Kõverus

M on muutkond, $E \xrightarrow{\pi} M$ on vektorkihtkond, $\dim M = m$, $\operatorname{rank} E = r$, indeksid $i, j, k = 1, 2, .., m$, $\mu, \nu, \sigma, \rho = 1, 2, .., r$. T^*M on kaaspuutujakihtkond, selle vektorkihtkonna kiht on puutujaruumi kaasruum $T_x M \to T_x^* M$. Moodustame vektorkihtkonna $\Lambda^p T^* M$. Selle vektorkihtkonna kiht punktis M on p-kovektorite vektorruum. Kui $x \in M$, siis kihi $\Lambda^p T_x^* M$ element on p-kovektor ω, st, kui $v_1, v_2, \ldots, v_p \in T_x M$ on muutkonna puutujavektorid, siis

$$\omega(\underbrace{v_1, v_2, .., v_p}_{\text{kaldsümmeetriline}}) \in \mathbb{R}$$

Kui võtame baasid

$$\partial_i = \frac{\partial}{\partial x^i}, \quad dx^i, \quad dx^i(\partial_j) = \delta^i_j$$

siis

$$\omega = \frac{1}{p!}\, \omega_{i_1 i_2 \ldots i_p}\, dx^{i_1} \wedge \ldots \wedge dx^{i_p}$$

Moodustame vektorkihtkonna

$$E^p = \Lambda^p T^* M \otimes E.$$

Selle kihtkonna lõige on E-väärtustega p-diferentsiaalvorm. Lokaalne struktuur

$$\omega = \frac{1}{p!}\, \omega^{\mu}_{i_1 \ldots i_p} (dx^{i_1} \wedge \ldots \wedge dx^{i_p}) \otimes e_\mu .$$

Kui $X_1, X_2, \ldots X_p$ on lokaalsed vektorväljad, siis

$$\omega(X_1, X_2, \ldots, X_p) = \underbrace{\omega^\mu(X_1, X_2, \ldots, X_p)}_{\text{funktsioonid}} e_\mu ,$$

st

$$(\omega^\mu_{i_1 \ldots i_p} \otimes e_\mu)(X_1, X_2, \ldots, X_p) = \omega^\mu_{i_1 \ldots i_p}(X_1, \ldots, X_p) \cdot e_\mu$$

Kuidas teiseneb seostuse maatriks? Olgu $f \to f \cdot g$, $e'_\mu = e_\nu g^\nu_\mu$. Kehtib

$$D e'_\mu = \tilde{\theta}^\nu_\mu \otimes e'_\nu = \tilde{\theta}^\nu_\mu \otimes (e_\sigma g^\sigma_\nu) =$$

$$= g^\sigma_\nu \tilde{\theta}^\nu_\mu \otimes e_\sigma .$$

Teiselt poolt

$$D e'_\mu = D(e_\nu \cdot g^\nu_\mu) = d g^\nu_\mu \otimes e_\nu + g^\nu_\mu D(e_\nu) =$$

$$= d g^\nu_\mu \otimes e_\nu + g^\nu_\mu (\theta^\rho_\nu \otimes e_\rho) =$$

$$= d g^\sigma_\mu \otimes e_\sigma + g^\nu_\mu \theta^\sigma_\nu \otimes e_\sigma = (d g^\sigma_\mu + \theta^\sigma_\nu g^\nu_\mu) \otimes e_\sigma$$

Seega

$$g^\sigma_\nu \tilde{\theta}^\nu_\mu = d g^\sigma_\mu + \theta^\sigma_\nu g^\nu_\mu$$

või

$$g \tilde{\theta} = dg + \theta \cdot g , \quad \boxed{\tilde{\theta} = g^{-1} \theta g + g^{-1} dg}$$

$$\boxed{\tilde{A}_\mu = g^{-1} A_\mu g + g^{-1} \partial_\mu g}$$

kalibratsiooni teisendus

Defineerime (lokaalselt)

$$\boxed{\Theta^\mu_\nu = d\vartheta^\mu_\nu + \vartheta^\mu_\sigma \wedge \vartheta^\sigma_\nu}$$

$\Theta = (\Theta^\mu_\nu)$ on seostuse D kõverus. Θ on r-järku ruutmaatriks, mille elemendid on 2-vormid. Leiame kuidas teiseneb kõveruse maatriks. Teistes kihi koordinaatides

$$\tilde{\Theta}_D = d\tilde{\vartheta} + \tilde{\vartheta}\wedge\tilde{\vartheta} =$$

$$= d(g^{-1}\vartheta g + g^{-1}dg) + (g^{-1}\vartheta g + g^{-1}dg)\wedge(g^{-1}\vartheta g + g^{-1}dg) =$$

$$= dg^{-1}\cdot\vartheta\cdot g + \underline{g^{-1}d\vartheta g} - \cancel{g^{-1}\vartheta\, dg} + dg^{-1}\wedge dg + \underline{g^{-1}\vartheta\wedge\vartheta g} +$$

$$\cancel{g^{-1}\vartheta\wedge dg} + (g^{-1}dg\, g^{-1})\wedge\vartheta g + g^{-1}dg\wedge g^{-1}dg =$$

$$d(g^{-1}g)=0,\quad dg^{-1}g + g^{-1}dg = 0,\quad dg^{-1} = -g^{-1}dg\, g^{-1}$$

$$= \cancel{-g^{-1}dg\, g^{-1}\vartheta\cdot g} - \cancel{g^{-1}dg\, g^{-1}\wedge dg} +$$

$$+\cancel{(g^{-1}dg\, g^{-1})\wedge\vartheta\cdot g} + \cancel{g^{-1}dg\wedge g^{-1}dg} + g^{-1}\Theta g$$

Seega

$$\tilde{\Theta}_D = g^{-1}\Theta_D\, g$$

Moodustame vektorkihtkonna $\mathrm{End}(E)$. Selle vektorkihtkonna kiht on vektorkihtkonna E kihi $\pi^{-1}(x) = E_x$ lineaarteisendused. Kõverus on 2-vorm väärtustega selles vektorkihtkonnas. Moodustame

$$\Lambda^2 T^*M \otimes \mathrm{End}(E)$$

$$\Theta_D \in \mathcal{E}^2(M, \mathrm{End}\, E), \quad \Theta_D \in \mathcal{E}(M, E^2).$$

$$(d+\theta)(d+\theta)(\xi f) = (d+\theta)(d\xi + \theta\xi) =$$

$$= d\theta\,\xi - \cancel{\theta\, d\xi} + \cancel{\theta\, d\xi} + \theta\wedge\theta\,\xi =$$

$$= (d\theta + \theta\wedge\theta)\xi = \Theta_D\, \xi.$$

$$d\Theta^\mu_\nu = d(d\theta^\mu_\nu + \theta^\mu_\sigma\wedge\theta^\sigma_\nu) = d\theta^\mu_\sigma\wedge\theta^\sigma_\nu - \theta^\mu_\sigma\wedge d\theta^\sigma_\nu =$$

$$= [\Theta, \theta]^\mu_\nu = (d\theta^\mu_\tau + \theta^\mu_\sigma\wedge\theta^\sigma_\tau)\wedge\theta^\tau_\nu - \theta^\mu_\tau\wedge(d\theta^\tau_\nu + \theta^\tau_\rho\wedge\theta^\rho_\nu)$$

$$= d\theta^\mu_\tau\wedge\theta^\tau_\nu + \cancel{\theta^\mu_\sigma\wedge\theta^\sigma_\tau\wedge\theta^\tau_\nu} - \theta^\mu_\tau\wedge d\theta^\tau_\nu - \cancel{\theta^\mu_\tau\wedge\theta^\tau_\rho\wedge\theta^\rho_\nu} =$$

$$= d\theta^\mu_\tau\wedge\theta^\tau_\nu - \theta^\mu_\sigma\wedge d\theta^\sigma_\nu.$$

Tõestatud Bianchi samasus

$$\boxed{d\Theta^\mu_\nu = [\Theta, \theta]^\mu_\nu.}$$

Maatriksite kommutaator, elemendid on diferentsiaalvormid. Definitsioon

$$[\omega, \theta]^\mu_\nu = \omega^\mu_\sigma\wedge\theta^\sigma_\nu - (-1)^{|\omega||\theta|}\,\theta^\mu_\sigma\wedge\omega^\sigma_\nu.$$

$$D: \mathcal{E}(M, E^p) \to \mathcal{E}(M, E^{p+1}),$$

$$D\rho = d\rho + \theta\wedge\rho \quad \text{(lokaalselt)}$$