Cherni klassid

Olgu $M_r(\mathbb{C})$ r-järku ruutmaatriksite vektorruum. Olgu $\varphi: M_r(\mathbb{C})\times\ldots\times M_r(\mathbb{C})$ (k korda) $\longrightarrow \mathbb{C}$ polülineaarne $\mathbb{C}$-väärtustega vorm. Vormi φ nimetatakse invariantseks kui $\forall g\in GL_r(\mathbb{C})$ korral kehtib

$$\varphi(g^{-1}A_1 g, g^{-1}A_2 g, \ldots, g^{-1}A_k g) = \varphi(A_1, A_2, \ldots, A_k),$$

kus $A_1, A_2, \ldots, A_k \in M_r(\mathbb{C})$. Erijuhul, kui $A_1 = \ldots = A_k = A$ saame

$$\varphi(A) = \varphi(A, A, \ldots, A).$$

On ilmne, et $\varphi(A)$ on maatriksi A elementide k-astme polünoom, mis rahuldab

$$\varphi(g^{-1}A g) = \varphi(A), \quad g\in GL_r(\mathbb{C}).$$

Sel juhul φ nim. invariantseks polünoomiks. Olgu $A\in M_r(\mathbb{C})$. Moodustame maatriksi $I+A$, kus I on $r\times r$-ühikmaatriks. Maatriksi $I+A$ determinandi kirjutame kujul

$$\mathrm{Det}(I+A) = \sum_{k=0}^{r} \Phi_k(A),$$

kus $\Phi_k(A)$ on maatriksi A elementide k-astme polünoom. Siis $\Phi_k(A)$, $k=0,1,\ldots,r$ on invariantsed polünoomid.

Tõepoolest, $A \to g^{-1} A g$, siis

$$\mathrm{Det}(I + g^{-1}Ag) = \sum_{k=0}^{r} \Phi_k(g^{-1}Ag)$$

$$\| $$

$$\mathrm{Det}(g^{-1}(I + A)g) = \mathrm{Det}(g^{-1})\,\mathrm{Det}(I+A)\cdot\mathrm{Det}(g) =$$

$$= 1$$

$$= \mathrm{Det}(I+A) = \sum_{k=0}^{r} \Phi_k(A).$$

Järelikult

$$\Phi_k(g^{-1}Ag) = \Phi_k(A)$$

Arvutame juhul, kui $r = 3$

$$\mathrm{Det}(I+A) = \begin{vmatrix} 1+a^1_1 & a^1_2 & a^1_3 \\ a^2_1 & 1+a^2_2 & a^2_3 \\ a^3_1 & a^3_2 & 1+a^3_3 \end{vmatrix} = (1+a^1_1)(1+a^2_2)(1+a^3_3) +$$

$$+ a^1_2 a^2_3 a^3_1 + a^1_3 a^2_1 a^3_2 - a^1_3(1+a^2_2)a^3_1 - a^1_2 a^2_1(1+a^3_3) -$$

$$-(1+a^1_1)a^2_3 a^3_2 = 1 + (a^1_1 + a^2_2 + a^3_3) + a^1_1 a^2_2 + a^1_1 a^3_3 + a^2_2 a^3_3 -$$

$$- a^1_3 a^3_1 - a^1_2 a^2_1 - a^2_3 a^3_2 + a^1_1 a^2_2 a^3_3 + a^1_2 a^2_3 a^3_1 + a^1_3 a^2_1 a^3_2 -$$

$$- a^1_3 a^2_2 a^3_1 - a^1_2 a^2_1 a^3_3 - a^1_1 a^2_3 a^3_2 = \underbrace{1}_{\Phi_0(A)} + \underbrace{\mathrm{Tr}\,A}_{\Phi_1(A)} + \underbrace{\tfrac{1}{2}\left((\mathrm{Tr}\,A)^2 - \mathrm{Tr}(A^2)\right)}_{\Phi_2(A)}$$

$$\mathrm{Tr}(A^2) - (\mathrm{Tr}\,A)^2 = \cancel{a^1_1 a^1_1} + a^1_2 a^2_1 + a^1_3 a^3_1 + a^2_1 a^1_2 + \cancel{(a^2_2)^2} + a^2_3 a^3_2 +$$

$$+ a^3_1 a^1_3 + a^3_2 a^2_3 + \cancel{(a^3_3)^2} - \cancel{(a^1_1)^2} - \cancel{(a^2_2)^2} - \cancel{(a^3_3)^2} -$$

$$-2a^1_1 a^2_2 - 2a^1_1 a^3_3 - 2a^2_2 a^3_3 = 2(a^1_2 a^2_1 + a^2_3 a^3_2 + a^1_3 a^3_1 -$$

$$- a^1_1 a^2_2 - a^1_1 a^3_3 - a^2_2 a^3_3)$$

$$+ \mathrm{Det}\, A. \quad \text{Seega}$$

$$\Phi_0(A) = 1,$$
$$\Phi_1(A) = \mathrm{Tr}\, A$$
$$\Phi_2(A) = \frac{1}{2}\left[(\mathrm{Tr}\, A)^2 - \mathrm{Tr}\,(A^2)\right]$$
$$\Phi_3(A) = \mathrm{Det}\, A.$$

üldine valem

$$\Phi_k(A) = \frac{1}{k!} \sum_{\substack{(i_1 \dots i_k) \\ (j_1 \dots j_k)}} \delta^{i_1 \dots i_k}_{j_1 \dots j_k} A^{j_1}_{i_1} A^{j_2}_{i_2} \dots A^{j_k}_{i_k},$$

kus $i_1, i_2, \dots, i_k$ on erinevad arvud hulgast $1, 2, \dots, r$,
$j_1, j_2, \dots, j_k$ on erinevad arvud hulgast $1, 2, \dots, r$,

$$\delta^{i_1 i_2 \dots i_k}_{j_1 j_2 \dots j_k} = \begin{cases} 1, & \text{kui } i_1, i_2, \dots, i_k \text{ on arvude } j_1, j_2, \dots, j_k \text{ paarispermutatsioon} \\ -1, & \dots \text{ paaritu permutatsioon} \\ 0, & \{i_1, i_2, \dots, i_k\} \neq \{j_1, j_2, \dots, j_k\}. \end{cases}$$

Kontrollime, kui $k = 2$. Siis

$$\delta^{i_2}_{j_1} A^{j_1}_{j_2} A^{j_2}_{i_2}$$

$$\Phi_2(A) = \frac{1}{2} \sum \delta^{i_1 i_2}_{j_1 j_2} A^{j_1}_{i_1} A^{j_2}_{i_2} =$$

$$= \frac{1}{2} \sum \left(\delta^{i_1}_{j_1} \delta^{i_2}_{j_2} - \delta^{i_1}_{j_2} \delta^{i_2}_{j_1} \right) A^{j_1}_{i_1} A^{j_2}_{i_2} =$$

$$= \frac{1}{2} \left(\mathrm{Tr}(A) \cdot \mathrm{Tr}(A) - \mathrm{Tr}(A^2) \right) = \frac{1}{2} \left((\mathrm{Tr}\, A)^2 - \mathrm{Tr}(A^2) \right)$$

Olgu $(\Theta^i_j) = \Theta_D$ seostuse D kõverusvorm. Näitame, et $\Phi_K(\Theta_D)$ on kinnine $2K$-vorm muutkonnal M.

Kehtib

$$\Phi_K(\Theta_D) = \frac{1}{K!}\sum_{(i)(j)} \delta^{i_1\dots i_K}_{j_1\dots j_K}\, \Theta^{j_1}_{i_1}\wedge\dots\wedge\Theta^{j_K}_{i_K}.$$

Näitame, et $\Phi_K(\Theta_D)$ on kinnine $2K$-vorm juhul, kui $K=2$.

Sellisel juhul

$$\Phi_K(\Theta_D) = \frac{1}{2}\left(\overline{Tr}\,\Theta_D\wedge\overline{Tr}\,\Theta_D - \overline{Tr}(\Theta_D\wedge\Theta_D)\right).$$

Arvutame välisdiferentsiaali

$$d\,\Phi_K(\Theta_D) = \frac{1}{2}\Big(d\,\overline{Tr}\,\Theta_D\wedge\overline{Tr}\,\Theta_D + \overline{Tr}\,\Theta_D\wedge d\,\overline{Tr}\,\Theta_D - d\,\overline{Tr}(\Theta_D\wedge\Theta_D)\Big)$$

Kehtib

$$d\,\overline{Tr} = \overline{Tr}\,d$$

Seega

$$d\,\Phi_K(\Theta_D) = \frac{1}{2}\Big(\overline{Tr}\,d\Theta_D\wedge\overline{Tr}\,\Theta_D + \overline{Tr}\,\Theta_D\wedge\overline{Tr}\,d\Theta_D - \overline{Tr}(d\Theta_D\wedge\Theta_D) - \overline{Tr}(\Theta_D\wedge d\Theta_D)\Big) =$$

$$= \frac{1}{2}\Big(\underbrace{\overline{Tr}\,[\Theta_D,\theta]\wedge\overline{Tr}\,\Theta_D} + \underbrace{\overline{Tr}\,\Theta_D\wedge\overline{Tr}\,[\Theta_D,\theta]} - \overline{Tr}([\Theta_D,\theta]\wedge\Theta_D) - \overline{Tr}(\Theta_D\wedge[\Theta_D,\theta])\Big)$$

$$\overline{Tr}\,[\Theta_D,\theta]\wedge\overline{Tr}\,\Theta_D = [\Theta_D,\theta]^\mu_\mu\wedge\Theta^\nu_\nu =$$

$$= (\Theta^\mu_\sigma\wedge\theta^\sigma_\mu - \theta^\mu_\sigma\wedge\Theta^\sigma_\mu)\wedge\Theta^\nu_\nu =$$

$$= (\Theta^\mu_\sigma\wedge\theta^\sigma_\mu - \Theta^\mu_\sigma\wedge\theta^\sigma_\mu)\wedge\Theta^\nu_\nu = 0$$

Analoogiliselt $\mathrm{Tr}\,\Theta_D \wedge \mathrm{Tr}[\Theta_D, \theta] = 0$

Nüüd

$$\mathrm{Tr}\left([\Theta_D, \theta] \wedge \Theta_D\right) = \Theta^{\mu}{}_{\nu} \wedge \theta^{\nu}{}_{\sigma} \wedge \Theta^{\sigma}{}_{\mu} - \theta^{\mu}{}_{\nu} \wedge \Theta^{\nu}{}_{\sigma} \wedge \Theta^{\sigma}{}_{\mu} =$$

$$= \theta^{\mu}{}_{\nu} \wedge \Theta^{\nu}{}_{\sigma} \wedge \Theta^{\sigma}{}_{\mu} - \theta^{\mu}{}_{\nu} \wedge \Theta^{\nu}{}_{\sigma} \wedge \Theta^{\sigma}{}_{\mu} = 0$$

Analoogiliselt $\mathrm{Tr}\left(\Theta_D \wedge [\Theta_D, \theta]\right) = 0$.