of the point R we obtain

$$\eta^i = \frac{2\, x_0^i}{1 + x_0^n}.$$

It follows that

$$\eta^i = \frac{1 - x_0^n}{1 + x_0^n}\, \xi^i. \tag{1.6}$$

Note that the vector $\overrightarrow{NP}$ is collinear with the vector $\overrightarrow{NQ}$. From this it follows that

$$\frac{x_0^1}{\xi^1} = \frac{x_0^2}{\xi^2} = \ldots = \frac{x_0^{n-1}}{\xi^{n-1}} = \frac{1 - x_0^n}{2}.$$

Therefore

$$x_0^i = \frac{1 - x_0^n}{2}\, \xi^i, \quad i = 1, 2, \ldots, n-1. \tag{1.7}$$

The point P lies on the sphere, hence its coordinates satisfy the equation $\sum_{i=1}^n (x_0^i)^2 = 1$. Substituting the right-hand sides of equalities (1.7) for the first $n-1$ coordinates, we obtain

$$\frac{1 - x_0^n}{1 + x_0^n} = \frac{4}{\sum_{i=1}^{n-1} (\xi^i)^2}.$$

Substituting the obtained formula into (1.6), we obtain the transition functions

$$\eta^i = \frac{4\xi^i}{\sum_{i=1}^{n-1} (\xi^i)^2}. \tag{1.8}$$

The domain of definition of these functions is the entire hyperplane $x = -1$, except for the zero point. The functions are rational functions, and the denominator does not vanish. Therefore, the transition functions are smooth. By completing the constructed atlas to a maximal one, we obtain a smooth structure on the $(n-1)$-dimensional unit sphere.